11 ottobre

Thm $u_0 \in H$

$\exists$ a (global) weak soluta such that

$$|u(t)|_{L^2}^2 + 2\int_0^t |\nabla u|_{L^2}^2 \, dt' \leq |u_0|_{L^2}^2$$

Pf We regularize the NS

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}(u \cdot \nabla u) = 0 \\ u|_{t=0} = u_0 \end{cases}$$

$\forall \varepsilon > 0$ $\quad \rho \in C_c^\infty(\mathbb{R}^d; [0,1])$ $\int \rho \, dx = 1$

$$\rho_\varepsilon = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$$

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}(\rho_\varepsilon * u \cdot \nabla u) = 0 \\ u(0) = \rho_\varepsilon * u_0 \end{cases}$$

$$u = e^{t\Delta} \rho_\varepsilon * u_0 - \Phi_\varepsilon(u)(t) \qquad (*)$$

$$\Phi_\varepsilon(u)(t) = \int_0^t e^{(t-t')\Delta}\,\mathbb{P}(\rho_\varepsilon * u \cdot \nabla u)\,dt'$$

<u>Lemma</u> Equation ⊛ has a unique maximal solution, It is global

$$u \in C^0([0,+\infty), L^2(\mathbb{R}^d)) \cap$$
$$\cap L^\infty([0,+\infty), L^2(\mathbb{R}^d))$$
$$\cap L^2([0,+\infty), \dot{H}^1(\mathbb{R}^d))$$

$\{u_\varepsilon\}_{\varepsilon>0}$ $\qquad \varepsilon_n \xrightarrow{n\to+\infty} 0$ has a subsequence $u_{\varepsilon_{n_k}} \xrightarrow{k\to+\infty} u$

<u>Lemma</u> X Banach

$B : X \times X \to X$ continuous bilinear

$$\|B\| = \sup_{\|x\|_X = \|y\|_X = 1} \|B(x,y)\|_X$$

Let $\alpha < \dfrac{1}{4\|B\|}$

and let $x_0 \in D_X(0, \alpha)$

Then the equation

$$x = x_0 + B(x, x)$$

has a solution $\bar{x} \in \overline{D_X(0, 2\alpha)}$,

and the solution is unique

Dim $x \to x_0 + B(x,x)$ ① we ~~the~~ frame as a fixed point problem.

First we show that $\overline{D_X(0, 2\alpha)}$ is invariant by ①

$$\|x_0 + B(x,x)\| \leq |x_0| + |B(x,x)|$$

$$< \alpha + \|B\| \, \|x\|^2 \leq \alpha + |B| \, 4\alpha^2$$

$$= \alpha \, (1 + \underbrace{4\alpha \, |B|}_{<1}) < 2\alpha$$

$$\alpha < \frac{1}{4|B|}$$

We check that 1 is a contraction

$$\left| \cancel{x_0} + B(x,x) - \cancel{x_0} - B(y,y) \right| =$$

$$= \left| \underbrace{B(x,x) - B(x,y)}_{B(x,x-y)} + \underbrace{B(x,y) - B(y,y)}_{B(x-y,y)} \right|$$

$$\leq |B|\,|x|\,|x-y| + |B|\,|y|\,|x-y|$$

$$\leq \underbrace{|B| 4 \alpha}_{<1} \, |x-y|$$

<u>Pf. of lemma for truncation of NS</u>

$T \in \mathbb{R}_+$

$$X = L^\infty([0,T], H) \cap L^2([0,T], \dot{H}^1)$$

$$B(u,v) = -\int_0^t e^{(t-t')\Delta} \mathbb{P}\left((\rho_\varepsilon * v) \cdot \nabla u \right) dt'$$

$$= -\int_0^t e^{(t-t')\Delta} \mathbb{P} \operatorname{div} \left((\rho_\varepsilon * v) \otimes u \right) dt'$$

$$|B(u,v)|_{L^\infty([0,T], L^2) \cap L^2([0,T], \dot{H}^1)} \leq$$

$$\leq C \; \| \mathbb{P} \operatorname{div}((\rho_\varepsilon * v) \otimes u) \|_{L^2([0,T], \dot{H}^{-1})}$$

$$\leq C \; \| (\rho_\varepsilon * v) \otimes u \|_{L^2([0,T], L^2)}$$

$$\leq C \sqrt{T} \, \| \rho_\varepsilon * v \|_{L^\infty([0,T], L^\infty)} \; \| u \|_{L^\infty([0,T], L^2)}$$

$$\leq C \sqrt{T} \; \| \rho_\varepsilon * v \|_{L^\infty([0,T], H^d)} \; \| u \|_{L^\infty([0,T], L^2}$$

$$\leq C_\varepsilon \sqrt{T} \quad \| v \|_{L^\infty([0,T], L^2)} \; \| u \|_{L^\infty([0,T], L^2)}$$

$$\| \rho_\varepsilon * v \|_{L^2} \leq C_d \; \| v \|_{L^2} \; \| \rho_\varepsilon \|_{L^1}$$

$$\frac{1}{2} + 1 = \frac{1}{2} + 1$$

$$u = e^{t\Delta} \rho_\varepsilon * u_0 + B(u, u)$$

$$\| e^{t\Delta} \rho_\varepsilon * u_0 \|_{L^\infty([0,T], L^2) \cap L^2([0,T], \dot{H}^1)}$$

$$\le |\rho_\varepsilon * u_0|_{L^2} \le |u_0|_{L^2} \le \alpha$$

$$|B| \le C_{\varepsilon d} \sqrt{T}$$

↳ $\alpha |B| < 1$

↳ $|u_0|_{L^2} \, C_{\varepsilon d} \sqrt{T} < 1 \qquad (2)$

We choose $T = T(|u_0|_{L^2})$ so that (2) holds.

Then we find there exists a solution of

$$u = e^{t\Delta} \rho_\varepsilon * u_0 + B(u,u)$$

in $L^\infty([0,T], L^2) \cap L^2([0,T], \dot{H}^1)$

Notice that

$$u \in C^0([0,T], L^2)$$

Let $u \in C^0([0,T^*), L^2)$

be a maximal solution, with lifespan T^*.

We show now that $T^* = +\infty$.

we claim that

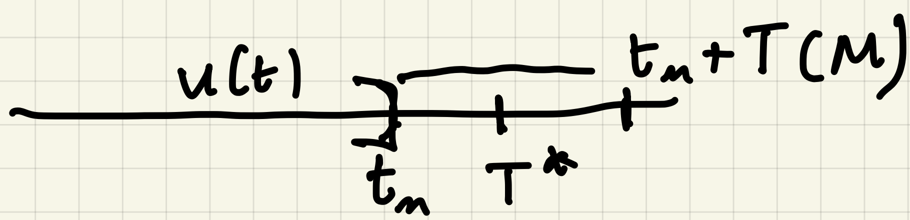
If $T^* < +\infty$ then

$$\lim_{t \to T^*} |u(t)|_{L^2} = +\infty$$

Suppose our claim is false

on t here exists $t_n \to T^*$

with $|u(t_n)|_{L^2} \le M$



Let n be such that $t_n + T(M) > T^*$

$$w(t) = \begin{cases} u(t) & t \le t_n \\ v(t-t_n) & t_n \le t \le t_n + T(M) \end{cases}$$

$$v(t) = e^{(t-t_n)\Delta} u(t_n) - \int_0^t e^{(t-t')\Delta} \mathbb{P}(\rho_\varepsilon * v \cdot \nabla v)\,dt'$$

w solves the equation and necessarily

$w = u$ in $[0, T^*)$

w is defined in $[0, t_n + T(M)]$

$\Rightarrow$ u is not a maximal solution.

contradiction

Energy equality excludes ~~$\nexists$~~

blow up

$$|u(t)|_{L^2}^2 + 2\int_0^t |\nabla u|_{L^2}^2\,dt'$$

$$= |\rho_\varepsilon * u_0|_{L^2}^2 - 2\int_0^t \underbrace{< \mathrm{div}((\rho_\varepsilon * v) \otimes u), u>}_{0} dt'$$

$$\leq |u_0|_{L^2}^2$$

$< \mathrm{div}(u \otimes v), v> = 0$ if $\mathrm{div}\, u = 0$

$$\mathrm{div}\,(\rho_\varepsilon * v) = \rho_\varepsilon * \underbrace{\mathrm{div}\, v}_{0} = 0$$

$\square$

u_ε $\rho_\varepsilon * u_0$

$\varepsilon_n \longrightarrow 0$

$u_n = u_{\varepsilon_n}$

$$\forall \psi \in C^\infty_{c\sigma}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$$

$$\langle u_n(t), \psi(t) \rangle = \int_0^t \Big(\langle u_n, \Delta\psi \rangle + \langle u_n, \partial_t \psi \rangle - \langle \mathbb{P} \operatorname{div}(\rho_{\varepsilon_n} * u_n \otimes u_n), \psi \rangle \Big) dt' + \langle \rho_{\varepsilon_n} * u_0, \psi(0) \rangle$$

$d = 3$

$$u_n \in L^\infty([0,T], L^2) \cap L^2([0,T], L^6)$$

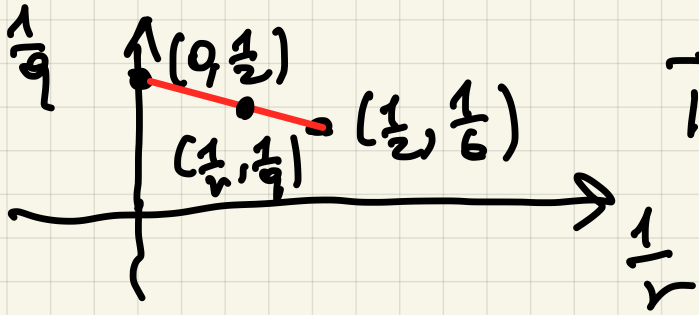
$$\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$$

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$$

$$u_n \in L^r([0,T], L^q(\mathbb{R}^3))$$

$$(r,q) < \begin{matrix} (\infty, 2) \\ (2, 6) \end{matrix}$$

$$\frac{3}{q} + \frac{2}{r} = \frac{3}{2} \quad \longleftarrow \quad q = r = \frac{10}{3}$$



$$\frac{9}{10} + \frac{6}{10} = \frac{15}{10}$$

$$0 \le t \le 1$$

$$\left(\frac{1}{r}, \frac{1}{q}\right) = (1-t)\left(0, \frac{1}{2}\right) + t\left(\frac{1}{2}, \frac{1}{6}\right)$$

$$|u_n|_{L^r(\mathbb{R}_+, L^q)} \le |u_n|^{1-t}_{L^\infty([0,T],L^2)} \, |u_n|^{t}_{L^2([0,T],L^6)}$$

$$\le |\rho_\epsilon * u_0|_{L^2} \le |u_0|_{L^2}$$

$$|u_n|_{L^{\frac{10}{3}}(\cancel{[0,T]}\,\mathbb{R}_+ \times \mathbb{R}^3)} \le |u_0|_{L^2}$$

$$\forall n$$

It is not reductive to conclude that there is $u \in L^{\frac{10}{3}}(\mathbb{R}_+ \times \mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $L^{\frac{10}{3}}(\mathbb{R}_+ \times \mathbb{R}^3)$