Oct. 6.

$$\boxed{|u(t)|^2_{\dot H^s} + 2\int_0^t |\nabla u(t')|^2_{\dot H^s}\, dt'}$$

$$= |u_0|^2_{\dot H^s} + 2\int_0^t \langle f(t'), u(t')\rangle_{\dot H^s}\, dt'$$

$$\leq |u_0|^2_{\dot H^s} + 2\int_0^t |f(t')|_{\dot H^{s-1}} |\nabla u|_{\dot H^s}\, dt'$$

$t \in [0,T]$

$$\langle |\xi|^s \hat f, |\xi|^s \hat u\rangle =$$

$$= \langle |\xi|^{s-1}\hat f, |\xi|^{s+1}\hat u\rangle$$

$$\leq |u_0|^2_{\dot H^s} + 2|f|_{L^2([0,T],\dot H^{s-1})} |\nabla u|_{L^2([0,T],\dot H^s)}$$

$$\leq \boxed{|u_0|^2_{\dot H^s} + \cancel{|\nabla u|^2_{L^2([0,T],\dot H^s)}} + |f|^2_{L^2([0,T],\dot H^{s-1})}}$$

$$\geq |u|^2_{L^\infty([0,T],\dot H^s)} + \cancel{2}|\nabla u|^2_{L^2([0,T],\dot H^s)}$$

$$|u|^2_{L^\infty([0,T],\dot H^s)} + |\nabla u|^2_{L^2([0,T],\dot H^s)}$$

$$\leq |u_0|^2_{\dot H^s} + |f|^2_{L^2([0,T],\dot H^{s-1})}$$

$$|u|_{L^\infty([0,T],\dot H^s)} +$$
$$\leq |u_0|_{\dot H^s} + |f|_{L^2([0,T],\dot H^{s-1})}$$

$$|u|_{L^2([0,T],\dot H^{s+1})}$$
$$\leq |u_0|_{\dot H^s} + |f|_{L^2([0,T],\dot H^{s-1})}$$

$\forall$

$2 \leq p \leq \infty$

$$|u|_{L^p([0,T],\dot H^{s+\frac{2}{p}})}$$
$$\leq |u_0|_{\dot H^s} + |f|_{L^2([0,T],\dot H^{s-1})}$$

$$\boxed{|u|_{\dot H^{s+\frac{2}{p}}} \leq |u|_{\dot H^s}^{1-\frac{2}{p}} \, |u|_{\dot H^{s+1}}^{\frac{2}{p}}}$$

$$s + \frac{2}{p} = \left(1 - \frac{2}{p}\right) s + \frac{2}{p}(s+1) =$$
$$= s + \frac{2}{p}$$

$$|u|_{L^p \dot H^{s+\frac{2}{p}}} = \left| \, |u|_{\dot H_x^{s+\frac{2}{p}}} \right|_{L^p_t}$$

$$\lesssim \left\| \|u\|_{\dot H^s}^{1-\frac{2}{p}} \|u\|_{\dot H^{s+1}}^{\frac{2}{p}} \right\|_{L^p_t}$$

$$\lesssim \|u\|_{L^\infty \dot H^s}^{1-\frac{2}{p}} \left\| \|u\|_{\dot H^{s+1}} \right\|_{L^2_t}^{\frac{2}{p}}$$

$$= \|u\|_{L^\infty \dot H^s}^{1-\frac{2}{p}} \|u\|_{L^2_t \dot H^{s+1}_x}^{\frac{2}{p}}$$

$$\lesssim \|u_0\|_{\dot H^s} + \|f\|_{L^2 \dot H^{s-1}}$$

$$\|u\|_{L^\infty([0,T], \dot H^s)}$$

$$V(t) \doteq \left(\int_{\mathbb{R}^d} |\xi|^{2s} \sup_{0 \le t' \le t} |\hat u(t', \xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

$$V(t) \le \|u_0\|_{\dot H^s} + \frac{1}{\sqrt{2}} \|f\|_{L^2([0,t], \dot H^{s-1})}$$

$$\hat u(t,\xi) = e^{-t|\xi|^2} \hat u_0(\xi) + \int_0^t dt' \, e^{-(t-t')|\xi|^2} \hat f(t', \xi)$$

$$|\hat u(t,\xi)| \le e^{-t|\xi|^2} |\hat u_0(\xi)|$$

$$+ \|\hat f\|_{L^2(0,t)} \underbrace{\|e^{-t|\xi|^2}\|_{L^2(0,+\infty)}}_{}$$

$$\int_0^{+\infty} e^{-2t|\xi|^2}\,dt = -\frac{e^{-2t|\xi|^2}}{2|\xi|^2}\Bigg]_0^{+\infty}$$

$$= \frac{1}{2|\xi|^2}$$

$$\sup_{0\le t'\le t} |\xi|^s\,|\hat{u}(t',\xi)| \le \cancel{e^{-t|\xi|^2}}\,|\hat{u}_0(\xi)|\,|\xi|^s$$

$$+ \|\hat{f}\|_{L^2(0,t)}\,\frac{1}{\sqrt{2}\,\cancel{|\xi|}} \qquad |\xi|^{s-1}$$

$V(t)$

$$\cancel{\sup_{0\le t'\le t} \big\|\,|\xi|^s\,\hat{u}(t',\xi)\big\|_{L^2_\xi}} \le \|u_0\|_{\dot{H}^s} + \frac{1}{\sqrt{2}}\|f\|_{L^2((0,t),\dot{H}^{s-1})}$$

Navier-Stokes Equation

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u = -\nabla p \\ \nabla\cdot u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

$$u\cdot\nabla u = u\cdot\nabla u_j\,\vec{e}_j = u_k\,\partial_k u_j\,\vec{e}_j$$

$$\boxed{u_k\,\partial_k u_j = \partial_k(u_k u_j)} - \underbrace{(\partial_k u_k)}_{0}\,u_j$$

$\mathrm{div}\,(u\otimes u)$

$$u\otimes v = \{u_k v_j\}_{k,j=1}^{d}$$

$$\mathrm{div}\,\{M_{kj}\} = \{\partial_k M_{kj}\}_{j=1}^{d}$$

$$\partial_t u - \Delta u + \mathrm{div}\,(u\otimes u) = -\nabla p \qquad .\,\mathbb{P}.$$

$(1-\mathbb{P})\,u = 0 \Rightarrow \partial_t u,\ \Delta u$ are divergence free

$$(1-\mathbb{P})\,\Delta\, u = \Delta\,(1-\mathbb{P})\,u = 0$$

$$(1-\mathbb{P})\,\mathrm{div}(u\otimes u) = -\nabla p$$

$$\partial_t u - \Delta u + \mathbb{P}\,\mathrm{div}(u\otimes u) = 0$$

Def Let $u_0 \in L^2(\mathbb{R}^d)$

A $u \in L^2_{loc}([0,\infty) \times \mathbb{R}^d, \mathbb{R}^d)$

such that $u \in C^0_w([0,+\infty), L^2(\mathbb{R}^d, \mathbb{R}^d))$

$\left(t \to \langle u(t), \phi \rangle \in C([0,+\infty), \mathbb{R}\right) \ \forall \phi$

and $\operatorname{div} u(t) \equiv 0$ is a weak solution $\forall \ \psi \in C^\infty_{c\sigma}([0,+\infty) \times \mathbb{R}^d, \mathbb{R}^d)$

$$\langle u(t), \psi(t) \rangle =$$

$$= \int_0^t \Big(\langle u(t'), \Delta\psi(t') \rangle + \langle u(t'), \partial_t \psi(t') \rangle$$
$$- \langle \operatorname{div}(u \otimes u), \psi \rangle \Big) \, dt'$$

$$+ \langle u_0, \psi(0) \rangle$$

$$\langle \operatorname{div}(u \otimes u), \psi \rangle =$$

$$= \langle \partial_k (u_k u_j), \psi_j \rangle = -\langle u_k u_j, \partial_k \psi_j \rangle$$

$$\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \qquad \langle \, , u \rangle$$

$$\langle \partial_t u, u \rangle + \overbrace{\langle u \cdot \nabla u, u \rangle}^{0} - \langle \Delta u, u \rangle$$

$$= - \underbrace{\langle \nabla p, u \rangle}_{0}$$

$$\langle \partial_j p, u_j \rangle = - \langle p, \overbrace{\partial_j u_j}^{0} \rangle = 0$$

$$\langle u_j \cdot \partial_j u_k, u_k \rangle = \langle \partial_j (u_j u_k), u_k \rangle$$

$$= - \langle u_j, u_k \, \partial_j u_k \rangle = - \frac{1}{2} \langle u_j, \partial_j (u_k u_k) \rangle$$

$$= \frac{1}{2} \langle \underbrace{\partial_j u_j}_{0}, u_k u_k \rangle$$

$$\langle \partial_t u, u \rangle - \langle \Delta u, u \rangle = 0$$

$$\frac{d}{dt} \| u \|_{L^2_x}^2 + 2 | \nabla u |_{L^2}^2 = 0$$

$$| u(t) |_{L^2_x}^2 + 2 \int_0^t | \nabla u(t') |_{L^2}^2 \, dt' =$$

$$= | u_0 |_{L^2_x}^2$$

Teor $u_0 \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ $d = 2, 3$, divergence free.

Then $\exists$ a weak solution

$$u \in L^\infty(\mathbb{R}_+, H) \cap L^2_{loc}(\mathbb{R}_+, V)$$

such that the following energy inequality holds:

$$|u(t)|^2_{L^2} + 2\int_0^t |\nabla u(t')|^2_{L^2}\, dt' \leq |u_0|^2_{L^2}$$

Thm (case $d=2$) The solution is unique, $u \in C^0([0, +\infty), L^2)$ and the energy identity is valid.

Lemma $d = 2, 3$

$$(u, v, \varphi) \in \left(C^\infty_c(\mathbb{R}^d, \mathbb{R}^d)\right)^3 \longrightarrow \langle \operatorname{div}(u \otimes v), \varphi \rangle \in \mathbb{R}$$

extends uniquely

$$\left(H^1(\mathbb{R}^d, \mathbb{R}^d)\right)^3 \longrightarrow \mathbb{R}$$

and $\exists$ C_d s.t.

$$\left| \langle \mathrm{div}(u \otimes v), \varphi \rangle \right| \leq C_d \, |\nabla u|_{L^2}^{\frac{d}{4}} \, |\nabla v|_{L^2}^{\frac{d}{4}} \, |u|_{L^2}^{1-\frac{d}{4}} \, |v|_{L^2}^{1-\frac{d}{4}} \, |\nabla \varphi|_{L^2}$$

and if $\mathrm{div}\, u = 0$ we have

$$\langle \mathrm{div}(u \otimes v), v \rangle = 0 .$$

$$\left| \langle \partial_k (u_k v_j), \varphi_j \rangle \right| = \left| - \langle u_k v_j, \partial_k \varphi_j \rangle \right|$$

$$\leq |u_k v_j|_{L^2} \, |\nabla \varphi|_{L^2}$$

$$\leq |u_k|_{L^4} \, |v_j|_{L^4} \, |\nabla \varphi|_{L^2}$$

$$\leq C_{GN}^2 \, |\nabla u_k|_{L^2}^{\frac{d}{4}} \, |u_k|_{L^2}^{1-\frac{d}{4}} \cdots \, |\nabla \varphi|_{L^2}$$