# Mathematics Education Approaches

## HOW TO DEVELOP STUDENTS' MATHEMATICAL SKILLS?

Lecture 5 – 12/10/23

| Content Area | Theoretical bases of learning | |
|---|---|---|
| | TRANSMISSIVE | CONSTRUCTIVIST |
| Nature of Knowledge | Mathematics as tools | Mathematics as processes |
| Mathematical Learning and Teaching | • The clarity in procedural solving<br>• Receptive learning through examples and demonstrations<br>• Automation of procedural techniques | • Learning autonomous discursive and intuitive<br>• Belief in students' mathematical thinking independency |

# Problematic facets of transmissive approach

Computational Facets

(CALCOLO)

Procedural Facets

(PROCEDURE)

RESULTS

(RISULTATI)

Didactics approach that present objects mechanically as facts to be learnt and therefore never questioned

LEARNING BASED ON MEMORIZING COMPUTATIONS AND PROCEDURE

To overcome the persistence of Maths difficulties in secondary and higher education, researchers profoundly revised the teaching approach to arithmetic and algebra, thinking of a trans-inter-domain of integration between them based on functional relations.

(D. W. Carraher and Schliemann, 2018; Kaput et al., 2008)

# EARLY ALGEBRA

AS A SPECIFIC DOMAIN OF MATHS TEACHING

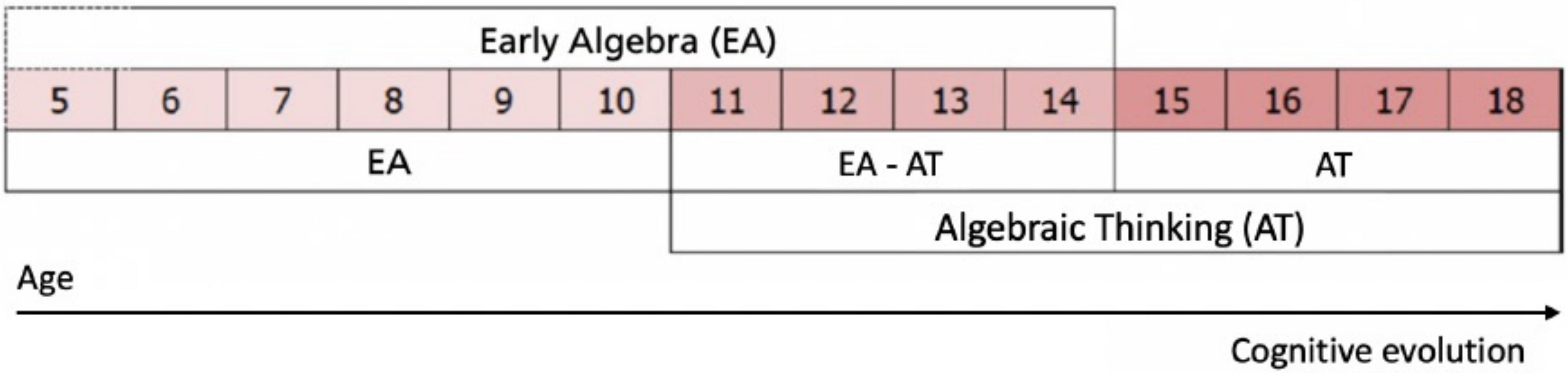
# EARLY ALGEBRA REFERS TO:

- the algebraic knowledge
- the algebraic thinking
- the (initially rather unconventionally) problem solving representations and technique

The focus on representing relationships among quantities affords its origins in the Russian-based approach developed by Davydov and his colleagues at the end of the past century (Davydov et al., 1999).

They emphasised the teaching of Algebra based not on its numerical foundations but on relationships among quantities. In this way, young students do not solve equations by thinking about “doing and undoing” numerical operations but by direct comparisons between quantities (Kieran et al., 2016).

There is no clear-cut break between *Early Algebra* and Algebra. *Early Algebra*, not to be referred to as “pre-algebra”, is not to be viewed as a bridge students cross after they have studied arithmetic and before they study Algebra.

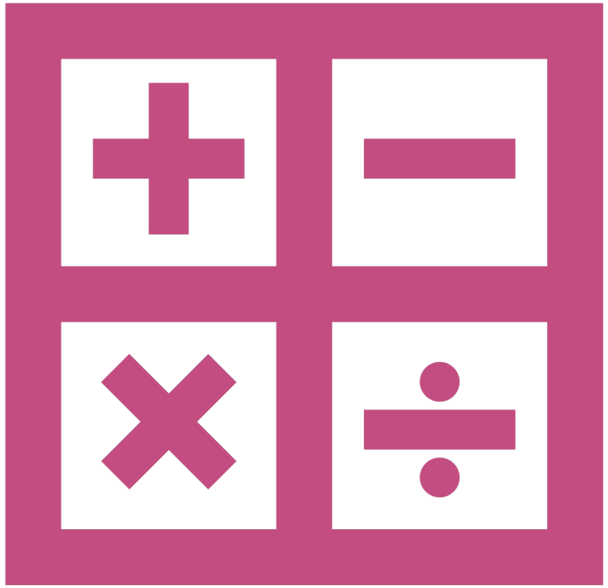



Temporal progression and intertwining of *Early Algebra* and the development of algebraic thinking (Navarra, 2019).

In principle, it can be developed and nurtured wherever there is arithmetic.

This is because arithmetic is inherently algebraic

(D. W. Carraher and Schliemann, 2018; D. Carraher and Schliemann, 2007).



The notion of functions and their representations provide powerful means for modelling physical attributes and measures, for justification or proof, and for adapting mathematical models when dealing with every day or science applications.

(D. W. Carraher and Schliemann, 2018)

# Discovering the role of functional relations, the algebraic nature of arithmetic is brought out for many reasons:

- arithmetic operations are functions;
- introductions of variables as placeholders for arbitrary members of sets and the extension of the classes of numbers supported by the concept of domains and range (or co-domain);
- multiple representations of functions are profitably employed in unison;
- comparison of two functions is inherently interpreted as equations and inequalities.

Algebraic reasoning is thus identified through formulating and operating upon relations, particularly towards functional relations.

With this framework, functions and relations become an unexploited resource for teaching and learning: this is a unique role in early algebraic thinking.

The conceptual development focus on evoking students' view about a problem involving relations among sets of quantities and gradually introducing new mathematical representations, conventions and tools.

Students discuss, represent and solve open-ended problems, focusing on relations between sets of quantities instead of performing computations on specific pairs of numbers.

The teacher acts following-up questions and suggestions built upon students' ideas and representations, introducing new ideas and representations.

In this way, students are engaged in classroom discourses which could be called "algebraic babbling".
(Malara, 1994; Malara and Navarra, 2018; Navarra, 2019, 2022).

Analogous to how children learn **natural language**, students learn to communicate in algebraic language by starting from its meaning and, through collective discussion, verbalisation, and argumentation, gradually become proficient in syntax
(Kieran, 2004; Kieran et al., 2016).

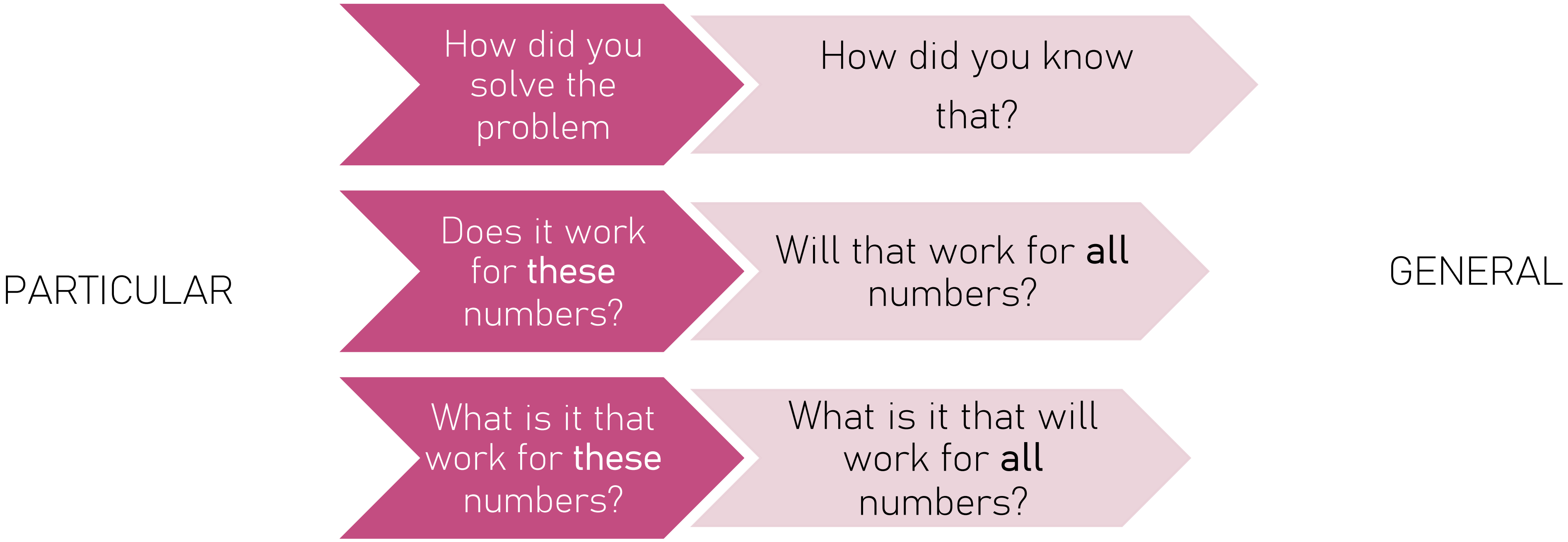
In order to establish a setting in which students engage in this manner, teacher needs to:

- set the expectation that students support their ideas with explanations,
- probing and challenging each other's ideas to make sure they follow classmates' reasoning;
- clarify questions to help students make the details of algebraic thinking explicitly;
- acknowledge and validate students' proposals to encourage classroom discussion;
- help students address contrasts in their thinking.

All these teachers' tasks involve students in meta-cognitive acts, reflecting on their observations and then moving to a level of generalisation and argument.
(Kieran et al., 2016)

Students improve their understanding by substituting the act of calculating with looking at oneself while calculating.
(Malara, 1994; Malara and Navarra, 2018; Navarra, 2022).

# Teachers support this perspective's shift-changing questions.




Process of generalisation

But the teacher only asks these questions if the algebraic goal is clear.

And this implies a change in teachers' dispositions, knowledge and skills in teaching Maths in an *Early Algebra* context.

To clarify what happens in the construction of knowledge of algebraic equations, we could consider the example where the equality-equivalence aspect is stressed.

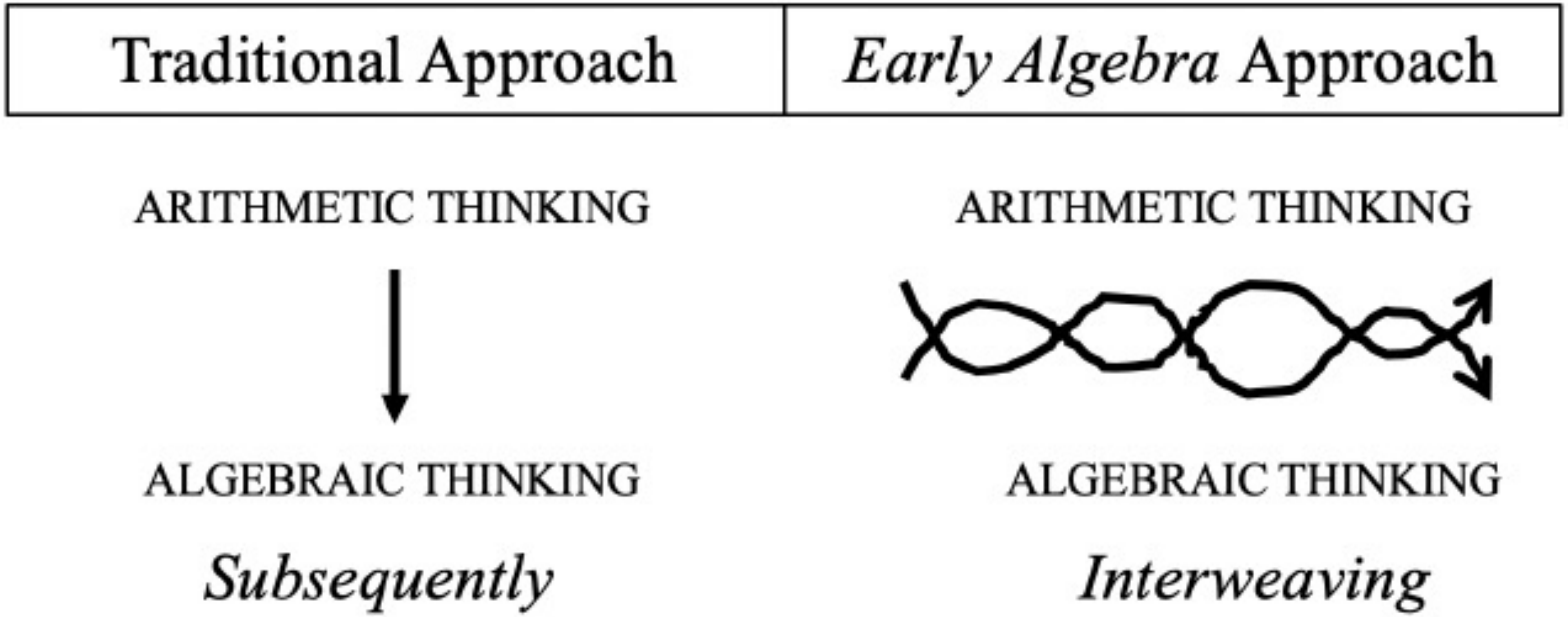
Treating the equal sign as a procedural symbol that announces the answer after a series of operations is so common that it hides its strong meaning of showing two different representations of the exact quantities.

Underlining, highlighting, expressing and emphasising this mutual role (equality-equivalence) enact the power of different representations instead of the direction of something to resolve to obtain a number (that, of course, it's the unique right solution). So, again the teachers' change is in the scaffolding questions: not more than "How much is it." or "How much does it make?" (that is a literal translation of a typical Italian Math teacher's question), but "Which is the process you did to represent this relation between quantities?".

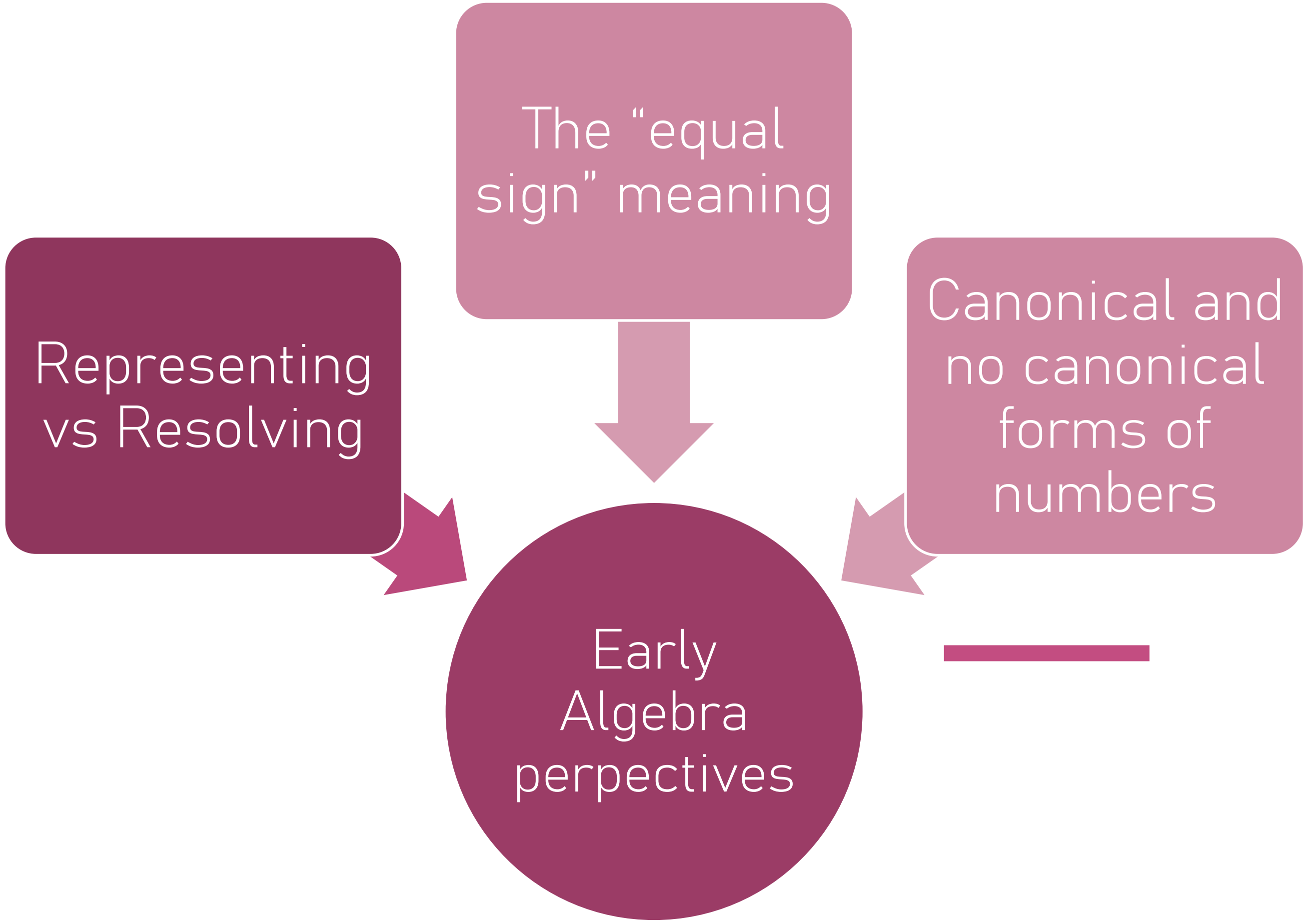
The focus is on the process, not the product (intending the result of a sequence of operations), on multiple representations and not on a single resolution (the result of computing or problem solving). In this way, the emphasis is on natural language and its role of paramount importance as a semantic facilitator.

**Treating Algebra with a language changes the teaching approach**

Algebra as a Language: approaches' differences (Navarra, 2019)



# The concepts involved



Su un ramo ci sono 3 corvi. Ne arrivano altri 5.
*Quanti sono i corvi rimasti sul ramo?*

Su un ramo ci sono 3 corvi. Ne arrivano altri 5.
***Rappresenta** la situazione in linguaggio matematico in modo che qualcun altro possa trovare il numero dei corvi sul ramo.*

Gli alunni propongono frasi come:

3+5 5+3 3+5=8 3+5= 8 3+5=n

Come si possono interpretare in relazione alla consegna?

| Prospettiva **aritmetica** | Prospettiva **algebrica** |
| --- | --- |

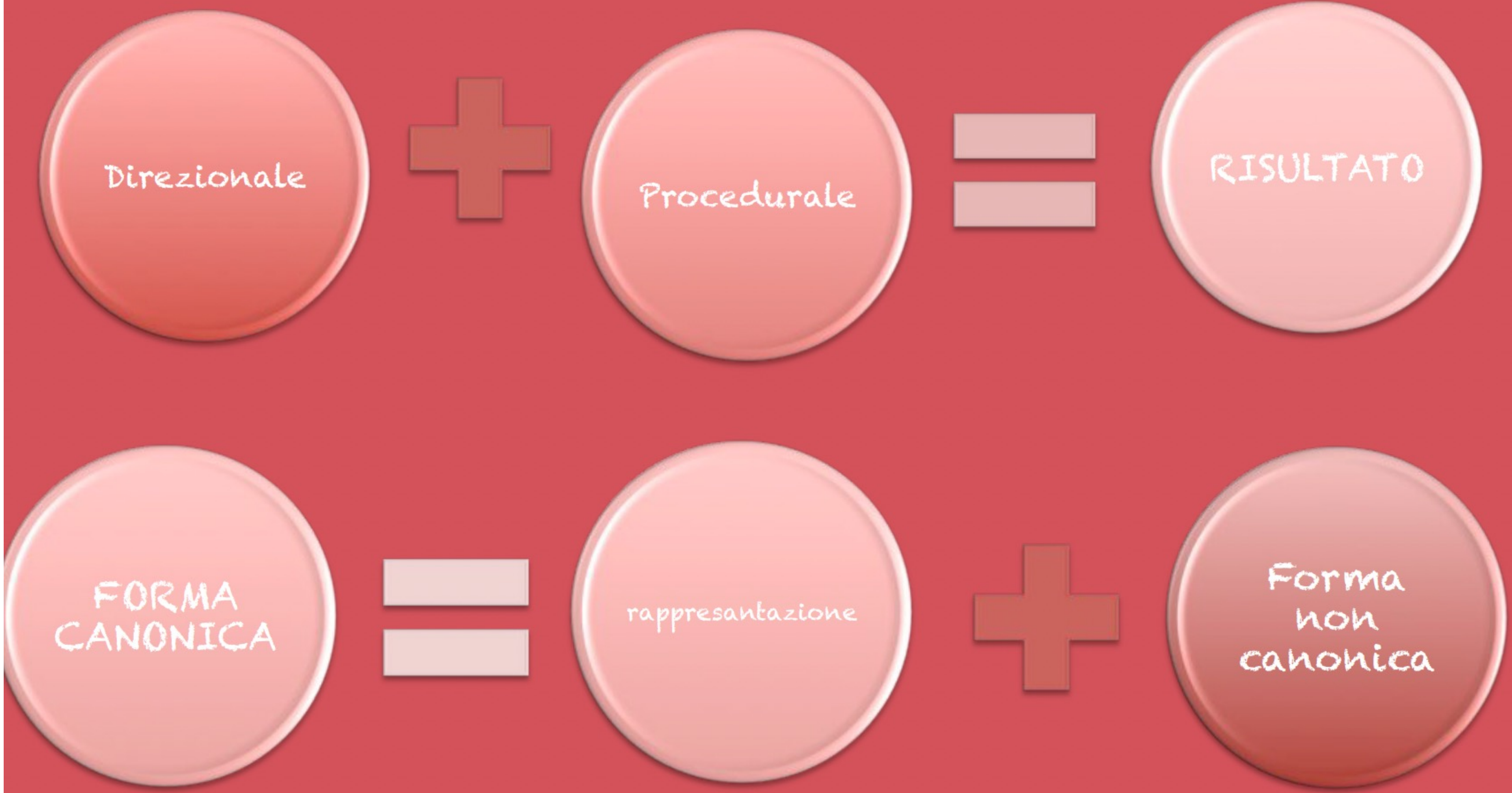
*Sul ramo ci sono 3 corvi. Ne arrivano altri 5*

| | |
| --- | --- |
| *Quanti sono in tutto?* | *Rappresenta la situazione in linguaggio matematico.* |
| Cercare il risultato | **Posporre** la ricerca del risultato |
| **Prodotto** 8 | **Processo** 3+5; 5+3; 3+5=8 |

$3+5=8$

| Forma non canonica | Forma canonica |
|---|---|
| Processo | Prodotto |
| Trasparente | Opaco |

# Il significato dell'UGUALE



Gli alunni stanno riflettendo su:

5+6=11 11=5+6

Piero osserva:
"E' giusto dire che 5 più 6 fa 11 ma non si può dire che 11 'fa' 5 più 6, quindi è meglio dire che 5 più 6 'è uguale a' 11 perché in questo caso è vero anche il contrario".

Cosa potete dire della frase di Piero?

**Piero sta discutendo il** significato relazionale **del segno uguale.**

Miriam rappresenta il numero di dolci: (3+4)×6.

Alessandro scrive: 7×6.

Lea scrive: 42.

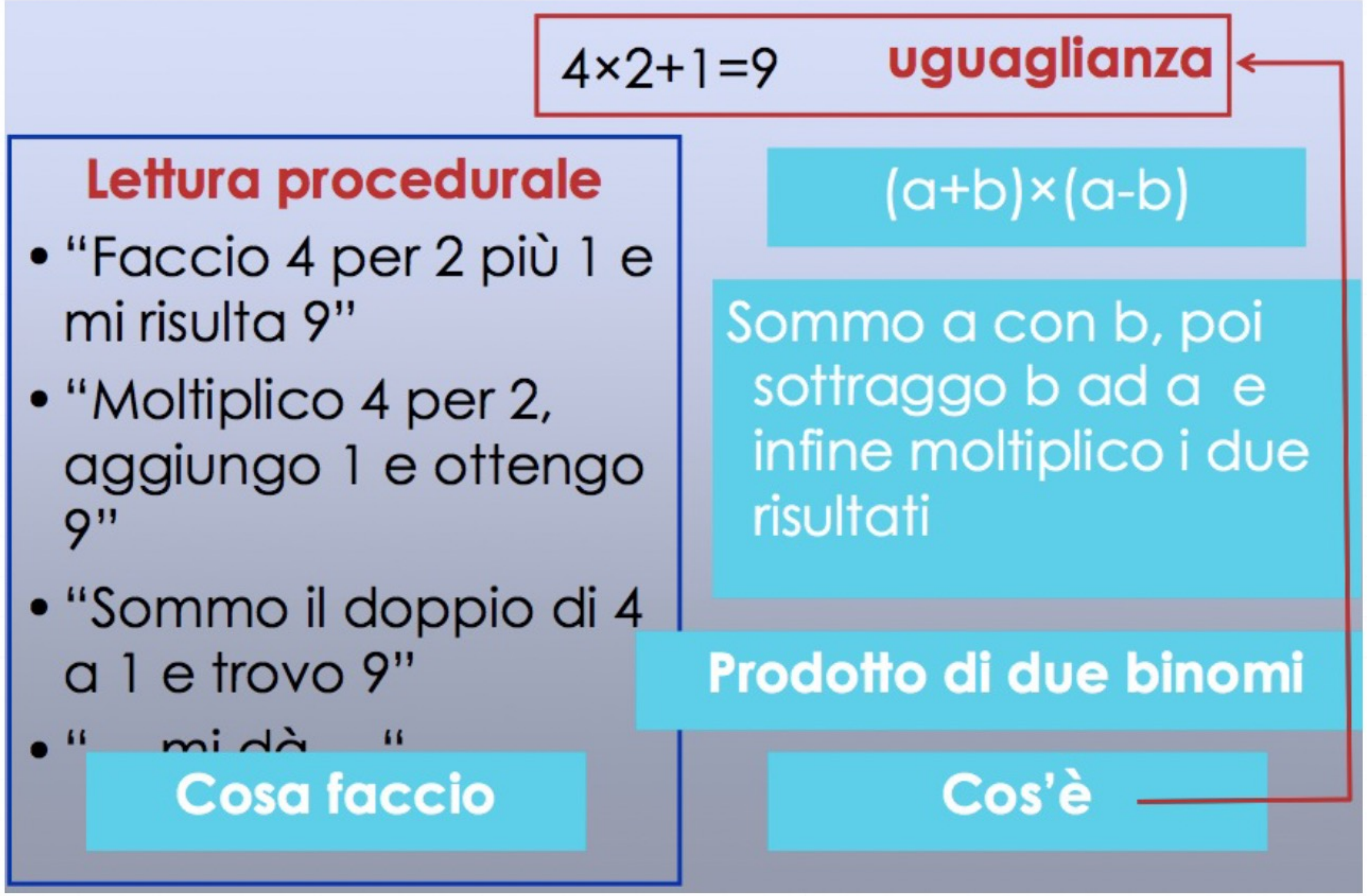
Miriam osserva: «Quello che ho scritto è più trasparente, le frasi di Alessandro di Lea sono opache. Opaco significa che non è chiaro, trasparente significa che è chiaro, che si capisce».

**Miriam riflette su come la rappresentazione non canonica di un numero aiuti a interpretare e illustrare la struttura di una situazione problematica**

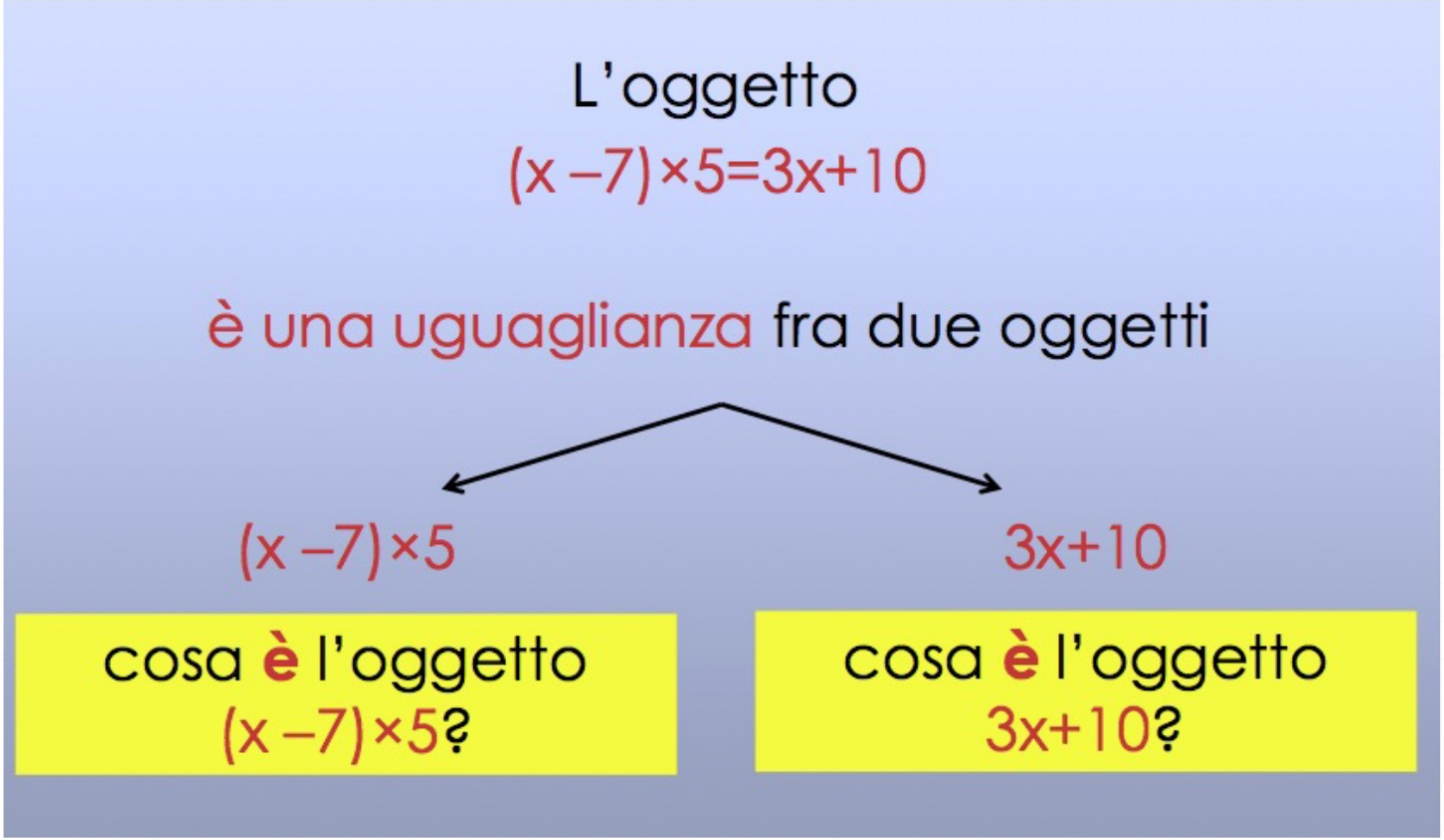
... ogni numero può essere rappresentato in diversi modi, attraverso una qualsiasi espressione equivalente ad esso: uno (ad esempio 12) è il suo nome, la cosiddetta forma **canonica**, tutti gli altri modi di nominarlo (3×4, (2+2)×3, 36/3, 10+2, 3×2×2, ...) sono forme **non canoniche**, e ognuna di loro riceverà un senso in relazione al contesto e al **processo** sottostante. Come Miriam osserva, la forma canonica, che rappresenta un **prodotto**, è **opaca** in termini di significati. La forma non canonica rappresenta un processo ed è **trasparente** in termini di significati.

Saper riconoscere e interpretare queste forme crea negli alunni la base semantica per accettare e comprendere, negli anni successivi, scritture algebriche come a-4p, ab, x2y, k / 3.
Il complesso processo che accompagna la costruzione di queste competenze dovrebbe essere sviluppato nel corso dei primi anni di scuola.
Il concetto di forma canonica / non-canonica comporta per gli alunni (e per i docenti) implicazioni essenziali per riflettere sui possibili significati attribuiti al segno di uguaglianza.

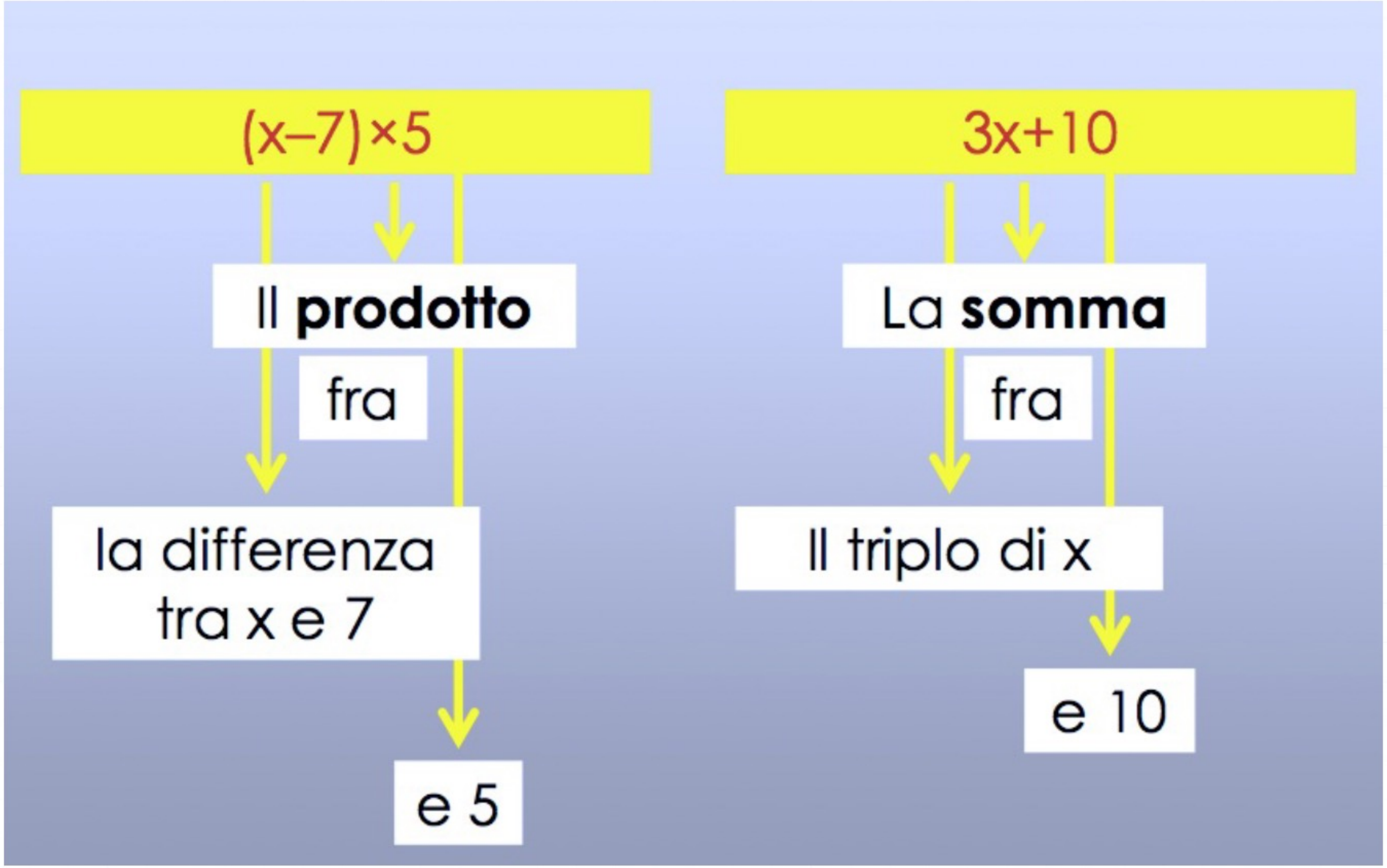
# Dalla lettura procedurale alla lettura relazionale



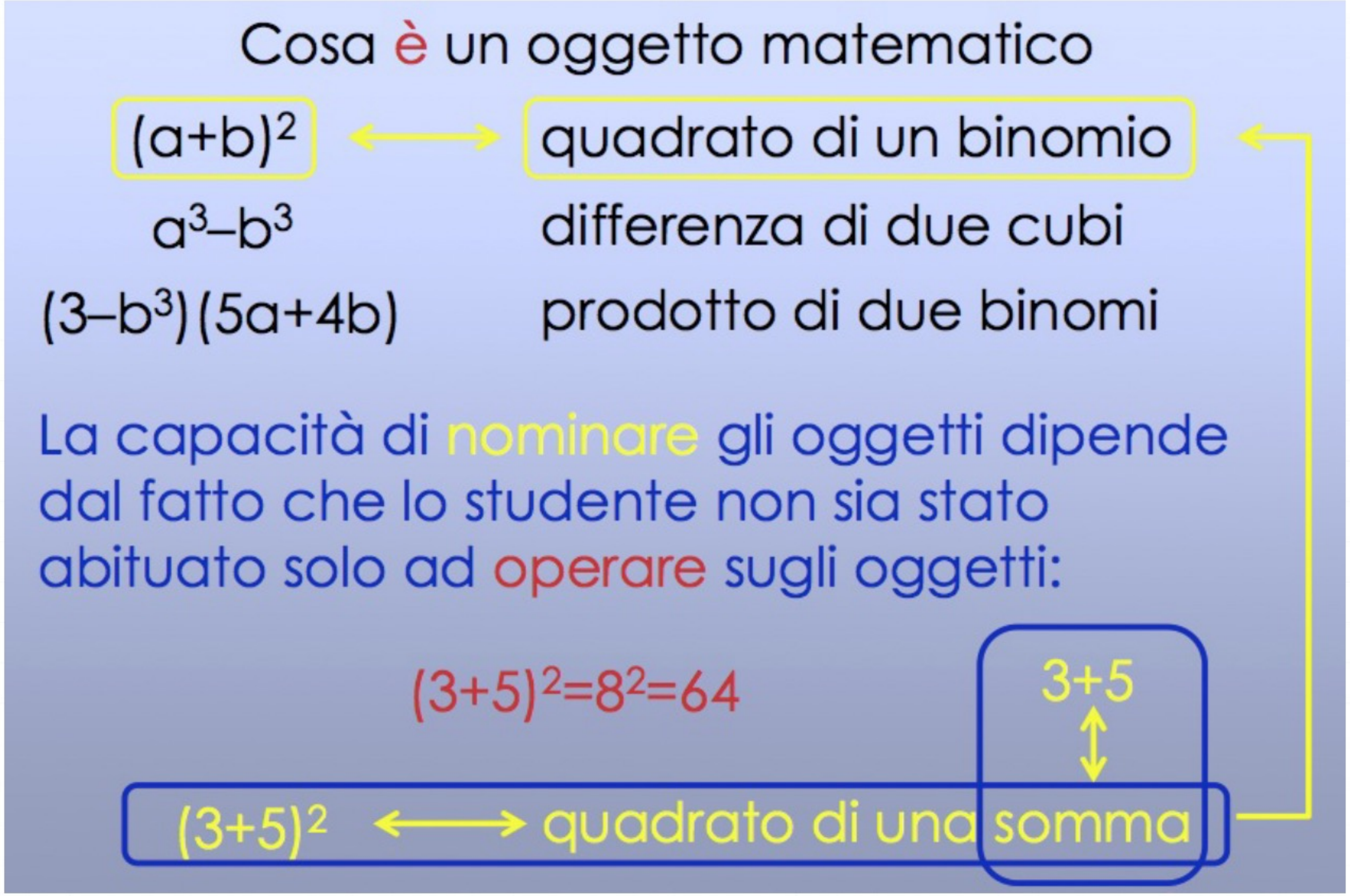
# Verso l'oggettivazione



# La lettura relazionale



# Verso il “*balbettio*” *algebrico*



## Sviluppo di competenze in ambito linguistico

- a livello **metalinguistico**, che comporta la comprensione del significato della consegna 'Traduci', che conduce alla categoria del **rappresentare**, contrapposta a quella del **risolvere**.
- a livello **linguistico**, relative all'interpretazione delle scritture e alla produzione delle traduzioni nei loro aspetti **semantici** e **sintattici**.

A1. Tradurre in linguaggio naturale un numero espresso in forma non canonica;
*Es: 3×2+5*

A2. Tradurre in linguaggio matematico un numero espresso attraverso una definizione procedurale;
*Es: Addiziona 4 a 15 e togli 9*

A3. Tradurre in linguaggio matematico un numero espresso attraverso una definizione relazionale.
*Es: Il doppio della somma fra 51 e 37*

A4. Esprimere in linguaggio naturale il confronto tra numeri scritti in forma canonica e non canonica, cogliendo le equivalenze senza calcoli scritti e argomentando le scelte

*Es:* $6 \times n - 4$ e $4 + n \times 3 \times 2$

A5. Ricavare scritture equivalenti ad una data esplicitando, dov'è possibile, le proprietà applicate

*Es:* $27 - \blacktriangle = 15$

A6. Completare frasi scritte in linguaggio matematico in cui un punto di domanda sostituisce un segno

Es: $5 \times 0$ ? $0:12$

# References


Davydov, V., Gorbov, S., Mukulina, T., Savelyeva, M., & Tabachnikova, N. (1999). *Mathematics*. Moscow Press.

Carraher, D. W., & Schliemann, A. D. (2018). Cultivating Early Algebraic Reasoning. In *Teaching and learning algebraic thinking with 5- 12- year-olds. the global evolution of an emerging field of research and practice. icme-13 monographs.* (pp. 107–138). Springer International Publishing.

Carraher, D., & Schliemann, A. (2007). Early algebra and algebraic reasoning. *Second handbook of research on mathematics teaching and learning*, *2*, 669–705.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it. *The Mathematics Educator*, *8*, 139–151.

Kaput, J., Carraher, D., & Blanton, M. (2008). *Algebra in the Early Grades* (1st ed.). Routledge.

Malara, N. A. (1994). Il pensiero algebrico: Come promuoverlo sin dalla scuola dell'obbligo limitandone le difficoltà? In *L'apprendimento della matematica: Dalla ricerca teorica alla pratica didattica* (pp. 67–77). Pitagora.

Malara, N. A., & Navarra, G. (2018). New words and concepts for early algebra teaching: sharing with teachers epistemological issues in early algebra to develop students' early algebra thinking. In *Teaching and learning algebraic thinking with 5- 12- year-olds. the global evolution of an emerging field of research and practice. icme-13 monographs.* (pp. 51–78). Springer International Publishing.

Navarra, G. (2019). Il progetto aral per un approccio relazionale all'insegnamento nell'area aritmetico-algebrica. *Didattica della matematica. Dalla ricerca alle pratiche d'aula*, (5), 70–94.

Navarra, G. (2022). *Aritmetica e algebra. un percorso intrecciato dai 5 ai 14 anni. ruoli dell'insegnante nella costruzione di una classe pensante.* UTET, Universitá.