Systems Dynamics

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Thomas Parisini
Gianfranco Fenu

University of Trieste
Department of Engineering and Architecture

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Lecture 2
Sampling and Reconstructing in Time Domain: Sampling and LTI Systems

Lecture 2: Table of Contents

Sampling and Reconstructing

From Continuous-Time to Discrete-Time

Remarks

- Till now we carried out a general treatment of dynamic systems considering both the continuous-time and the discrete-time cases
- Since the course is intended to cover **data-based** system dynamics, analysis and estimation, from now on only the discrete-time case will be dealt with
- However, before doing this, the issue of **conversion of a continuous-time into a discrete-time by sampling** has to be dealt with in some detail

Sampling and Reconstructing

Sampling and Reconstructing in Time Domain

From Continuous-Time to Discrete-Time: Signal Taxonomy

Continuous-time vs. discrete-time signals

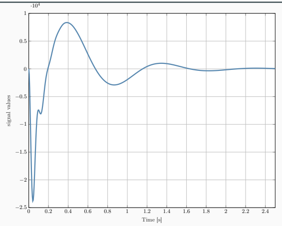
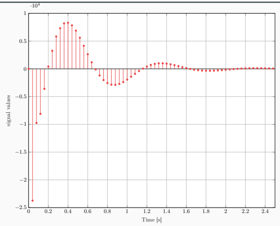
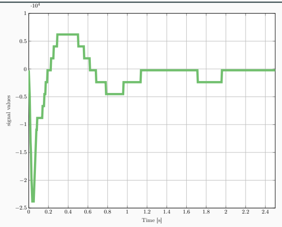
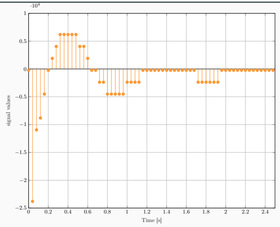
- **continuous-time signal**: a *function of time* (independent variable) $x = x(t)$, such that the independent variable **time is continuous**
  - the domain of the function $x = x(t)$ has the cardinality of the real numbers set $\mathbb{R}$.
- **discrete-time signal**: a signal $y = y(k)$, **specified** only **for discrete values of time** (the independent variable)
  - the domain of the function $y = y(k)$ has the cardinality of the integer numbers set $\mathbb{Z}$.
  - a discrete-time signal is usually called *sequence*

Signal Taxonomy (cont.)

Analog vs. digital signals

- **analog signal**: the **amplitude** of the signal may vary in a **continuous** range
  - an analog signal can be both continuous-time and discrete-time signal.
- **digital signal**: a signal whose **amplitude is quantized**, i.e. the amplitude of a digital signal can take only a finite number of values.
  - a digital signal can be both continuous-time and discrete-time signal.

Signal Taxonomy: Graphical Summary

	continuous-time	discrete-time
analog		
digital		

Sampling & Digital Coding: Main Issues

$$e(t) \rightsquigarrow e_k = e(k\,T_s) \rightsquigarrow 0\ 1\ 1\ 0\ 1\ 0 \ldots$$

The conversion of an analog, continuous-time signal $e = e(t)$ to a digital, discrete-time sequence is subject to two main issues:

- **loss of information**, due to the conversion from continuous-time to discrete-time (more details later)
- **quantisation noise and distortion**, due to the analog to digital conversion process

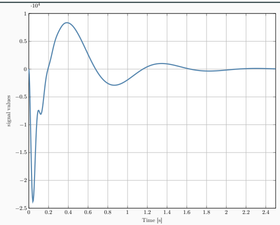
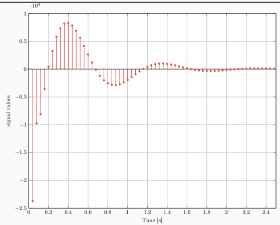
Sampling issues taken into account

- sampling and the loss of information, a glimpse on the theoretical motivations of, and how to cope with this issue are discussed topics
- quantisation and coding issues are not taken into account

Sampling & Digital Coding: Main Issues (cont.)

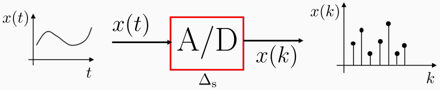
From now on, consider the **sampling procedure** simply as a **conversion** from an analog, **continuous-time signal** to an analog, **discrete-time signal**.

Moreover, hereafter each time-based signal will be labelled just as continuous-time or discrete-time signal.

	continuous-time	**discrete-time**
analog		

The Ideal Sampler

How to convert a continuous-time signal to a discrete-time one?



Periodic sampling using an *ideal sampler*

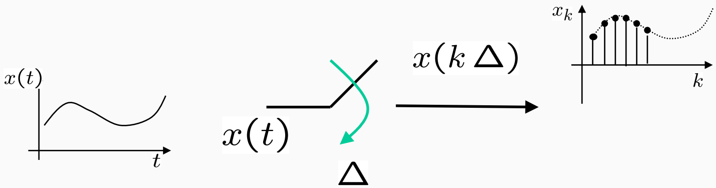
- the aim of the A/D converter is to transform a continuous-time signal $x(t)$ into a discrete-time sequence $x(k)$
- given a time interval Δ, called **sampling period**, applying a *periodic sampling* means to extract and collect, creating a *sequence*, values of the signal corresponding to time instants, integer multiples of the sampling period

$$\{x(k)\}_{k\in\mathbb{Z}} \Longrightarrow \{x(t)\ :\ t = k\,\Delta\,,\ k\in\mathbb{Z}\}$$

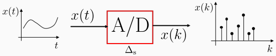
The Ideal Sampler (cont.)

An ideal sampler acts as an **ideal** electrical **switch**

- the switch commutes between the two states “open” and “closed”, driven by a periodic pulse signal (called the *clock signal*), with the time period equal to the sampling period Δ;
- when a clock pulse occurs, the switch closes instantaneously, the actual sample of the input signal can be “copied” into the sampler output and then the switch commutes (instantaneously) to the “open” state, waiting for the next clock pulse.



The Ideal Sampler (cont.)



Sampling rate
Given the sampling period Δ, let's define the *rate of conversion* from continuous to discrete time using

- **sampling angular frequency**

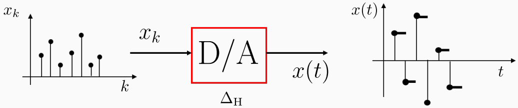
$$\Omega_s = \frac{2\pi}{\Delta} \quad [\text{rad/s}]$$

- **sampling frequency**

$$f_s = \frac{1}{\Delta} \quad [\text{Hz}]$$

The Reconstructor

Consider now the backward operation: how to characterize the conversion of a discrete-time signal to a continuous-time one?



Reconstruction using a data-holder

- the purpose of the D/A subsystem is to reconstruct the sampled signal into a form that resembles the original signal, before sampling.
- the simplest D/A subsystem [indeed the most common one] is the so-called **zero-order-hold (ZOH)**.

The Reconstructor (cont.)

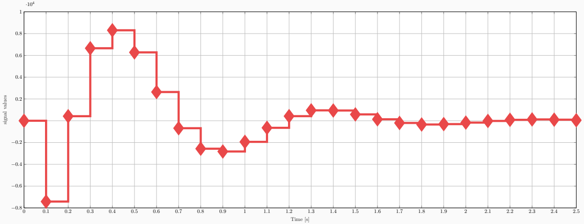
Reconstruction using a D/A converter (cont.)

- the ZOH clamps the output signal to a value corresponding to that of the input sequence at the current clock pulse, until the next clock pulse arrives.

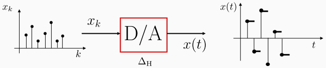
$$x(t) = x(k)\,,\;\; k\,\Delta_{\mathsf{H}} \leq t < (k+1)\;\Delta_{\mathsf{H}} \quad k \in \mathbb{Z}$$

- the time period Δ_{H} is called **holding period**.

Note that the output signal of a ZOH is a stair-wise signal



The Reconstructor (cont.)



Holding rate
Given the holding period Δ_H, let's define the *rate of conversion* for a D/A device using

- **holding angular frequency**

$$\Omega_H = \frac{2\pi}{\Delta_H} \quad [\mathrm{rad/s}]$$

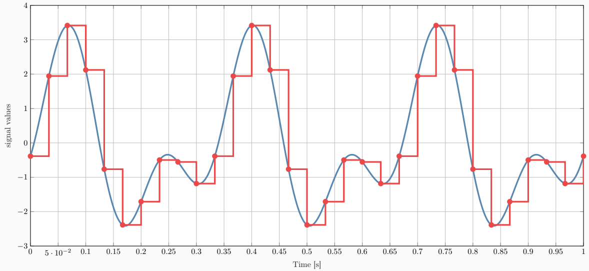
- **holding frequency**

$$f_H = \frac{1}{\Delta_H} \quad [\mathrm{Hz}]$$

Usually the sampling and holding frequencies have the same value.

Sampling and Reconstructing

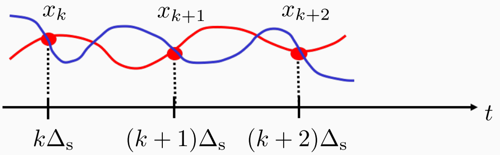
- What happens if a continuous-time signal is firstly sampled and then reconstructed? How is the output signal of the ZOH w.r.t the original continuous-time signal? The same or?
- Indeed, the output of the ZOH is a stair-wise signal, so **the reconstructed signal is different from the original one**: sampling and reconstruction are just approximately the opposite function of each other.



Sampling and Loss of Information

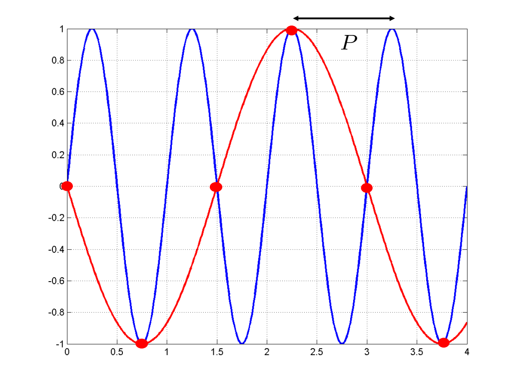
$$\left.\begin{array}{c} x_k\ ,\ \ k \in \mathbb{Z} \\ + \\ \text{a priori knowledge} \\ \text{of the signal features} \end{array}\right\} \longrightarrow x(t) = \text{?}$$

- In general, **reconstructing** the continuous-time signal starting from the samples is an **ill-posed problem**: the reconstruction may be ambiguous.



Sampling a Sinusoidal Signal

Consider the signal $x(t) = \sin\left(\bar{\Omega} t\right) \quad P = \dfrac{2\pi}{\bar{\Omega}}$



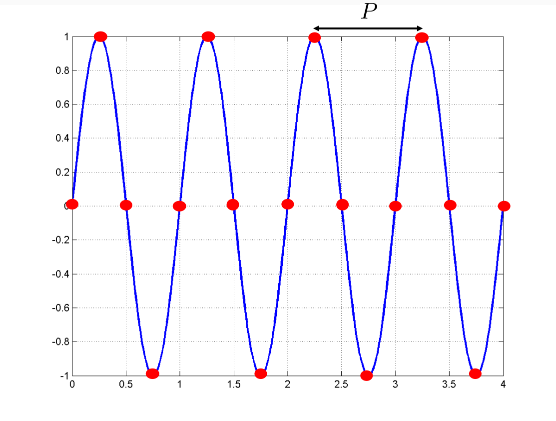
Select as sampling period the value

$$\Delta = \frac{3}{4} P = \frac{3\pi}{2\bar{\Omega}}$$

Indeed, it's easy to determine sinusoidal signals, with period $\hat{P} > P$, that may generate the same values, obtained by sampling $x(t)$.

Note: the frequency of the ambiguous signal is lower than the frequency of the original signal. This effect is called **frequency aliasing** (or *frequency fold-over*).

Sampling a Sinusoidal Signal (cont.)



Reducing the sampling period (i.e. increasing the sampling frequency) the ambiguity disappears: no more frequency fold-over effect.

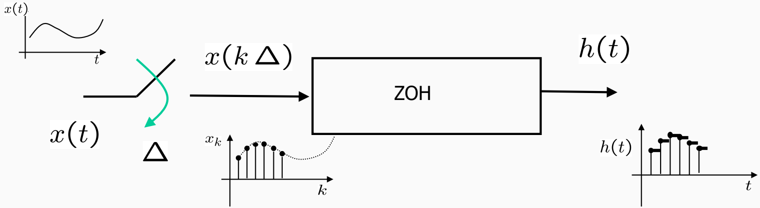
$$\Delta = \frac{P}{4} = \frac{\pi}{2\,\bar{\Omega}}$$

By choosing properly the sampling period, the frequency aliasing effect has been avoided. Note: the effective sampling frequency is much higher than the signal time frequency.

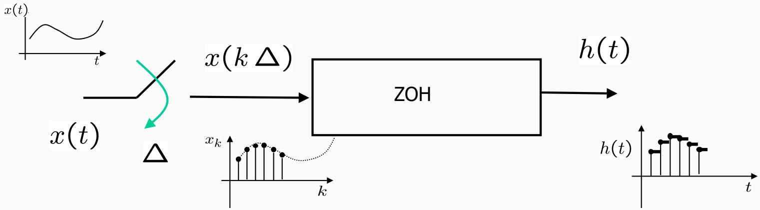
Ideal Sampler & ZOH: Mathematical Model

- So far, it has been illustrated by examples that, when sampling a simple sinusoidal signal, choosing properly the sampling period grants to avoid the aliasing effect.
- How to generalize? What is the effect of the sampling procedure? How does the choice of the sampling period influence the frequency aliasing effect?

The influence of the sampling period on the aliasing effect will be explained by modelling the direct connection of an ideal sampler to a ZOH (Δ is the sampling period)



Ideal Sampler & ZOH: Mathematical Model (cont.)



The output of the ZOH is a continuous-time signal, expressed as

$$h(k\Delta + \tau) = x(k\Delta)\,, \quad 0 \le \tau < \Delta\,,\; k = 0,\, 1,\, 2,\, 3\, \cdots$$ a stair-wise signal

$$\begin{aligned} h(t) &= x(0)\,[1(t) - 1(t-\Delta)] + x(\Delta)\,[1(t-\Delta) - 1(t-2\Delta)] + \cdots \\ &= \sum_{k=0}^{+\infty} x(k\Delta)\,[1(t-k\Delta) - 1\,(t-(k+1)\Delta)] \end{aligned}$$

Ideal Sampler & ZOH: Mathematical Model (cont.)

Applying the Laplace transform

$$\mathcal{L}\{1(t-k\Delta)\} = \frac{e^{-k\Delta s}}{s}$$

$$\mathcal{L}\{h(t)\} = H(s) = \sum_{k=0}^{+\infty} x(k\Delta)\,\frac{e^{-k\Delta s} - e^{-(k+1)\Delta s}}{s}$$

$$= \frac{1-e^{-\Delta s}}{s} \cdot \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}$$

function only of Δ

function of input signal $x(t)$ and sampling period Δ

Ideal Sampler & ZOH: Mathematical Model (cont.)

a transfer function model for the ZOH

the Laplace transform of ideal sampler's output as continuous-time signal

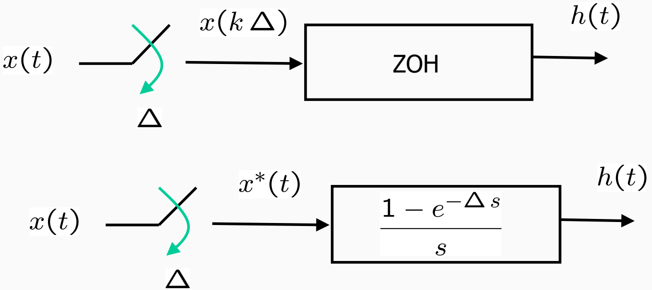
$$H(s) = \frac{1 - e^{-\Delta s}}{s} \cdot \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s} = G_{\text{ZOH}}(s)\, X^*(s)$$

where

$$G_{\text{ZOH}}(s) = \frac{1 - e^{-\Delta s}}{s} \qquad X^*(s) \triangleq \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}$$

Ideal Sampler & ZOH: Mathematical Model (cont.)

So far, we demonstrated the equivalence between the following two structures



where $x^*(t) = \mathcal{L}^{-1}\left\{X^*(s)\right\}$ $\qquad X^*(s) \triangleq \sum_{k=0}^{+\infty} x(k\Delta)\,e^{-k\Delta s}$

Ideal Sampler as Impulse Modulator

Note: $x^*(t)$ is a continuous-time representation of the ideal sampler output (indeed a sequence of samples)

$$x^*(t) = \mathcal{L}^{-1}\left\{X^*(s)\right\} = \mathcal{L}^{-1}\left\{\sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}\right\}$$

Now, recalling the main properties of the Dirac *delta function*

$$\mathcal{L}^{-1}\left\{e^{-k\Delta s}\right\} = \delta\left(t - k\Delta\right) \qquad \delta(t) = \begin{cases} 0 & \forall t \neq 0 \\ \text{undef.} & t = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \qquad \int_{-\infty}^{+\infty} f(t)\, \delta(t-\tau)\, dt = f(\tau)$$

$x^*(t)$ can be expressed as

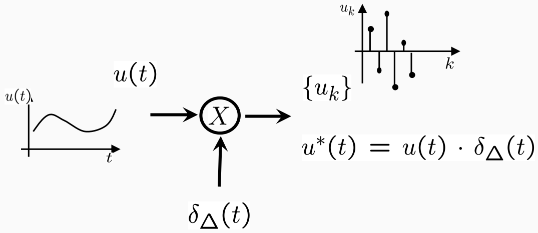
$$x^*(t) = \mathcal{L}^{-1}\left\{\sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta s}\right\} = \sum_{k=0}^{+\infty} x(k\Delta)\, \delta(t - k\Delta)$$

Ideal Sampler as Impulse Modulator (cont.)

$$
\begin{aligned}
x^*(t) &= \sum_{k=0}^{+\infty} x(k\Delta)\,\delta(t-k\Delta) \\
&= \sum_{k=0}^{+\infty} x(t)\,\delta(t-k\Delta) \\
&= x(t)\cdot \sum_{k=0}^{+\infty} \delta(t-k\Delta) \\
&= x(t)\cdot \delta_\Delta(t)
\end{aligned}
$$

where

$$\delta_\Delta(t) = \sum_{k=0}^{+\infty} \delta(t-k\Delta)$$



- $x^*(t)$ can be expressed as the result of the modulation of the original signal $x(t)$ with a train of Dirac impulses
- owing to this result, the ideal sampler is also referred as an **impulse modulator**

Sampling and Reconstructing

Sampling and Reconstructing using Laplace- and Z- Transform

Laplace- & Z- Transform of Ideal Sampler Output

- Since the output of the impulse modulator may be described as a continuous-time signal $x^*(t)$ but also as a discrete-time sequence $x(k\Delta)$, how to correlate such representations?
- Consider the Laplace-transform of $x^*(t)$ and the Z-transform of the sequence $x(k\Delta)$

$$\mathcal{L}\left\{x^*(t)\right\} = X^*(s) = \sum_{k=0}^{+\infty} x(k\Delta)e^{-k\Delta s}$$

$$\mathcal{Z}\left\{x(k\Delta)\right\} = X(z) = \sum_{k=0}^{+\infty} x(k\Delta)\, z^{-k}$$

It's easy to find that using the substitutions

$$z = e^{s\,\Delta} \qquad \Longleftrightarrow \qquad s = \frac{1}{\Delta}\,\log z$$

the Laplace transform may be rewritten as Z-transform and vice-versa.

Properties of $X^*(s)$: $X^*(s)$ vs $X(s)$

Definition: starred transform

The function $X^*(s) = \mathcal{L}\left\{x^*(t)\right\}$ is usually called the starred transform.

Property 1: the starred transform $X^*(s)$ vs. $X(s)$

The starred transform may be expressed as a scaled summation of infinite copies of the Laplace transform of the original analog signal $X(s) = \mathcal{L}\left\{x(t)\right\}$, shifted each other by $j\Omega_s$ (where $\Omega_s = \dfrac{2\pi}{\Delta}$ and Δ is the sampling period)

$$X^*(s) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\left(s - jk\Omega_s\right), \quad \Omega_s = \frac{2\pi}{\Delta}, \quad X(s) = \mathcal{L}\left\{x(t)\right\}$$

1^{st} Property of Starred Transform - Sketch of Proof

Proof - a sketch.

Recall the ideal sampler output expression

$$x^*(t) = \sum_{k=0}^{+\infty} x(k\Delta)\,\delta(t-k\Delta)$$

Remember: the original, analog signal $x(t)$ is a *causal signal*. Owing to this property, the summation may be modified

$$x(t) \equiv 0 \;\forall t < 0 \quad \Longrightarrow \; x^*(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta(t-k\Delta)$$

According to this modification, let's redefine also the *impulse train*

$$\delta_\Delta(t) = \sum_{k=-\infty}^{+\infty} \delta(t-k\Delta)$$

1st Property of Starred Transform - Sketch of Proof (cont.)

Now represent the ***impulse train*** as **Fourier series**

$$\delta_\Delta(t) = \sum_{k=-\infty}^{k=+\infty} C_\Delta(k) e^{jk\Omega_s t} \qquad \Omega_s = \frac{2\pi}{\Delta}$$

$$C_\Delta(k) = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \delta_\Delta(t)\, e^{-jk\Omega_s t}\, dt$$

$$= \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \delta(t)\, e^{-jk\Omega_s t}\, dt = \frac{1}{\Delta}$$

Thus

$$\delta_\Delta(t) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} e^{jk\Omega_s t}$$

1^{st} Property of Starred Transform - Sketch of Proof (cont.)

By substitution of the impulse train expression into the ideal sampler output $x^*(t)$, we obtain

$$x^*(t) = x(t) \cdot \delta_\Delta(t) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} x(t)\, e^{jk\Omega_s t}$$

Applying the Laplace transform

$$X^*(s) = \mathcal{L}\left\{x^*(t)\right\} = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} \int_{-\infty}^{+\infty} \left[x(t)\, e^{jk\Omega_s}\right] e^{-st}\, dt$$

Let's apply the bilateral Laplace transform to $x^*(t)$: remember, we rewrote $x^*(t)$ as it is non-causal signal

1st Property of Starred Transform - Sketch of Proof (cont.)

Thus

$$X^*(s) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} \int_{-\infty}^{+\infty} \left[x(t)\, e^{jk\Omega_s}\right] e^{-st}\, dt$$

Recall the Laplace transform property

$\mathcal{L}\left\{e^{kt}\, f(t)\right\} = F\,(s-k) \quad \forall k \in \mathbb{C}\,, \qquad F(s) = \mathcal{L}\left\{f(t)\right\}$

Finally

$$X^*(s) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\,(s - jk\Omega_s)\,, \quad k \in \mathbb{Z}\,, \quad \Omega_s = \frac{2\pi}{\Delta}$$

Properties of $\mathrm{X}^*(\mathrm{s})$: Periodicity of the Starred Transform

Property 2: the starred transform is periodic in s, with period $j\Omega_s$

$$X^*(s) = X^*\left(s + jn\Omega_s\right)\,, \quad n \in \mathbb{N}\,,\ \Omega_s = \frac{2\pi}{\Delta}$$

Proof.

$$X^*\left(s + jn\Omega_s\right) = \sum_{k=0}^{+\infty} x(k\Delta)\, e^{-k\Delta(s+jn\Omega_s)}$$

Since $\Omega_s \cdot \Delta = 2\pi$, applying the Euler's relationship $e^{j\theta} = \cos\theta + j\sin\theta$

$$e^{-jnk\Delta\Omega_s} = e^{-jnk2\pi} = 1 \quad \forall n\,,\ k \in \mathbb{N}$$

thus

$$X^*\left(s + jn\Omega_s\right) = \sum_{k=0}^{+\infty} x(k\Delta)e^{-k\Delta s} = X^*\left(s\right)$$

□

Properties of $\mathrm{X}^*(\mathrm{s})$: Poles of the Starred Transform

Property 3: poles of the starred transform vs poles of $\mathrm{X}(\mathrm{s})$

If $X(s)$ has a pole at $s = \hat{s}$,
then $X^*(s)$ must have poles at $s = \hat{s} + jk\Omega_s \ , \quad k \in \mathbb{Z}$

Proof.

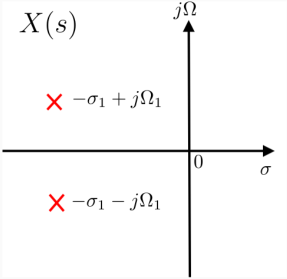
Rewrite the result of "Property 1"

$$
\begin{aligned}
X^*(s) =& \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\left(s - jk\Omega_s\right) \\
=& \frac{1}{\Delta} \Big[X(s) + X\left(s - j\Omega_s\right) + X\left(s - 2j\Omega_s\right) + \cdots \\
& + X\left(s + j\Omega_s\right) + X\left(s + 2j\Omega_s\right) + \cdots \Big]
\end{aligned}
$$

If $X(s)$ has a pole at $s = \hat{s}$, then each term of the latter expression will contribute with a pole at $s = \hat{s} - jk\Omega_s \ , \quad k \in \mathbb{Z}$.

Properties of $X^*(s)$: Poles of the Starred Transform (cont.)

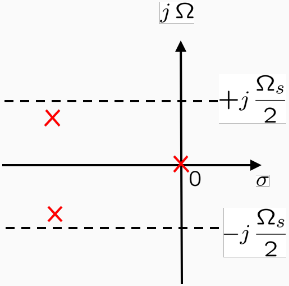
Poles map of the starred transform



- if $X(s)$ has a pole in $s = -\sigma_1 + j\Omega_1$, then the sampling operation will generate poles for $X^*(s)$ in
  $s = -\sigma_1 + j\Omega_1 \pm jk\Omega_s \;, \quad k \in \mathbb{Z}$
- on the contrary, if $X(s)$ has a pole in
  $s = -\sigma_1 + j(\Omega_1 + \Omega_s)$, then $X^*(s)$ will have a pole in
  $s = -\sigma_1 + j\Omega_1$
- pole locations in $X(s)$ at $s = -\sigma_1 + j(\Omega_1 \pm k\Omega_s) \;, \quad k \in \mathbb{Z}$ will result in identical pole locations in $X^*(s)$

Properties of $X^*(s)$: Poles of the Starred Transform (cont.)

Primary and complementaries strips in the s-plane

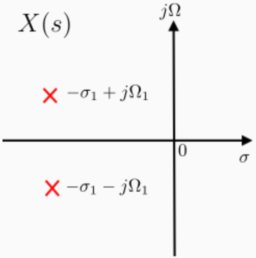


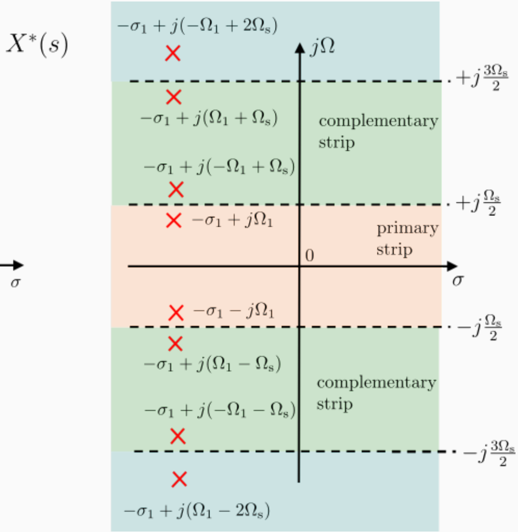
- consider the s-plane of the starred transform and divide it into strips
  - the **primary strip** is defined as the strip for which

$$\left\{ s:\ s \in \mathbb{C}\,,\ s = \sigma + j\Omega\,,\ -\frac{\Omega_s}{2} \leq \Omega \leq +\frac{\Omega_s}{2} \right\}$$

- if the pole-zero locations for the starred transform are known in the primary strip, then the pole-zero locations for $X^*(s)$ in the entire s-plane are known.

Properties of $X^*(s)$: Poles Map of Starred Transform





Properties of $X^*(s)$: Poles of The Starred Transform (cont.)

What about zeros of starred transform?

Indeed, the zeros of $X(s)$ **do not uniquely determine** the location of zeros of the starred transform $X^*(s)$. However, the zero locations of $X^*(s)$ are periodic, with period $j\Omega_s$ (Property 2).

Sampling and Reconstructing

Sampling, Reconstructing and Aliasing in the Frequency Domain

Laplace & Fourier Transform of a Causal, Continuous-time Signal

Consider a causal, continuous-time signal $x(t)$. The **unilateral Laplace transform** of such a signal is defined as

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^{+\infty} x(\tau)e^{-s\tau}\,d\tau$$

whereas the **Fourier transform** is

$$\mathcal{F}\{x(t)\} = X(\Omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\Omega\tau}\,d\tau$$

Exploiting the signal causality, the Fourier transform may be rewritten as

$$\mathcal{F}\{x(t)\} = X(\Omega) = \int_0^{+\infty} x(\tau)e^{-j\Omega\tau}\,d\tau = \mathcal{L}\{x(t)\}\big|_{s=j\Omega}$$

provided that both transforms exist.

Sampling and Aliasing in the Frequency Domain

Starred transform result in the frequency domain

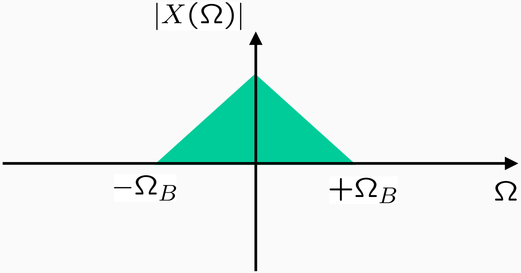
Analysing the starred signal $x^*(t)$ by applying **the Fourier transform** (instead of the Laplace one), provides the same result:

the Fourier transform of the starred signal may be expressed as a scaled summation of infinite copies of the Fourier transform of the original analog signal

$$X^*(\Omega) = \frac{1}{\Delta} \sum_{k=-\infty}^{k=+\infty} X\left(\Omega - k\Omega_{\mathsf{s}}\right) \,, \quad \Omega_{\mathsf{s}} = \frac{2\pi}{\Delta} \,, \quad X(\Omega) = \mathcal{F}\left\{x(t)\right\}$$

Band-limited Signals

Suppose that the signal $x(t)$ is a so-called **band-limited signal**, i.e. the amplitude spectrum $|X(\Omega)|$ of the signal is non zero only if $|\Omega| \leq \Omega_B$ (where $X(\Omega) = \mathcal{F}\{x(t)\}$).



What happens when we acquire such a signal by sampling? In particular, what if $\Omega_s > 2\Omega_B$, $\Omega_s = 2\Omega_B$ or $\Omega_s < 2\Omega_B$?

Band-limited Signals (cont.)

Band-limited signal

In rigorous terms, a signal is called a *band-limited signal* if

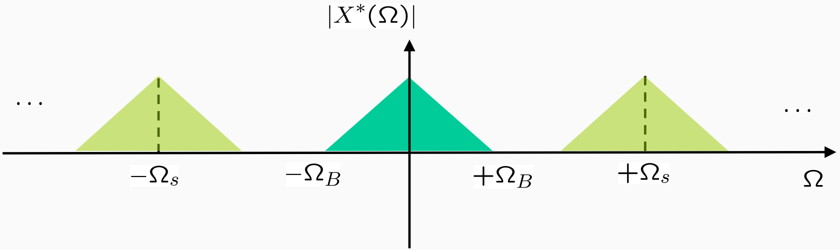
$$x(t) = \sum_{k=1}^{k=N} \alpha_k \sin\left(\Omega_k t + \varphi_k\right) \ , \quad \Omega_k \leq \Omega_{\mathsf{B}} \quad \forall k$$

or

$$x(t) = \int_0^{\Omega_{\mathsf{B}}} \alpha(\Omega) \sin\left[\Omega t + \varphi(\Omega)\right] \, d\Omega \ , \quad \Omega \in [0 \ , \quad \Omega_{\mathsf{B}}]$$

Sampling and Aliasing in the Frequency Domain (cont.)

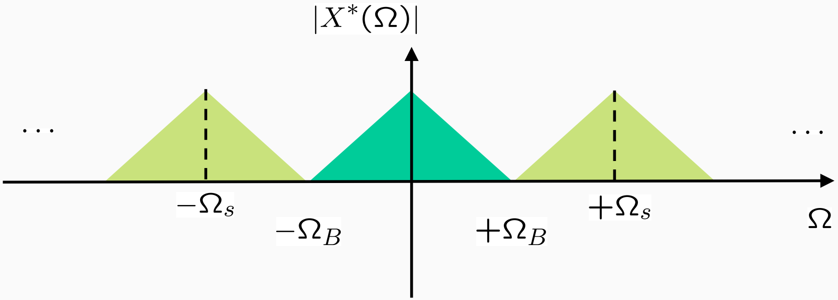
$$\Omega_s > 2\Omega_B$$



- no overlapping of spectra, so **no aliasing**
- to **reconstruct** the original signal (to isolate the original spectrum) a **realizable** (causal) **low-pass filter** is needed

Sampling and Aliasing in the Frequency Domain (cont.)

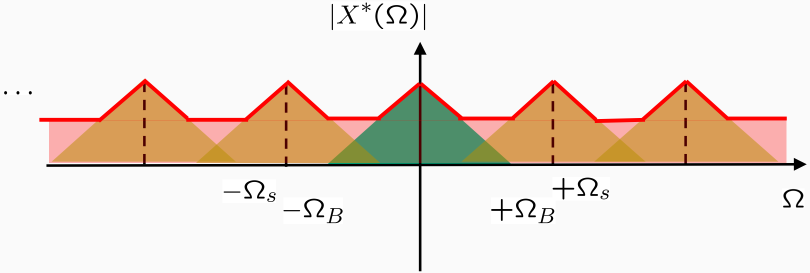
$$\Omega_s = 2\Omega_B$$



- still no overlapping of spectra, so **no aliasing**
- to reconstruct the original signal (to isolate the original spectrum) an **ideal (non-causal) low-pass filter** is needed

Sampling and Aliasing in the Frequency Domain (cont.)

$$\Omega_s < 2\Omega_B$$



- overlapping of spectra, so **aliasing**
- **no way to reconstruct** the original signal (to isolate the original spectrum)

Sampling and Reconstructing

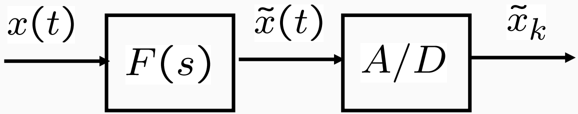
The Sampling Theorem

Nyquist-Shannon theorem

A continuous-time signal which contains no frequency components greater than Ω_{B} rad/s, is uniquely determined by the signal samples

$$\left\{ x_k = x(k\Delta)\ , \quad k \in \mathbb{Z}\ , \quad \Delta:\ \Omega_{\mathsf{s}} = \frac{2\pi}{\Delta} > 2\Omega_{\mathsf{B}} \right\}$$

Anti-aliasing Filter



- How to guarantee the band limitedness of a signal?
- From a practical point of view, how to restrict the bandwidth of the signal to the band of interest, with the aim to satisfy the sampling theorem?
- **anti-aliasing filter**: a realizable low-pass filter

$$F(s) = \frac{1}{1 + \frac{s}{\bar{\Omega}}} \,, \quad B = [0 \,, \bar{\Omega}] \quad \Omega_s = \frac{2\pi}{\Delta} > 2\bar{\Omega}$$

Matlab live script

The impact of the sampling process on the reconstruction of the original continuous-time signal starting from the sampled data, the effect of the aliasing, the tuning of the anti-aliasing low-pass filters can be experimented by means of a **Matlab live script**.

Steps to retrieve the live script:

- Download as a ZIP archive the whole contents of the folder named ”**L2_Sampling_Aliasing**,” available in the ”**Class Materials**” file area of the MS Teams course team, and uncompress it in a preferred folder.
- Add the chosen folder and subfolders to the Matlab path.
- Open the live script using the Matlab command:

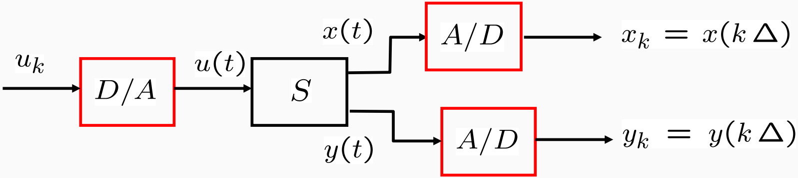
```
open('Sampling_Aliasing_IdealSampler.mlx');
```

Sampling and LTI Systems: from Continuous-Time to Discrete-Time Systems

Sampling and LTI Systems: from Continuous-Time to Discrete-Time Systems

The Step-Invariant Transform

C2d with Sampler & Hold



Consider the scheme

- How to obtain a discrete-time description of a linear, time-invariant, continuous-time dynamic system?
- Both state variables and outputs are sampled by means of an **ideal sampler**
- The inputs to the LTI system are converted from discrete- to continuous-time using a **ZOH**

C2d with Sampler & Hold (cont.)

- Consider a LTI dynamic system, described by means of state equations

$$\begin{cases} \dot{x}(t) = & A_c x(t) + B_c u(t) \\ y(t) = & C_c x(t) + D_c u(t) \end{cases}$$

- The following expression holds

$$x(t) = e^{A_c(t-t_0)} x\left(t_0\right) + \int_{t_0}^{t} e^{A_c(t-\tau)} B_c u(\tau)\, d\tau$$

(from "*Fundamentals in Control*") where

$$e^{A_c t} = \mathcal{L}^{-1}\left\{(sI - A_c)^{-1}\right\} = I + A_c t + \frac{A_c^2 t^2}{2} + \frac{A_c^3 t^3}{3!} + \cdots$$

C2d with Sampler & Hold (cont.)

- Remember the stairwise behaviour of the output of a ZOH device

$$u(t) = u_k = u(k\Delta)\,,\; k\,\Delta \le t < (k+1)\;\Delta \quad k \in \mathbb{Z}$$

- Evaluate the state movement expression in a time interval between two successive sampling instants $k\Delta$ and $(k+1)\Delta$

$$x\left[(k+1)\Delta]\right) = e^{A_c\Delta} x\left(k\Delta\right) + \left\{ \int_{k\,\Delta}^{(k+1)\,\Delta} e^{A_c(t-\tau)} B_c)\, d\tau \right\} u\left(k\Delta\right)$$

the input $u(t)$ is a constant signal during the considered time interval

C2d with Sampler & Hold (cont.)

- Substitute $r = (k+1)\Delta - \tau$ into the integral term and rewrite the last expression,

$$x\left[(k+1)\Delta]\right) = e^{A_c \Delta}\, x\,(k\Delta) + \left\{ \int_0^{\Delta} e^{A_c r} B_c\, dr \right\} u\,(k\Delta)$$

- By comparison with the expression of the discrete-time state equations for the dynamic system considered

$$\begin{cases} x\left[(k+1)\Delta\right] = A_d\, x\,(k\Delta) + B_d\, u\,(k\Delta) \\ \\ y\,(k\Delta) = C_d\, x\,(k\Delta) + D_d\, u\,(k\Delta) \end{cases}$$

finally we obtain the **continuous to discrete-time conversion rule, applying ZOH** (the so-called *step–invariant transform*)

C2d with Sampler & Hold (cont.)

Step–invariant transform

Starting from a continuous-time LTI dynamic system

$$\begin{cases} \dot{x}(t) = A_c x(t) + B_c u(t) \\ y(t) = C_c x(t) + D_c u(t) \end{cases}$$

the corresponding discrete-time description, using a ZOH for inputs and ideal samplers for state and output signals is given by

$$A_d = e^{A_c \Delta} \qquad B_d = \int_0^{\Delta} e^{A_c r} B_c \, dr$$

$$C_d = C_c \qquad D_d = D_c$$

C2d with Sampler & Hold: an Example

Consider

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} x + \begin{bmatrix} 0 \\ K \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

and let's determine the discrete-time description, by sampling with ZOH and ideal samplers.

$$(sI - A_c)^{-1} = \frac{1}{s(s+a)} \begin{bmatrix} s+a & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+a)} \\ 0 & \frac{1}{s+a} \end{bmatrix}$$

C2d with Sampler & Hold: an Example (cont.)

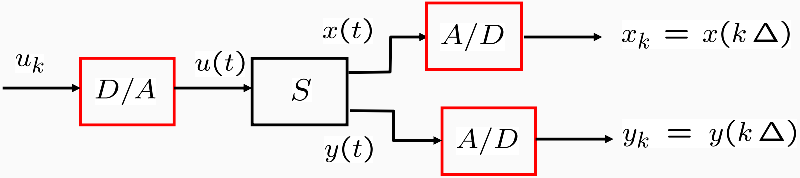
Applying the step-invariant transform

$$e^{A_c\,t} = \mathcal{L}^{-1}\left\{(sI - A_c)^{-1}\right\} = \begin{bmatrix} 1(t) & \frac{1}{a}\cdot 1(t) - \frac{1}{a}e^{-at}\cdot 1(t) \\ 0 & e^{-at}\cdot 1(t) \end{bmatrix}$$

$$e^{A_c\,\Delta} = \begin{bmatrix} 1 & \frac{1}{a}\left(1 - e^{-a\,\Delta}\right) \\ 0 & e^{-a\,\Delta} \end{bmatrix}$$

$$B_d = \int_0^{\Delta} e^{A_c\,r}\,B_c\,dr = \begin{bmatrix} \int_0^{\Delta} \frac{K}{a}\left(1 - e^{-a\,r}\right)\,dr \\ \int_0^{\Delta} K e^{-a\,r}\,dr \end{bmatrix}$$

Why Do They Call It the Step-Invariant Transform?



Step response of the sampled-time LTI system

The considered conversion technique from continuous-time to discrete-time LTI systems is usually called **step invariant transform**, due to a peculiar feature of the conversion rule itself:

- **the conversion rule preserves the step response** of the dynamic system, i.e. the values of the step response of the discrete-time description of the LTI system are exactly the samples of the step response of the effective continuous-time LTI system. [Hint: what is the output of a ZOH if the input is a discrete-time step sequence?]

Why Do They Call It the Step-Invariant Transform? (cont.)

Continuous- and discrete-time step responses: a comparison

- Let's consider the continuous-time LTI system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} \cdot u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x(t) \end{cases}$$

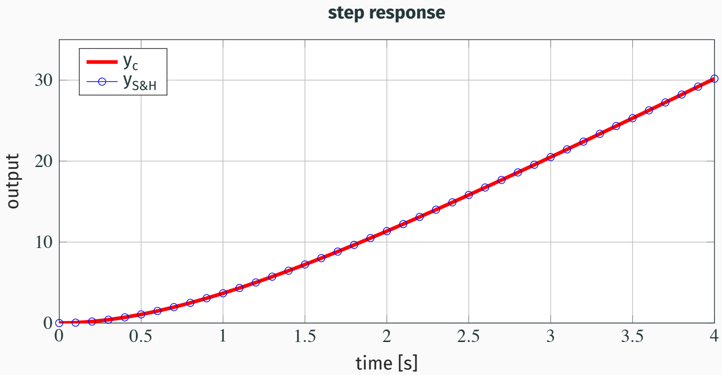
- Chosen as sampling period the value $\Delta = 0.1$ s, the discrete-time description of the considered LTI system is

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & 0.09516 \\ 0 & 0.9048 \end{bmatrix} \cdot x(k) + \begin{bmatrix} 0.04837 \\ 0.9516 \end{bmatrix} \cdot u(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x(k) \end{cases}$$

Why Do They Call It the Step-Invariant Transform? (cont.)

Continuous- and discrete-time step responses: a comparison (cont.)

- Let's compare visually the step responses of the continuous-time LTI system and of the discrete-time description



The Step-Invariant Transform: Hands-On

Matlab live script

How to use Matlab to apply the step-invariant transform?

A **Matlab live script** is available.

Steps to retrieve the live script:

- Download as a ZIP archive the whole contents of the folder named "**L2_Step_Invariant_Transform**," available in the "**Class Materials**" file area of the MS Teams course team, and uncompress it in a preferred folder.
- Add the chosen folder and subfolders to the Matlab path.
- Open the live script using the Matlab command:

```
open('StepInvariantTransform.mlx');
```

Sampling and LTI Systems: from Continuous-Time to Discrete-Time Systems

Practical Issues

C2d with Sampler & Hold: Practical Issues

$$A_d = e^{A_c \Delta} \qquad B_d = \int_0^{\Delta} e^{A_c r} B_c \, dr$$

$$C_d = C_c \qquad D_d = D_c$$

- How does one **determine in practice** the matrices described into the step-invariant transform?
- Are exact solutions or approximate expressions available?

C2d with Sampler & Hold: Practical Issues (cont.)

Exact formulas for the step-invariant transform

$$A_d = e^{A_c \Delta} \quad \Longleftarrow \quad e^{A_c t} = \mathcal{L}^{-1}\left\{(sI - A_c)^{-1}\right\}$$

$$B_d = \int_0^{\Delta} e^{A_c r} B_c \, dr = A_c^{-1} \cdot \left[e^{A_c \Delta} - I\right] \cdot B_c$$ if A_c is nonsingular

Approximate expressions

$$A_d = e^{A_c \Delta} \approx I + A_c \Delta + \frac{A_c^2 \Delta^2}{2!} + \frac{A_c^3 \Delta^3}{3!} + \cdots$$

$$B_d = \int_0^{\Delta} e^{A_c r} B_c \, dr \approx \left[\Delta + \frac{A_c \Delta^2}{2!} + \frac{A_c^2 \Delta^3}{3!} + \cdots\right] \cdot B_c$$

267MI –Fall 2023

Lecture 2

Sampling and Reconstructing in Time Domain:

Sampling and LTI Systems

END