Systems Dynamics

Course ID: 267MI – Fall 2023

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Lecture 4

Stability of Discrete-Time Dynamic Systems

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Stability of Linear Discrete-Time Systems

Stability of Linear Discrete-Time Systems

Analysis of the Free State Movement

Stability of Linear Systems via Analysis of the Free State Movement

- Given the linear time-invariant discrete-time dynamic system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

- In **equilibrium** conditions:

$$\begin{aligned} & x(0) = \bar{x} \\ & u(k) = \bar{u},\ k \geq 0 \\ & \qquad \Longrightarrow x(k) = A^k\,\bar{x} + \sum_{i=0}^{k-1} A^{k-i-1}\,B\bar{u} = \bar{x},\ \forall k \geq 0 \end{aligned}$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

- **Perturbing the equilibrium** conditions:

$$\begin{aligned} x(0) &= \bar{x} + \delta\bar{x} \\ u(k) &= \bar{u},\ k \geq 0 \end{aligned} \quad \Longrightarrow \quad \begin{gathered} x(k) \neq \bar{x}, k \geq 0 \\ \text{perturbed state movement} \end{gathered}$$

$$\begin{aligned} \Longrightarrow x(k) &= A^k\ (\bar{x} + \delta\bar{x})\ + \sum_{i=0}^{k-1} A^{k-i-1}\,B\bar{u} \\ &= \bar{x} + A^k\,\delta\bar{x} \end{aligned}$$

Hence:

$$\delta x(k) = A^k \delta\bar{x}$$

- Also, recall that:

$$x_l(k) = A^k\,x(0)$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Stability and A^k

- The stability properties do not depend on the specific value taken on by the equilibrium state $\bar{x}$
- Hence, the stability properties are a structural property of the linear dynamic system as a whole
- The stability properties depend on the time-behaviour of the $n \times n$ elements of the matrix A^k:
  - Stability $\iff$ all elements of A^k are bounded $\forall\, k \geq 0$
  - Asymptotic stability $\iff$ $\lim_{k \to \infty} A^k = 0$
  - Instability $\iff$ at least one element of A^k diverges

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Recall that the matrix A^k can be expressed as a sum of the so-called **response modes** (**Part 2**):

- Let $\lambda_1, \ldots, \lambda_\sigma$ the **distinct** eigenvalues of A and n_i the **algebraic multiplicity** of such eigenvalues (with $\sum_{i=1}^{\sigma} n_i = n$).
- If $\lambda_i \neq 0\,,\ i = 1\,, \ldots \sigma$ then

$$A^k = \sum_{i=1}^{\sigma} \left[A_{i0} \lambda_i^k 1(k) + \sum_{l=1}^{n_i - 1} A_{il} l! \begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l} 1(k-l) \right]$$

- if $\lambda_j \ = \ 0\,,\ \lambda_j \in \{\lambda_1, \ldots, \lambda_\sigma\}$ then the corresponding response modes are

$$A_{j0} \cdot \delta(k) + \sum_{l=1}^{n_j - 1} A_{jl}\, l!\, \delta(k-l)$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

- The matrices A_{il} can be determined as

$$A_{il} = \frac{1}{l!}\frac{1}{(n_i-1-l)!}\lim_{z\to\lambda_i}\left\{\frac{d^{n_i-1-l}}{dz^{n_i-1-l}}\left[(z-\lambda_i)^{n_i}(zI-A)^{-1}\right]\right\}$$

where $l = 0,\ 1,\ 2,\ \ldots,\ n_i - 1$.

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Stability and A^k

Using the response modes

$$A^k = \sum_{i=1}^{\sigma} \sum_{l=0}^{n_i-1} \left[A_{il} l! \begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l} 1(k-l) \right]$$

For the stability analysis, the **boundedness of the free-state movement** has to be ascertained. Since the matrices A_{jl} does not depend on k, it suffices to **analyse the boundedness of the terms**

$$\begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l} 1(k-l) \qquad l = 0,\ 1,\ 2,\ \ldots,\ n_i - 1$$

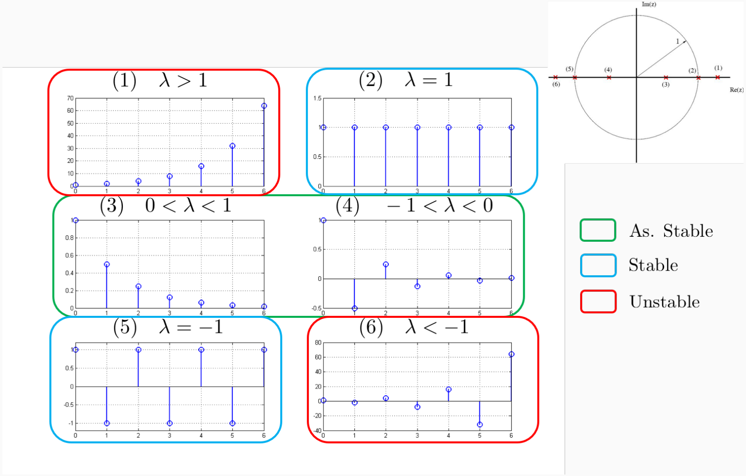
where n_i is the **algebraic multiplicity** of the eigenvalue λ_i.

Stability of Linear Discrete-Time Systems

Stability Criterion Based on Eigenvalues

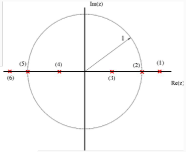
Stability & Qualitative Behaviour of Response Modes

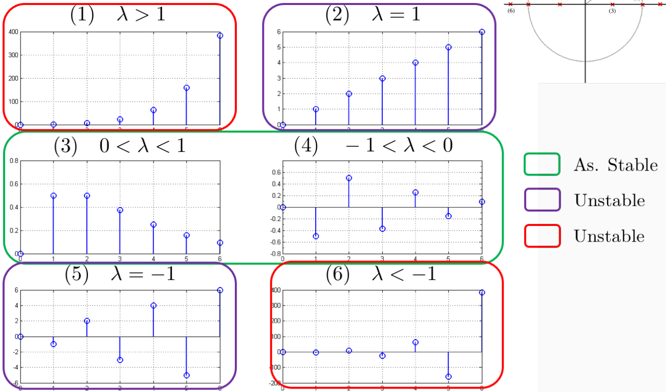
- $\begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l}$ with $\lambda_i \in \mathbb{R}$, multiplicity $n_i = 1$ (so $l = 0$).



Stability & Qualitative Behaviour of Response Modes

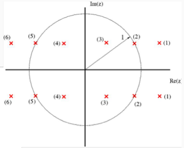
- $\begin{pmatrix} k \\ l \end{pmatrix} \lambda_i^{k-l}$ with $\lambda_i \in \mathbb{R}$, mult. $n_i > 1$ ($l = 0,\ 1,\ \ldots n_i - 1$).

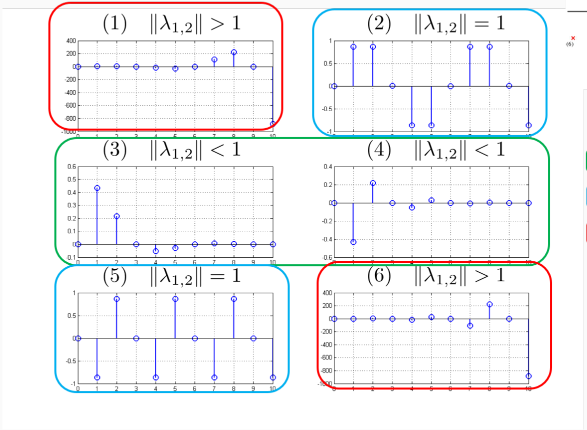




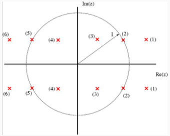
Stability & Qualitative Behaviour of Response Modes

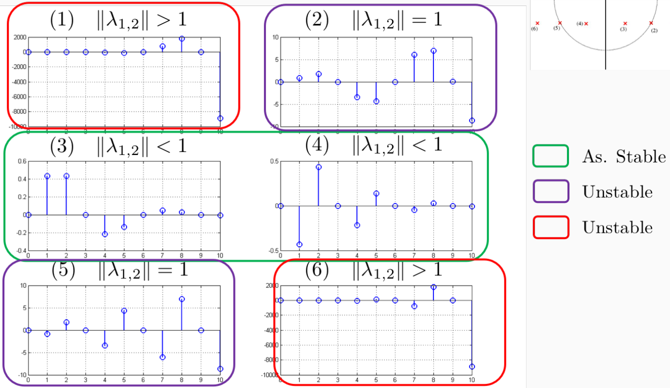
- $\binom{k}{l} \lambda_i^{k-l}$ with $\lambda_i \in \mathbb{C}$, multiplicity $n_i = 1$





- $\binom{k}{l} \lambda_i^{k-l}$ with $\lambda_i \in \mathbb{C}$, multiplicity $n_i > 1$



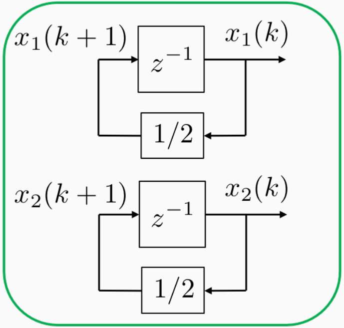


Asymptotically Stable

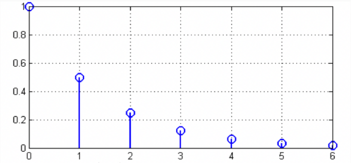
$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = \frac{1}{2}$$

$$A^k = \begin{bmatrix} (1/2)^k & 0 \\ 0 & (1/2)^k \end{bmatrix}$$



Response modes for $x_1(k)$ and $x_2(k)$



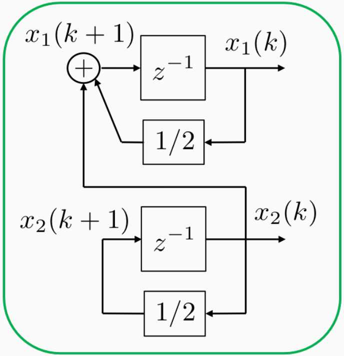
Stability & Behaviour of Response Modes: Example 2

Asymptotically Stable

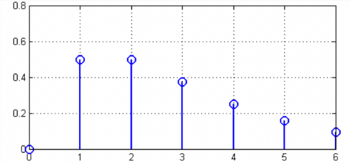
$$A = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = \frac{1}{2}$$

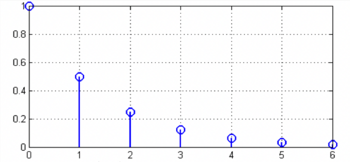
$$A^k = \begin{bmatrix} (1/2)^k & k(1/2)^{k-1} \\ 0 & (1/2)^k \end{bmatrix}$$



Response mode for $x_1(k)$



Response mode for $x_2(k)$



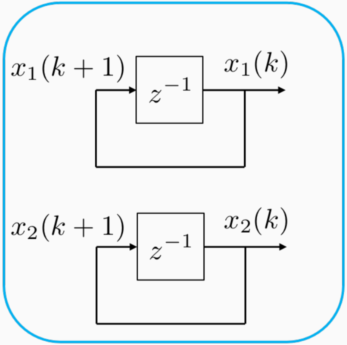
Stability & Behaviour of Response Modes: Example 3

Stable (not asymptotically)

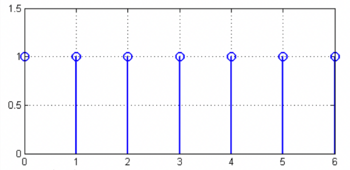
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 1$$

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Response modes for $x_1(k)$ and $x_2(k)$



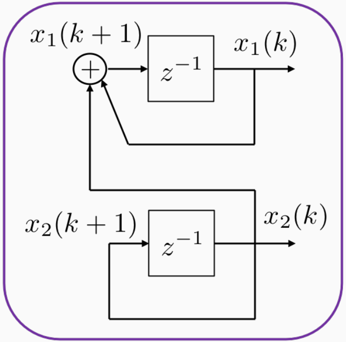
Stability & Behaviour of Response Modes: Example 4

Unstable

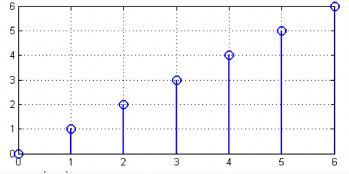
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 1$$

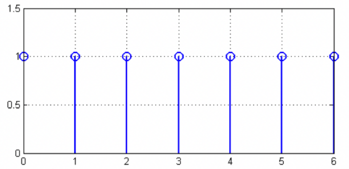
$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



Response mode for $x_1(k)$



Response mode for $x_2(k)$



Algebraic vs Geometrical Multiplicity of an Eigenvalue

- Let $\bar{\lambda}$ be an **eigenvalue** of A.
- The **eigenvectors** of A associated with $\bar{\lambda}$ are the nonzero vectors in the **nullspace** of $A - \bar{\lambda}\, I$, called **the eigenspace** of A for $\bar{\lambda}$ and denoted by

$$\text{null}\left(A - \bar{\lambda}\, I\right) = \mathcal{E}_A\left(\bar{\lambda}\right)$$

- The **geometric multiplicity** of the eigenvalue $\bar{\lambda}$ of A is the dimension of $\mathcal{E}_A\left(\bar{\lambda}\right)$.
- The **algebraic multiplicity** of the eigenvalue $\bar{\lambda}$ of A is the multiplicity of $\bar{\lambda}$ as a root of the characteristic polynomial of A $p_A(z) = \det\left(zI - A\right)$.

Algebraic and Geometrical Multiplicity of an Eigenvalue (cont.)

Diagonalisable Matrices – Algebraic vs Geometrical Multiplicity of an Eigenvalue

- In general, an eigenvalue's algebraic and geometric multiplicity can differ. However, the geometric multiplicity can never exceed the algebraic one.
- Let $\lambda_1, \dots, \lambda_\sigma$ the distinct eigenvalues of A and n_i the algebraic multiplicity of such eigenvalues. Of course $\sum_{i=1}^{\sigma} n_i = n$
- If for every eigenvalue of A, the geometric multiplicity equals the algebraic multiplicity, then A is said to be **diagonalisable**.

Complete Stability Criterion Based on Eigenvalues of A

Stability Criterion

Given the system $x(k+1) = Ax(k)$ and denoting by $\lambda_i,\ i = 1,\dots n$ the eigenvalues of matrix A.

- $|\lambda_i| < 1,\ \forall\, i = 1,\dots n \quad \Longleftrightarrow$ The system is as. stable
- $\exists\, i,\ 1 \le i \le n:\ |\lambda_i| > 1 \quad \Longrightarrow$ The system is unstable
- $\left.\begin{array}{l} |\lambda_i| \le 1,\ \forall\, i = 1,\dots n \\ \exists\, j,\ 1 \le j \le n:\ |\lambda_j| = 1 \end{array}\right\} \Longrightarrow$ The system is not as. stable
  - $\lambda_j : |\lambda_j| = 1$ have algebraic multiplicity $= 1$, then the system is stable (not as.)
  - $\lambda_j : |\lambda_j| = 1$ have algebraic multiplicity > 1 and the same value as geometrical multiplicity, then the system is stable (not as.)
  - $\lambda_j : |\lambda_j| = 1$ have algebraic multiplicity > 1, but the geometrical multiplicity is different, then the system is unstable

Stability of Linear Discrete-Time Systems

Analysis of the Characteristic Polynomial

Stability by Analysing the Characteristic Polynomial

- The previous complete stability criterion requires checking whether the eigenvalues of matrix A belong to the unit circle in the complex plane
- A number of techniques exist to perform the check above **without explicitly calculating** the eigenvalues of matrix A
- Considering the characteristic polynomial

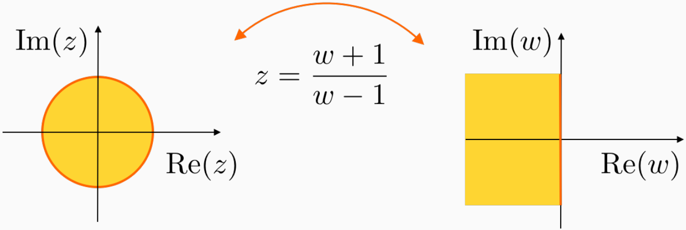
$$p_A(z) = \det(zI - A) = \varphi_0 z^n + \varphi_1 z^{n-1} + \cdots + \varphi_{n-1} z + \varphi_n$$

  a suitable **bi-linear transformation** allows to reduce the problem of checking whether the roots of polynomial $p_A(z)$ belong to the unit circle in the complex plane to an **equivalent problem** of checking whether the roots of a suitable polynomial $q_a(w)$ belong to the complex left half-plane
- This equivalent problem can then be solved by using the **Routh-Hurwitz** technique (see the course *Fundamentals of Automatic Control*)

Use of the Bi-linear Transformation

$$z = \frac{w+1}{w-1}, \; z, w \in \mathbb{C}$$

$$|z| < 1 \iff \mathrm{Re}(w) < 0$$
$$|z| = 1 \iff \mathrm{Re}(w) = 0$$
$$|z| > 1 \iff \mathrm{Re}(w) > 0$$



Use of the Bi-linear Transformation (cont.)

Substitute

$$z = \frac{w+1}{w-1}, \; z, \, w \in \mathbb{C}$$

into

$$p_A(z) = \varphi_0 z^n + \varphi_1 z^{n-1} + \cdots + \varphi_{n-1} z + \varphi_n$$

thus obtaining

$$q_A(w) = \; (w-1)^n \left[\varphi_0 \frac{(w+1)^n}{(w-1)^n} + \varphi_1 \frac{(w+1)^{n-1}}{(w-1)^{n-1}} + \cdots \right.$$
$$\left. + \varphi_{n-1} \frac{(w+1)}{(w-1)} + \varphi_n \right]$$

and hence one gets

$$q_A(w) = q_0 \, w^n + q_1 \, w^{n-1} + \cdots + q_{n-1} \, w + q_n$$

with suitable coefficients $q_0, \; q_1, \ldots, q_n$.

Use of the Bi-linear Transformation. Example 1

Given

$$p_A(z) \;=\; z^3 + 2\,z^2 + z + 1$$

one gets

$$q_A(w) \;=\; (w-1)^3 \left[\frac{(w+1)^3}{(w-1)^3} + 2\,\frac{(w+1)^2}{(w-1)^2} + \frac{w+1}{w-1} + 1 \right]$$

and after some algebra

$$q_A(w) = 5\,w^3 + w^2 + 3\,w - 1$$

3	5	3
2	1	-1
1	8	
0	-1	

Hence, there is one root of $q_A(w)$ on the complex right-half plane which in turn implies that one of the roots of $p_A(z)$ lies outside the unit circle.

Use of the Bi-linear Transformation. Example 2

Given

$$p_A(z) = z^2 + az + b$$

with $a, b \in R$. Thus, one gets:

$$q_A(w) \; = \; (w-1)^2 \left[\frac{(w+1)^2}{(w-1)^2} + a \, \frac{(w+1)}{(w-1)} + b \right]$$

and after some easy algebra

$$q_A(w) = (1+b+a)w^2 + 2(1-b)w - a + 1 + b$$

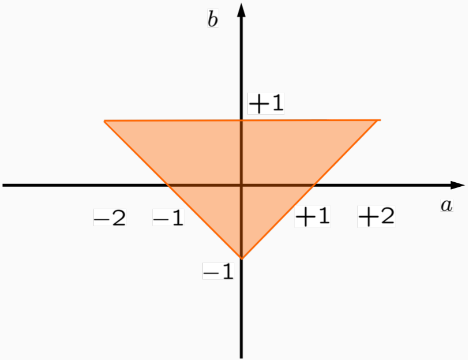
$$\begin{array}{c|cc} 2 & (1+b+a) & (1+b-a) \\ 1 & 2(1-b) & \\ 0 & (1+b-a) & \end{array}$$

$$\begin{cases} 1+b+a > 0 \\ 2(1-b) > 0 \\ 1+b-a > 0 \end{cases} \Longrightarrow \begin{cases} b > -a-1 \\ b < 1 \\ b > a-1 \end{cases}$$

Use of the Bi-linear Transformation. Example 2 (cont.)

The stability condition has a nice geometric interpretation:

$$\begin{cases} b > -a - 1 \\ b < 1 \\ b > a - 1 \end{cases}$$



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