Systems Dynamics

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Lecture 3

State and Output Movement of Linear Discrete-Time Systems

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3. State and Output Movement of Linear Discrete-Time Systems

State-Space Solution: the Time-Invariant Case

State-Space Solution: the Time-Invariant case

- In the time-invariant case, matrices $A(k), B(k), C(k), D(k)$ do not depend on time-index k, that is they are **constant** matrices A, B, C, D.
- Hence, when considering a linear discrete-time free (no inputs) time-invariant dynamic system:

$$x(k+1) = Ax(k)\,, \quad x(k_0) = x_0$$

one gets:

$$x(k) = \varphi(k, k_0, x_0) = \Phi(k, k_0)x_0$$

where the **discrete-time state-transition matrix** now takes on the form

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A = A^{(k-k_0)}\,, \quad k > k_0\,; \quad \Phi(k_0, k_0) = I$$

- With some abuse of notation, we denote $\Phi(k-k_0)$ to highlight the dependence on $(k-k_0)$ instead of k and k_0 separately.

State-Space Solution: the Time-Invariant case (cont.)

Now, consider a linear discrete-time time-invariant dynamic system with inputs:

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0$$

Therefore, using

$$\Phi(k-k_0) = A^{(k-k_0)}\,, \quad k > k_0\,; \quad \Phi(k_0, k_0) = I$$

one gets

$$\begin{aligned} x(k) &= \varphi(k, k_0, x_0, \{u(k_0), \ldots, u(k-1)\}) \\ &= A^{(k-k_0)} x_0 + \sum_{j=k_0}^{k-1} A^{k-(j+1)} Bu(j)\,, \quad k > k_0 \end{aligned}$$

The explicit form $\Phi(k-k_0) = A^{(k-k_0)}$ will be used later on to determine the state and output evolution over time in **closed-form**.

State-Space Solution: the Time-Invariant case (cont.)

- **Free state movement.** Setting $u(k) = 0, \forall\, k \geq k_0$ gives:

$$x(k) = \varphi(k, k_0, x_0, 0) = \varphi_L(k) = A^{(k-k_0)} x_0\,, \quad k > k_0$$

- **Forced state movement.** Setting $x_0 = 0$ gives:

$$\begin{aligned} x(k) &= \varphi(k, k_0, 0, \{u(k_0), \ldots, u(k-1)\}) = \varphi_F(k) \\ &= \sum_{j=k_0}^{k-1} A^{k-(j+1)} B u(j)\,, \quad k > k_0 \end{aligned}$$

The **total state movement** is thus given by:

$$\varphi(k, k_0, x_0, \{u(k_0), \ldots, u(k-1)\}) = \varphi_L(k) + \varphi_F(k)$$

which is a direct consequence of the **linearity** of the dynamic system.

State-Space Solution: the Time-Invariant case (cont.)

Now, by adding the output equation:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

one gets:

$$y(k) = \quad CA^{(k-k_0)}x_0 + \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k)\,, \quad k > k_0$$

- **Free output movement.** Setting $u(k) = 0,\ \forall\, k \geq k_0$ gives:

$$y(k) = y_L(k) = CA^{(k-k_0)}x_0\,,\ k > k_0$$

- **Forced output movement.** Setting $x_0 = 0$ gives:

$$y(k) = y_F(k) = \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k)\,,\ k > k_0$$

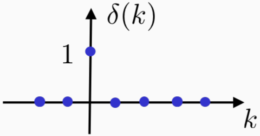
The **total output movement** is thus given by: $y(k) = y_L(k) + y_F(k)$

Input-Output Dynamic Description for Linear Systems

Preliminaries

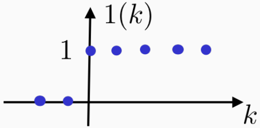
Discrete-time unit **impulse** sequence

$$\delta(k) = \begin{cases} 0, & k \neq 0,\ k \in \mathbb{Z} \\ 1, & k = 0 \end{cases}$$



Discrete-time unit **step** sequence

$$1(k) = \begin{cases} 0, & k < 0,\ k \in \mathbb{Z} \\ 1, & k \geq 0,\ k \in \mathbb{Z} \end{cases}$$



$$\Longrightarrow \quad \delta(k) = 1(k) - 1(k-1)\,; \quad 1(k) = \begin{cases} \sum_{j=0}^{\infty} \delta(k-j), & k \geq 0 \\ 0, & k < 0 \end{cases}$$

Moreover, an arbitrary sequence $\{x(k)\}$ can be expressed as $x(k) = \sum_{j=-\infty}^{\infty} x(j)\delta(k-j)$

Input-Output Dynamic Description of Linear Systems (cont.)

- Consider a linear discrete-time system with scalar input and output

- Moreover, consider the ”external” input/output relationship

$$y(k) = \sum_{j=-\infty}^{\infty} h(k,j)u(j) \quad (\star)$$

Assumption. The sequences $\{h(k,j)\}$ for any given k and $\{u(j)\}$ are such that the relationship $(\star)$ is well-defined. For example, $\{h(k,j)\} \in l_2$ and $\{u(j)\} \in l_2$.

- Under the above assumption, relationship $(\star)$ is **linear**.

Input-Output Dynamic Description of Linear Systems (cont.)

- Denote by $h(k,j)$ the output response at time k produced by a unit impulse $\delta(j)$ applied at time j
- By linearity, the output response at time k produced by a impulse of amplitude $u(j)$ applied at time j is $h(k,j)u(j)$
- By linearity, the output response at time k produced by two impulses of amplitude $u(j_1)$ and $u(j_2)$ applied at times j_1 and j_2, respectively, is $h(k,j_1)u(j_1) + h(k,j_2)u(j_2)$

Input-Output Model

At time k, the system output $y(k)$ produced by the input sequence $\{u(j)\}$ is given by

$$y(k) = \sum_{j=-\infty}^{\infty} h(k,j)u(j)$$

where $h(k,j)$ denotes the output response at time k produced by a unit impulse $\delta(k-j)$ applied at time j

Input-Output Dynamic Description of Linear Systems (cont.)

Properties

- Due to **causality**, the response to an input sequence has to be **identically zero before the input sequence is applied**. Hence:

$$h(k,j) = 0\,, \quad \forall\, j, \forall\, k < j$$

Hence:

$$y(k) = \sum_{j=-\infty}^{k} h(k,j)u(j)$$

$$\implies \quad y(k) = \sum_{j=-\infty}^{k_0-1} h(k,j)u(j) + \sum_{j=k_0}^{k} h(k,j)u(j)$$

$$= Y(k;k_0-1) + \sum_{j=k_0}^{k} h(k,j)u(j)$$

Input-Output Dynamic Description of Linear Systems (cont.)

- The system is **at rest** at time k_0 if

$$u(k) = 0, \, \forall \, k \geq k_0 \quad \Longrightarrow \quad y(k) = 0, \, \forall \, k \geq k_0$$

  and this implies $Y(k; k_0 - 1) = 0$.
- Hence, if the system is **at rest** at time k_0, it follows that

$$y(k) = \sum_{j=k_0}^{\infty} h(k, j) u(j)$$

  and due to causality, one gets

$$y(k) = \sum_{j=k_0}^{k} h(k, j) u(j)$$

Input-Output Dynamic Description of Linear Systems (cont.)

- If the system is **time-invariant**, denoting by $\{h(k,0)\}$ the response to $\{\delta(k)\}$, it follows that $\{h(k-j,0)\}$ is the response to $\{\delta(k-j)\}$
- Letting (with some abuse of notation)

$$h(k-j) := h(k-j,0)$$

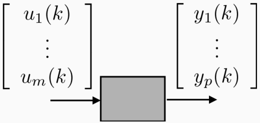
one gets the well-known **convolution formula**:

$$y(k) = u(k) * h(k) = \sum_{j=-\infty}^{\infty} h(k-j)u(j)$$

or equivalently (via a change of variables)

$$y(k) = h(k) * u(k) = \sum_{i=-\infty}^{\infty} h(i)u(k-i)$$

Input-Output Dynamic Description of Linear Systems (cont.)



- Consider a linear discrete-time system with **vector input and output**

- The scalar case (with all properties) can be generalised as:

$$y(k) = \sum_{j=-\infty}^{\infty} H(k,j)u(j)$$

$$H(k,j) = \begin{bmatrix} h_{11}(k,j) & h_{12}(k,j) & \cdots & h_{1m}(k,j) \\ h_{21}(k,j) & h_{22}(k,j) & \cdots & h_{2m}(k,j) \\ \cdots & \cdots & \cdots & \\ h_{p1}(k,j) & h_{p2}(k,j) & \cdots & h_{pm}(k,j) \end{bmatrix}$$

where $h_{rs}(k,j)$ denotes the r-th component of the response at time k produced by a unit impulse applied at time j on the s-th component of the input, while all other input components are set to zero.

Relationship between State-Space and Input-Output Dynamic Descriptions

Consider a state-space description with **initial state set to zero**:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k)\,, & x(k_0) = 0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

Recalling that

$$y(k) = \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k)\,, \quad k > k_0$$

one gets immediately

$$H(k,j) = \begin{cases} C(k)\Phi(k, j+1)B(j)\,, & k > j \\ D(k) & k = j \\ 0 & k < j \end{cases}$$

which, in the **time-invariant** case, becomes

$$H(k-j) = \begin{cases} CA^{k-(j+1)}B\,, & k > j \\ D & k = j \\ 0 & k < j \end{cases}$$

Determination of the State/Output Movement

Determination of the State/Output Movement

Response Modes

Determination of the State/Output Movement

In the **time-invariant** case, recall that the solution specialises as follows:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

one gets:

$$\begin{aligned} y(k) = \quad & CA^{(k-k_0)}x_0 \\ & + \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k)\,, \quad k > k_0 \end{aligned}$$

where the **state-transition matrix** now is given by:

$$\Phi(k-k_0) = A^{(k-k_0)}\,, \quad k > k_0\,; \quad \Phi(k_0, k_0) = I$$

Response Modes

- Without loss of generality we let $k_0 = 0$ and we "expand" matrix $A^{k-k_0} = A^k$ in "matrix partial fractions".
- Clearly

$$\det(zI - A) = \prod_{i=1}^{\sigma} (z - \lambda_i)^{n_i}$$

  where $\lambda_1, \ldots, \lambda_\sigma$ are the **distinct** eigenvalues of A and n_i is the **algebraic multiplicity** of such eigenvalues.
- Of course $\displaystyle\sum_{i=1}^{\sigma} n_i = n$.

Response Modes (cont.)

- Let's assume $\lambda_i \neq 0\,,\; i = 1\,, \ldots \sigma$.
- It can be shown that:

$$A^k = \sum_{i=1}^{\sigma} \left[A_{i0} \lambda_i^k 1(k) + \sum_{l=1}^{n_i - 1} A_{il} k(k-1) \cdots (k-l+1) \lambda_i^{k-l} 1(k-l) \right]$$

where

$$A_{il} = \frac{1}{l!} \frac{1}{(n_i - 1 - l)!} \lim_{z \to \lambda_i} \left\{ \frac{d^{n_i - 1 - l}}{dz^{n_i - 1 - l}} \left[(z - \lambda_i)^{n_i} (zI - A)^{-1} \right] \right\}$$

Response Modes (cont.)

Hence:

- A^k can be expressed as a sum of terms $A_{il} l! \begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l}$ which are called **Response Modes**
- If an eigenvalue λ_i has algebraic multiplicity n_i, then, in general, n_i response modes

$$A_{il} l! \begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l}, \; l = 0, 1, \ldots, n_i - 1$$

  can be associated to λ_i.
- When all eigenvalues of A are distinct, one has $\sigma = n; \; n_i = 1, i = 1, \ldots, n$ and

$$A^k = \sum_{i=1}^{n} A_i \lambda_i^k$$

with

$$A_i = \lim_{z \to \lambda_i} \left[(z - \lambda_i)(zI - A)^{-1} \right]$$

Response Modes: A particular Case

- If an eigenvalue λ_j of A is zero, then the terms in the expression of A^k, corresponding to the zero eigenvalue must be modified.
- The terms corresponding to $\lambda_j = 0$ are

$$A_{j0} \cdot \delta(k) + \sum_{l=1}^{n_j-1} A_{jl}\, l!\, \delta(k-l)$$

where

$$A_{jl} = \frac{1}{l!} \frac{1}{(n_j - 1 - l)!} \lim_{z \to 0} \left\{ \frac{d^{n_j-1-l}}{dz^{n_j-1-l}} \left[z^{n_j} (zI - A)^{-1} \right] \right\}$$

Response Modes: A different Characterisation

In the special case of **distinct eigenvalues** of A:

- In such a case: $\det(zI - A) = \prod_{i=1}^{n}(z - \lambda_i)$ and $A^k = \sum_{i=1}^{n} A_i \lambda_i^k$
- It can be shown that $A_i = v_i \tilde{v}_i^\top$ where:
  - $(\lambda_i I - A)v_i = 0$: v_i right eigenvector associated with λ_i
  - $\tilde{v}_i^\top(\lambda_i I - A) = 0$: $\tilde{v}_i^\top$ left eigenvector associated with λ_i

  In fact:

$$Q := [v_1 \,|\, v_2 \,|\, \cdots \,|\, v_n] \implies P = Q^{-1} = \begin{bmatrix} \tilde{v}_1^\top \\ \vdots \\ \tilde{v}_n^\top \end{bmatrix}; \; \tilde{v}_i^\top v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

  and then

$$\begin{aligned}(zI - A)^{-1} &= [zI - Q\mathrm{diag}\,[\lambda_1, \ldots, \lambda_n]Q^{-1}]^{-1} \\ &= Q[zI - \mathrm{diag}\,[\lambda_1, \ldots, \lambda_n]]^{-1}Q^{-1} \\ &= Q\mathrm{diag}\,[(z-\lambda_1)^{-1}, \ldots, (z-\lambda_n)^{-1}]Q^{-1} = \sum_{i=1}^{n} v_i \tilde{v}_i^\top (z - \lambda_i)^{-1}\end{aligned}$$

Response Modes: A different Characterisation (cont.)

- If the initial state vector x_0 is "parallel" to eigenvector v_j of A, then the only response mode showing up int the state movement is λ_j^k:

$$x_0 = \alpha v_j \implies x(k) = A^k x_0 = v_1 \tilde{v}_1^\top x_0 \lambda_1^k + \cdots + v_n \tilde{v}_n^\top x_0 \lambda_n^k = \alpha v_j \lambda_j^k$$

Example: consider $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$; $\lambda_1 = -1,\ \lambda_2 = 1$

$$\implies Q = [v_1 \,|\, v_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ Q^{-1} = \begin{bmatrix} \tilde{v}_1^\top \\ \tilde{v}_2^\top \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A^k = v_1 \tilde{v}_1^\top \lambda_1^k + v_2 \tilde{v}_2^\top \lambda_2^k = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} (-1)^k + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} 1^k$$

and thus, if $x_0 = \alpha v_1 = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then the response mode 1^k **does not show up** in the free state response starting from such an initial state x_0

Calculation of A^k by Similarity Transformation

Consider:

- $x(k+1) = Ax(k)\,,\ x(0) = x_0 \implies x(k) = A^k x_0$
- $T \in \mathbb{R}^{n\times n},\ \det(T) \neq 0 \implies x = T\hat{x},\quad \hat{x} = T^{-1}x$

Hence $\hat{x}(k+1) = T^{-1}Ax(k) = T^{-1}AT\hat{x}(k),\ \hat{x}_0 = T^{-1}x_0$ which yields

$$\hat{x}(k) = (T^{-1}AT)^k T^{-1} x_0$$

Letting $\hat{A} := T^{-1}AT$, one gets the closed-form expression for the free-state response expressed in the original state coordinates

$$x(k) = T\hat{A}^k T^{-1} x_0$$

Calculation of A^k by Similarity Transformation (cont.)

The *matrices* A and $\hat{A}$ are *similar*, and T is called a *similarity transformation*.

$$\hat{A} := T^{-1}AT \qquad \Longleftrightarrow \qquad A = T\hat{A}T^{-1}$$

$$\hat{A}^k = T^{-1}A^kT \qquad \Longleftrightarrow \qquad A^k = T\hat{A}^kT^{-1}$$

Note: also the matrices A^k and $\hat{A}^k$ are *similar*.

Question: may we take advantage of a suitable similarity transformation in evaluating A^k?

Calculation of A^k by Similarity Transformation (cont.)

Case 1. Suppose that matrix A admits the construction of a basis of n linearly-independent eigenvectors v_i associated with the eigenvalues λ_i, $i = 1, \dots, n$ (**not necessarily distinct**).

Thus:

$$T = [v_1 | v_2 | \cdots | v_n] \implies D = T^{-1} A T = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Hence:

$$D^k = \begin{bmatrix} {\lambda_1}^k & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & {\lambda_n}^k \end{bmatrix}$$

$$\implies x(k) = T D^k T^{-1} x_0 = T \begin{bmatrix} {\lambda_1}^k & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & {\lambda_n}^k \end{bmatrix} T^{-1} x_0$$

Calculation of A^k by Similarity Transformation (cont.)

Case 2. Consider the general case in which matrix A has multiple eigenvalues. It is always possible[1] to construct a basis of n linearly-independent vectors v_i such that:

$$T = [v_1|v_2|\cdots|v_n] \implies J = T^{-1}AT = \begin{bmatrix} J_0 & \cdots & \cdots & 0 \\ \vdots & J_1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & J_s \end{bmatrix}$$

where J is the so called **Jordan canonical form** of the matrix A.

Matrix J is block-diagonal and its special structure makes it possible to compute A^k in **closed-form**.

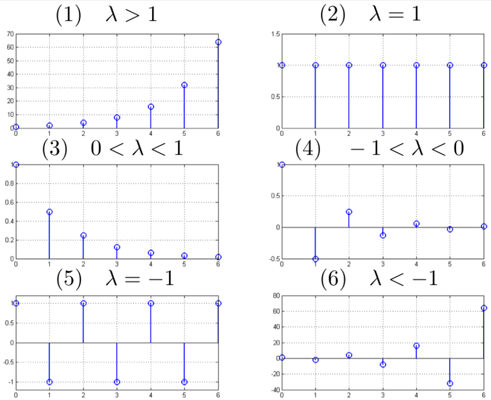
[1] For details, see the Ch. 2 in the ref. book: Antsaklis P. J. and Michel A. N. , *Linear Systems*, Birkhäuser, 2006. Refer also to the course **Control Theory**, taught by Prof. Felice A. Pellegrino.

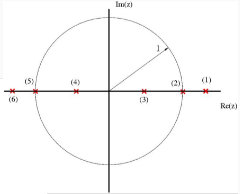
Determination of the State/Output Movement

Qualitative Behaviour of Response Modes

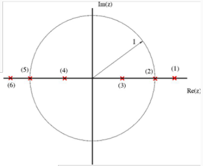
Qualitative Behaviour of Response Modes

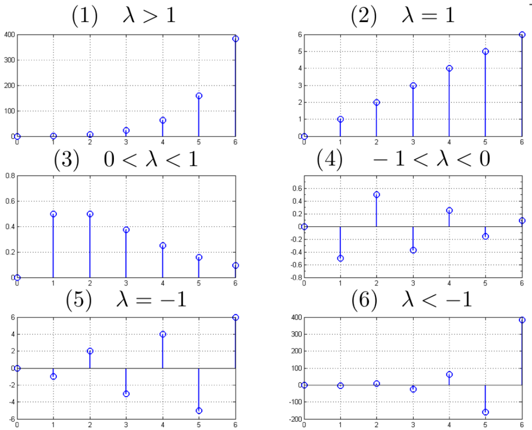
- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{R}$, multiplicity $= 1$





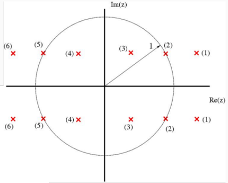
- $\binom{k}{n_i} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{R}$, multiplicity > 1

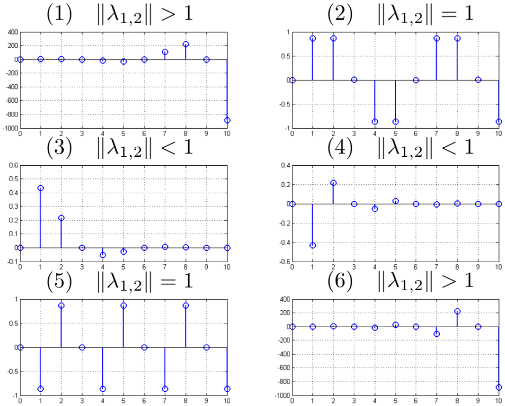




Qualitative Behaviour of Response Modes

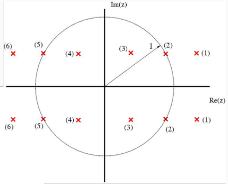
- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{C}$, multiplicity $= 1$

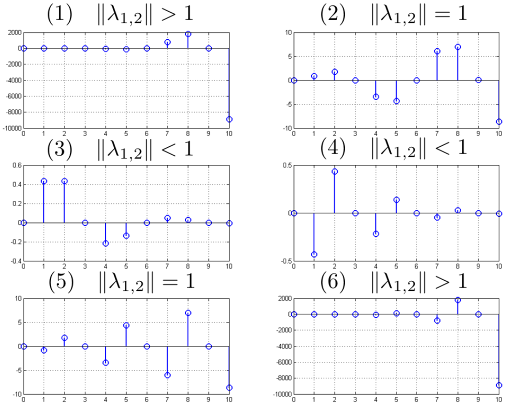




Qualitative Behaviour of Response Modes

- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{C}$, multiplicity > 1





Response Modes & MATLAB: Hands-On

Matlab live script

How to compute the response mode matrices using MATLAB, either in the scenario of all distinct eigenvalues, either in the case of multiple eigenvalues can be experimented by means of a **Matlab live script**.

Steps to retrieve the live script:

- Download as a ZIP archive the whole contents of the folder named "**L3_Response_Modes**," available in the "**Class Materials**" file area of the MS Teams course team, and uncompress it in a preferred folder.
- Add the chosen folder and subfolders to the Matlab path.
- Open the live script using the Matlab command:

```
open('LTI_movements_response_modes.mlx');
```

External Description of LTI Dynamic Systems: Transfer Function

External Description of LTI Dynamic Systems: Transfer Function

Recall the relationship between the state space description and the impulse response **(an external description)**:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k)\,, & x(k_0) = 0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

Recalling that

$$y(k) = \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k)\,, \quad k > k_0$$

one gets immediately

$$H(k,j) = \begin{cases} C(k)\Phi(k,j+1)B(j)\,, & k > j \\ D(k) & k = j \\ 0 & k < j \end{cases}$$

which, in the **time-invariant** case, becomes

$$H(k-j) = \begin{cases} CA^{k-(j+1)}B\,, & k > j \\ D & k = j \end{cases}$$

Transfer Function

Consider the time-invariant dynamic system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = 0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Applying the $\mathcal{Z}$ Transform to both sides one gets:

$$\begin{aligned} z\left[X(z) - x_0\right] &= AX(z) + BU(z) \\ \Longrightarrow (zI - A)X(z) &= z\,x_0 + BU(z) \\ \Longrightarrow &\begin{cases} X(z) = (zI-A)^{-1} z\,x_0 + (zI-A)^{-1}BU(z) \\ \\ Y(z) = CX(z) + DU(z) \end{cases} \\ \Longrightarrow Y(z) &= C(zI-A)^{-1} z\,x(0) + \left[C(zI-A)^{-1}B + D\right] U(z) \end{aligned}$$

Letting $x_0 = 0$, it follows that:

$$Y(z) = \left[C(zI-A)^{-1}B + D\right]U(z) = H(z)U(z)$$

and $H(z)$ is called **transfer function**.

Transfer Function (cont.)

Let's analyse the structure of the transfer function:

$$H(z) = \begin{bmatrix} H_{11}(z) & \cdots & H_{1m}(z) \\ \vdots & & \vdots \\ H_{i1}(z) & \cdots & H_{im}(z) \\ \vdots & & \vdots \\ H_{p1}(z) & \cdots & H_{pm}(z) \end{bmatrix}$$

$H(z)$ is a $p \times m$ matrix where the i-th component of the output vector is given by:

$$Y_i(z) = \sum_{j=1}^{m} H_{ij}(z)U_j(z) = H_{i1}(z)U_1(z) + H_{i2}(z)U_2(z) + \cdots$$

Hence:

$$\begin{aligned} x(0) &= x_0 \\ u_r(k) &= 0, \; r \neq j \end{aligned} \quad \Longrightarrow \quad H_{ij}(z) = \frac{Y_i(z)}{U_j(z)}$$

Transfer Function of equivalent dynamic systems

Recall:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Let $\hat{x} := T^{-1}x$, where $T \in \mathbb{R}^{n\times n}$ is a generic non-singular $n \times n$ matrix ($\det(T) \neq 0$).
Then, the equivalent state-space description is given by:

$$\begin{cases} \hat{x}(k+1) = T^{-1}x(k+1) = T^{-1}AT\hat{x}(k) + T^{-1}Bu(k) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = CT\hat{x}(k) + Du(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

Hence:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \iff \begin{cases} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

Transfer Function of equivalent dynamic systems (cont.)

$$
\begin{aligned}
\hat{H}(z) &= \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} \\
&= C\left[T\left(zI - T^{-1}AT\right)^{-1}T^{-1}\right]B + D \\
&= C\left[T\left(zT^{-1}T - T^{-1}AT\right)^{-1}T^{-1}\right]B + D \\
&= C\left[T\left(T^{-1}(zI - A)T\right)^{-1}T^{-1}\right]B + D \\
&= C\left[TT^{-1}\left(zI - A\right)^{-1}TT^{-1}\right]B + D \\
&= C\left[(zI - A)^{-1}\right]B + D \\
&= H(z)
\end{aligned}
$$

Hence: the transfer function does not depend on the specific choice of the state variables

Transfer Function: Properties

Consider the scalar case, that is, $u(k) \in \mathbb{R}$, $y(k) \in \mathbb{R}$:

$$H(z) = C\left[(zI - A)^{-1}\right]B + D$$

and

$$(zI - A)^{-1} = \begin{bmatrix} z - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & z - a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & z - a_{nn} \end{bmatrix}^{-1}$$

Transfer Function: Properties (cont.)

The inverse can be expressed as:

$$(zI - A)^{-1} = \frac{1}{\det(zI - A)} K(z)$$

where $K(z)$ is the matrix of the algebraic complements.

Clearly:

- $\det(zI - A)$ is a polynomial with degree n
- $K(z) = [k_{ij}(z);\, i, j = 1, \ldots, n]$ where $k_{ij}(z)$ is a polynomial of degree $< n, \forall\, i, j$
- $C(zI - A)^{-1} B = \frac{1}{\det(zI - A)} CK(z)B = \frac{M(z)}{\varphi(z)}$ where $M(z)$ is a polynomial of degree $< n$,

Transfer Function: Properties (cont.)

Therefore:

$$H(z) = C\,(zI - A)^{-1}\,B + D = \frac{M(z)}{\varphi(z)} + D$$
$$= \frac{M(z) + D\varphi(z)}{\varphi(z)} = \frac{N(z)}{\varphi(z)}$$

where:

- $N(z)$ in general is a polynomial of degree n
- In case of a **strictly proper** system, that is $D = 0$, $N(z)$ in general is a polynomial of degree $< n$
- All the above holds if **no common factors** between $N(z)$ and $\varphi(z)$ are present

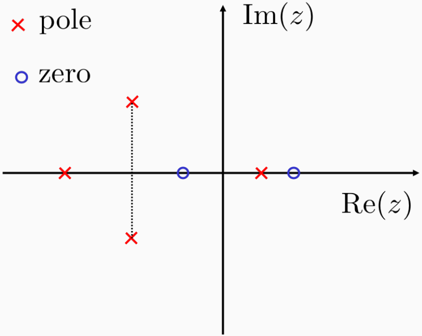
Transfer Function: Properties (cont.)

In the presence of common factors between $N(z)$ and $\varphi(z)$:

$$H(z) = \frac{\overline{N}(z)}{\overline{\varphi}(z)}$$

- $\overline{\varphi}(z)$ is a factor of $\varphi(z)$ of degree $\nu < n$
- $\overline{N}(z)$ has degree $m < \nu$ and has degree ν only if $D \neq 0$ (non strictly proper systems)

Transfer Function: Poles and Zeros (scalar case)



- **Poles**: roots of polynomial $\varphi(z)$
- **Zeros**: roots of polynomial $N(z)$

- The poles are eigenvalues of A
- An eigenvalue of A might not belong to the set of poles when common factors are present
- In case of more then one input and/or more than one output extra-care has to be exercised

Transfer Function: Example

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\ \\ y(k) = [0 \;\; 1]\, x(k) \end{cases} \qquad n = 2$$

Hence:

$$G(z) = [0 \;\; 1] \begin{bmatrix} z-1 & -1 \\ 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [0 \;\; 1] \frac{1}{(z-1)(z+1)} \begin{bmatrix} z+1 & 1 \\ 0 & z-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

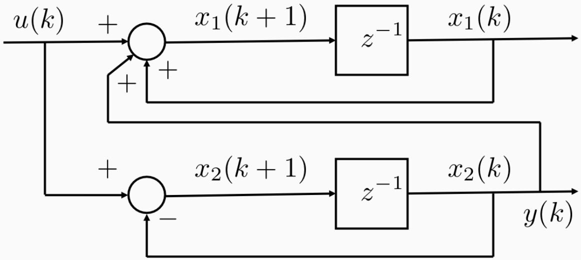
$$= \frac{(z-1)}{(z-1)(z+1)} = \frac{1}{z+1}$$

Thus: $\overline{\varphi}(z) = z + 1$ is a factor of $\varphi(z) = (z-1)(z+1)$

Transfer Function: Example (cont.)

The state equations have the form:

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) + u(k) \\ x_2(k+1) = -x_2(k) + u(k) \\ y(k) = x_2(k) \end{cases}$$



Only the dynamics $\begin{cases} x_2(k+1) = -x_2(k) + u(k) \\ y(k) = x_2(k) \end{cases}$ shows up in the transfer function

$G(z) = \dfrac{1}{z+1}$ and the time-evolution of $x_1(k)$ is not influencing the output $y(k)$.

Transfer Function: Example in the Non-Scalar Case

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} u(k) \\ y(k) = [-3 \;\; 3]\, x(k) \end{cases}$$

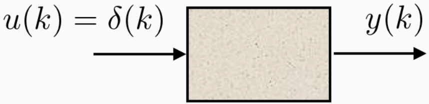
Hence, one gets:

$$\begin{aligned} H(z) &= [-3 \;\; 3] \begin{bmatrix} z & -1 \\ 1 & z+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= [-3 \;\; 3] \frac{1}{(z+1)^2} \begin{bmatrix} z+2 & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= \left[-\frac{3}{z+1} \;\; \frac{3(z-1)}{(z+1)^2} \right] \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} = \left[\frac{3(z-1)}{(z+1)^2} \;\; \frac{3}{z+1} \right] \end{aligned}$$

The notion of zeros and poles of a transfer function in the non-scalar case is more complicated (and less useful though)

Transfer Function: Alternative Definition in the Scalar Case

$$
\begin{aligned}
&x(0) = 0 \\
&u(k) = \delta(k) \\
&\quad \Longrightarrow U(z) = \mathcal{Z}[\delta(k)] = 1
\end{aligned}
$$



Therefore:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{Y(z)}{1} = Y(z)$$

that is:

$$H(z) = \mathcal{Z}[\,\text{Impulse Response}\,]$$

Determination of Response Modes: Examples

Determination of Response Modes: Example 1

Consider:

$$\begin{cases} x(k+1) & = \begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} u(k) \\ y(k) & = \begin{bmatrix} 2 & -1.5 \end{bmatrix} x(k) \end{cases}$$

Determine the free-state movement

$$x_l(k) = A^k \, x(0)$$

starting from the initial state

$$x(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

The free-state movement is given by

$$x(k) = A^k\, x(0) + \sum_{i=0}^{k-1} A^{k-i-1}\, Bu(i)$$

We are going to determine the free-state movement in two ways:

- by the $\mathcal{Z}$ transform
- by calculating the matrix A^k.

Calculation by the $\mathcal{Z}$ transform

$$x_l(k) = A^k\, x(0) \implies X_l(z) \;=\; z\,(z\,I \,-\, A)^{-1}\, x(0)$$

$$(z\,I \,-\, A) = \begin{bmatrix} z + 0.5 & -2 \\ 0 & z - 0.1 \end{bmatrix} \implies (z\,I \,-\, A)^{-1} = \begin{bmatrix} \dfrac{2}{2\,z + 1} & \dfrac{40}{(2\,z + 1)\,(10\,z - 1)} \\ 0 & \dfrac{10}{10\,z - 1} \end{bmatrix}$$

Hence:

$$X_l(z) \;=\; \begin{bmatrix} \dfrac{20\,z\,(10\,z - 21)}{(10\,z - 1)\,(2\,z + 1)} \\ -\dfrac{100\,z}{10\,z - 1} \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

First, we proceed with the inverse $\mathcal{Z}$ transform:

$$X_l(z) = \begin{bmatrix} X_{l\,1}(z) \\ X_{l\,2}(z) \end{bmatrix} = \begin{bmatrix} \dfrac{20\,z\,(10\,z - 21)}{(10\,z - 1)\,(2\,z + 1)} \\ -\dfrac{100\,z}{10\,z - 1} \end{bmatrix}$$

Hence:

$$X_{l\,1}(z) = \frac{20\,z\,(10\,z - 21)}{(10\,z - 1)\,(2\,z + 1)}$$

$$\implies \frac{X_{l\,1}(z)}{z} = \frac{20\,(10\,z - 21)}{(10\,z - 1)\,(2\,z + 1)} = \frac{C_1}{z - \frac{1}{10}} + \frac{C_2}{z + \frac{1}{2}}$$

$$C_1 = \lim_{z \to \frac{1}{10}} \frac{20\,(10\,z - 21)}{10\,(2\,z + 1)} = -\frac{100}{3}\,;\; C_2 = \lim_{z \to -\frac{1}{2}} \frac{20\,(10\,z - 21)}{2\,(10\,z - 1)} = \frac{130}{3}$$

thus getting: $X_{l\,1}(z) = -\dfrac{100}{3}\dfrac{z}{\left(z - \frac{1}{10}\right)} + \dfrac{130}{3}\dfrac{z}{\left(z + \frac{1}{2}\right)}$

Determination of Response Modes: Example 1 (cont.)

Then, it follows that:

$$X_l(z) = \begin{bmatrix} -\frac{100}{3}\frac{z}{\left(z-\frac{1}{10}\right)} + \frac{130}{3}\frac{z}{\left(z+\frac{1}{2}\right)} \\ -10\frac{z}{\left(z-\frac{1}{10}\right)} \end{bmatrix}$$

and thus:

$$x_l(k) = \begin{bmatrix} \left\{-\frac{100}{3}\left(\frac{1}{10}\right)^k + \frac{130}{3}\left(-\frac{1}{2}\right)^k\right\} \cdot 1(k) \\ -10\left(\frac{1}{10}\right)^k \cdot 1(k) \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

Now, as alternative technique, we proceed with calculating the matrix A^k.

- $A = \begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix}$
- Eigenvalues: $\lambda_1 = -0.5$, $\lambda_2 = 0.1$. Hence, matrix A admits a diagonal similar matrix because the eigenvalues are distinct
- The characteristic polynomial is given by:

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda + 0.5)(\lambda - 0.1)$$

- A basis of linearly independent eigenvectors is now determined.

Determination of Response Modes: Example 1 (cont.)

- $Az = \lambda_1 z \quad \text{with} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$\begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = -0.5 \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies \begin{cases} -0.5z_1 + 2z_2 & = & -0.5z_1 \\ 0.1z_2 & = & -0.5z_2 \end{cases}$$

For example: $z_2 = 0 \implies z^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- $Az = \lambda_2 z$

$$\begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.1 \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies \begin{cases} -0.5z_1 + 2z_2 & = & 0.1z_1 \\ 0.1z_2 & = & 0.1z_2 \end{cases}$$

For example: $z_2 = \frac{3}{10} z_1 \implies z^{(2)} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

Determination of Response Modes: Example 1 (cont.)

One now proceeds with calculating the equivalent state-space representation of matrix A:

$$T = \left[z^{(1)} \middle| z^{(2)} \right] = \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} \implies T^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix}$$

thus obtaining:

$$\tilde{A} = T^{-1} A T = \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 2 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{10} \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

The calculation of A^k is now straightforward:

$$A^k = M\tilde{A}^k M^{-1} = M \begin{bmatrix} \left(-\frac{1}{2}\right)^k & 0 \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix} M^{-1}$$

$$= \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \left(-\frac{1}{2}\right)^k & 0 \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{1}{2}\right)^k & \left(-\frac{10}{3}\left(-\frac{1}{2}\right)^k + \frac{10}{3}\left(\frac{1}{10}\right)^k\right) \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

Finally, from

$$A^k = \begin{bmatrix} \left(-\frac{1}{2}\right)^k & \left(-\frac{10}{3}\left(-\frac{1}{2}\right)^k + \frac{10}{3}\left(\frac{1}{10}\right)^k\right) \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix}$$

and $x(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$, one gets:

$$x_l(k) = \begin{bmatrix} \left\{-\frac{100}{3}\left(\frac{1}{10}\right)^k + \frac{130}{3}\left(-\frac{1}{2}\right)^k\right\} \cdot 1(k) \\ -10\left(\frac{1}{10}\right)^k \cdot 1(k) \end{bmatrix}$$

Determination of Response Modes: Example 2

Consider:

$$\begin{cases} x_1(k+1) & = & x_1(k) + 4\,x_2(k) \\ x_2(k+1) & = & x_1(k) + x_2(k) \end{cases}$$

Setting $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, show in two different ways that

$$\lim_{k \to \infty} \frac{x_1(k)}{x_2(k)} = 2$$

We are going to determine the free-state movement yielding $x_1(k), x_2(k), \forall k \geq 0$ in two ways:

- by the $\mathcal{Z}$ transform
- by calculating the matrix A^k.

Determination of Response Modes: Example 2 (cont.)

Using the $\mathcal{Z}$ transform:

$$\begin{cases} zX_1(z) - z = X_1(z) + 4X_2(z) \\ zX_2(z) - z = X_1(z) + X_2(z) \end{cases} \implies \begin{cases} X_1(z) = \dfrac{z(z+3)}{(z+1)(z-3)} \\ X_2(z) = \dfrac{z^2}{(z+1)(z-3)} \end{cases}$$

Hence:

$$\begin{cases} x_1(k) = \left[\left(-\dfrac{1}{2}\right)(-1)^k + \dfrac{3}{2}\,3^k\right]1(k) \\ x_2(k) = \left[\dfrac{1}{4}(-1)^k + \dfrac{3}{4}\,3^k\right]1(k) \end{cases}$$

$$\implies \lim_{k\to\infty} \frac{x_1(k)}{x_2(k)} = \lim_{k\to\infty} \frac{\left(\dfrac{3}{2}\right)3^k}{\left(\dfrac{3}{4}\right)3^k} = 2$$

Determination of Response Modes: Example 2 (cont.)

Using the calculation of A^k:

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \Longrightarrow \det(\lambda I - A) = \lambda^2 - 2\lambda - 3 = 0 \Longrightarrow$$ distinct eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$

$$\begin{aligned} \ker(A - 3I) &= \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \\ \ker(A + I) &= \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \end{aligned} \Longrightarrow \begin{aligned} T &= \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \\ T^{-1} &= -\frac{1}{4} \begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix} \end{aligned}$$

Thus

$$\tilde{A} = T^{-1} A T = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

Determination of Response Modes: Example 2 (cont.)

By some algebra:

$$A^k = T\,\tilde{A}^k\,T^{-1} = \begin{bmatrix} \frac{1}{2}\,3^k + \frac{1}{2}\,(-1)^k & 3^k - (-1)^k \\ \frac{1}{4}\,\left(3^k - (-1)^k\right) & \frac{1}{2}\,3^k + \frac{1}{2}\,(-1)^k \end{bmatrix}$$

and then:

$$x(k) = A^k x(0) = \begin{cases} x_1(k) & = \left[\left(-\frac{1}{2}\right)\,(-1)^k + \frac{3}{2}\,3^k\right] 1(k) \\ x_2(k) & = \left[\frac{1}{4}\,(-1)^k + \frac{3}{4}\,3^k\right] 1(k) \end{cases}$$

$$\implies \quad \lim_{k\to\infty} \frac{x_1(k)}{x_2(k)} = \lim_{k\to\infty} \frac{\left(\frac{3}{2}\right)\,3^k}{\left(\frac{3}{4}\right)\,3^k} = 2$$

267MI –Fall 2023

Lecture 3

State and Output Movement of Linear Discrete-Time Systems

END