$(\Omega, \mathcal{F}, P)$ X v.a.

$\sigma(X)$

Y $\sigma(X)$-MISURABILE

$$Y : \underset{\sigma(X)}{\Omega} \longrightarrow \underset{\mathcal{B}}{\mathbb{R}}$$

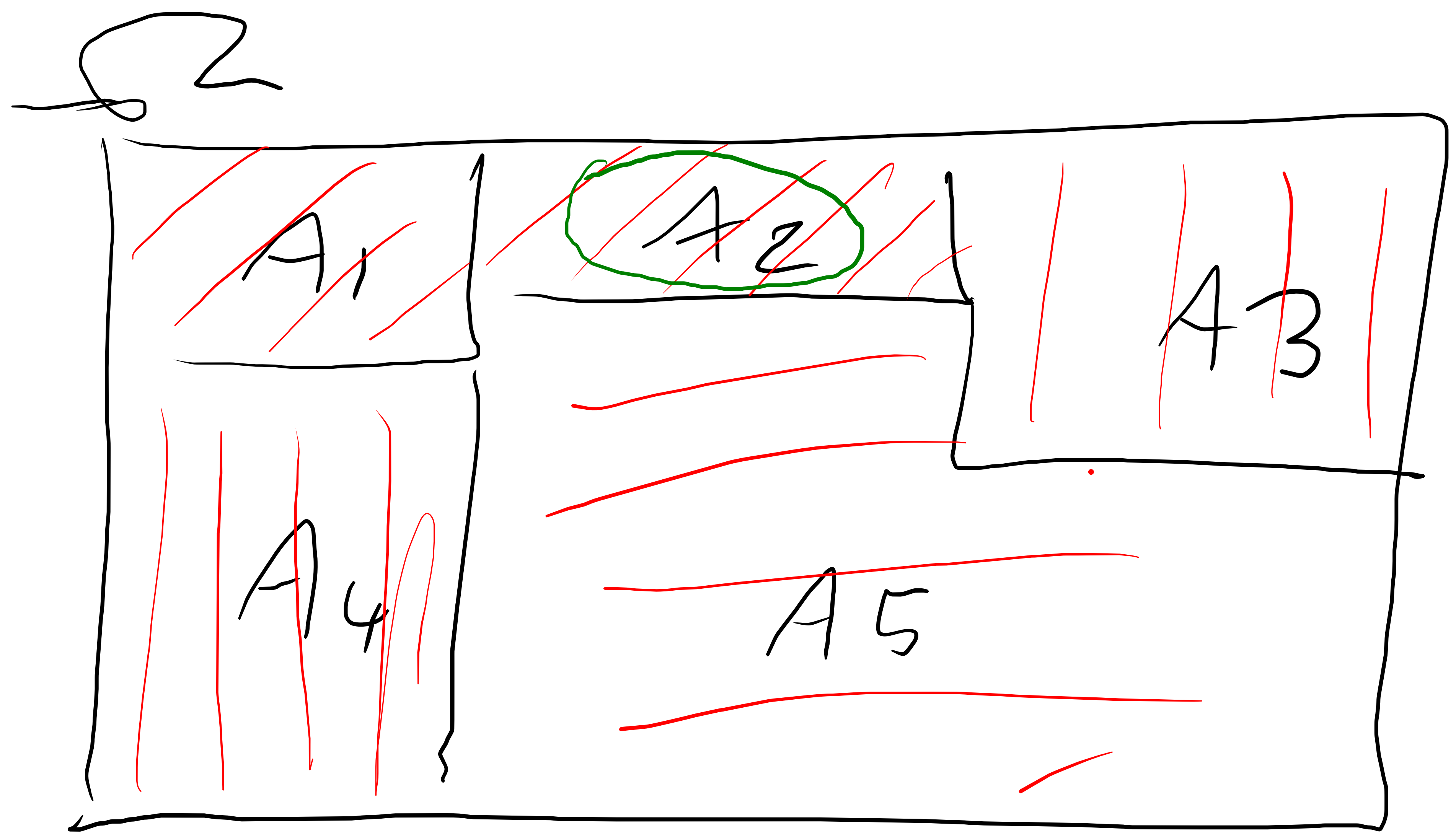
CHE RELAZIONE C'È TRA Y E X?

Y È FUNZIONE DI X

$\mathbb{P}$ PARTIZIONE DISCRETA

$\mathbb{P} = (A_n)_{n \geq 1}$

$$\mathcal{F} = \sigma(\mathbb{P}) = \left\{ \bigcup_{i \in I} A_i \;\middle|\; I \subset \mathbb{N}_+ \right\}$$



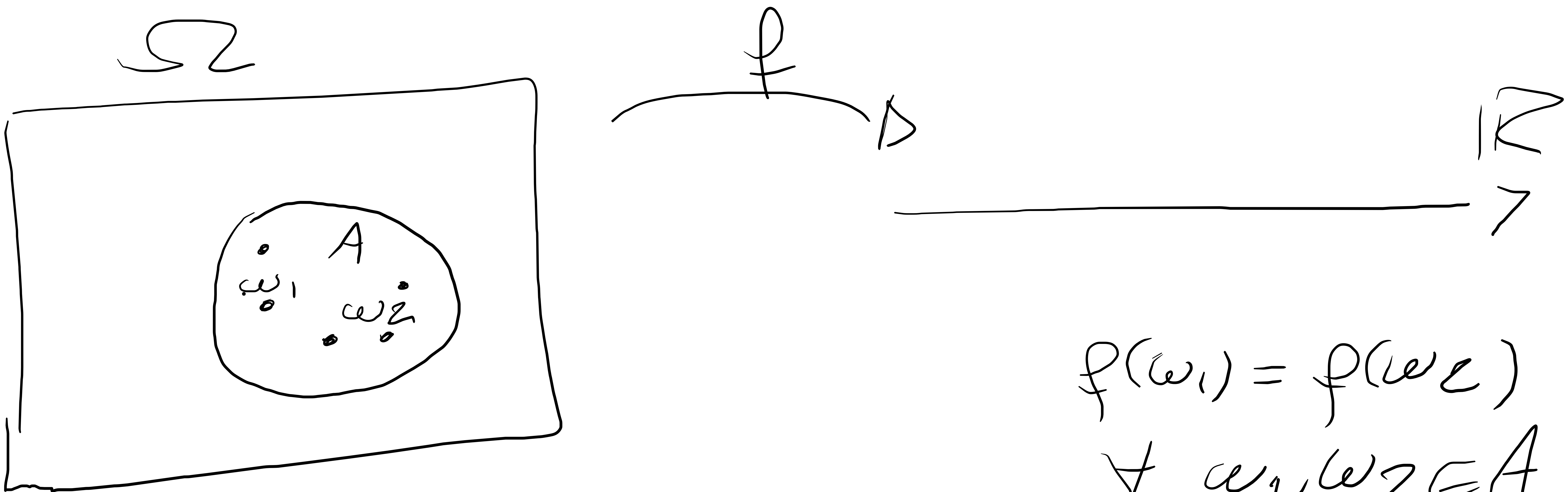
A_i È UN ATOMO DI $\mathcal{F} = \sigma(\mathbb{P})$

$B \in \mathcal{F}$, $B \subset A_i$

$\Rightarrow B = \emptyset$ OPPURE $B = A_i$

$$f : \underset{\mathcal{F}}{\Omega} \longrightarrow \underset{\mathcal{B}}{\mathbb{R}}$$

$$\left.\emptyset \neq A \text{ ATOMO DI } \mathcal{F}\right\} \Rightarrow f \text{ È COSTANTE SU } A$$



$$f(\omega_1) = f(\omega_2)$$
$$\forall\, \omega_1, \omega_2 \in A$$

SIA $\bar{\omega} \in A$ $\quad f(\bar{\omega}) = x$

CONSIDERIAMO

$$\underbrace{f^{-1}(\{x\})}_{\in \mathcal{B}} \in \mathcal{F}$$

$$f^{-1}(\{x\}) = \{\omega \in \Omega \mid f(\omega) = x\}$$

$$\bar{\omega} \in f^{-1}(\{x\})$$

$$\underbrace{A \cap f^{-1}(\{x\})}_{\neq \emptyset} \subset A \implies A = A \cap f^{-1}(\{x\})$$

$$\implies A \subset f^{-1}(\{x\})$$

CIOÈ f COSTANTE SU A

$(\Omega, \mathcal{F}, P)$ X LANCIO DI UN DADO

$$\mathbb{P}_1 = \{ \{X \text{ PARI}\}, \{X \text{ DISPARI}\} \}$$

$$\mathbb{P}_2 = \{ \{X = 1\}, \ldots, \{X = 6\} \}$$

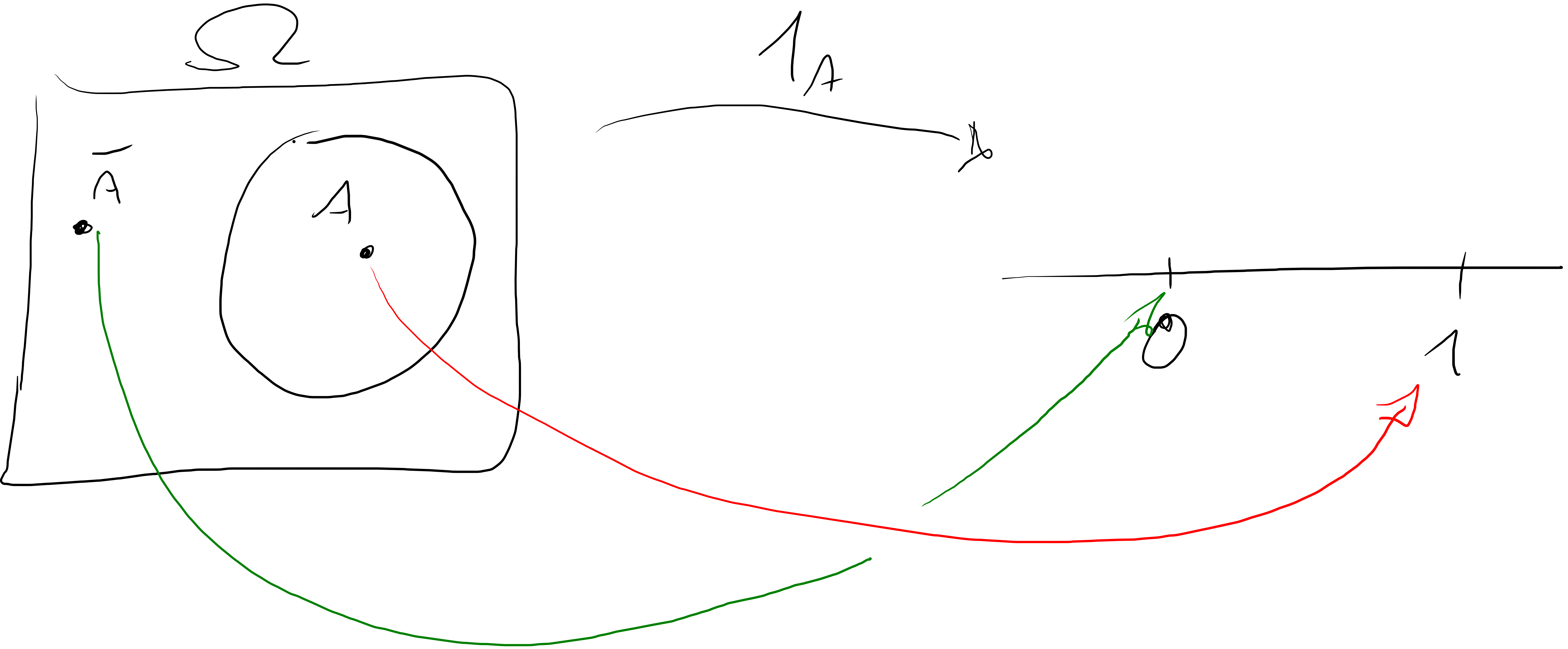
$$\mathcal{F}_1 = \sigma(\mathbb{P}_1) = \{ \emptyset, \{X \text{ PARI}\}, \{X \text{ DISPARI}\}, \Omega \}$$

$$\mathcal{F}_2 = \sigma(\mathbb{P}_2) = \{ \bigcup_{i \in I} \{X = i\} \mid I \subset \{1, \ldots, 6\} \}$$

$\{x \text{ PARI}\}$ È UN ATOMO DI $\mathcal{F}_1$

NON È UN ATOMO DI $\mathcal{F}_2$

$$\{x=2\} \subset \underbrace{\{x \text{ PARI}\}}_{\{x=2,4 \text{ o } 6\}}$$

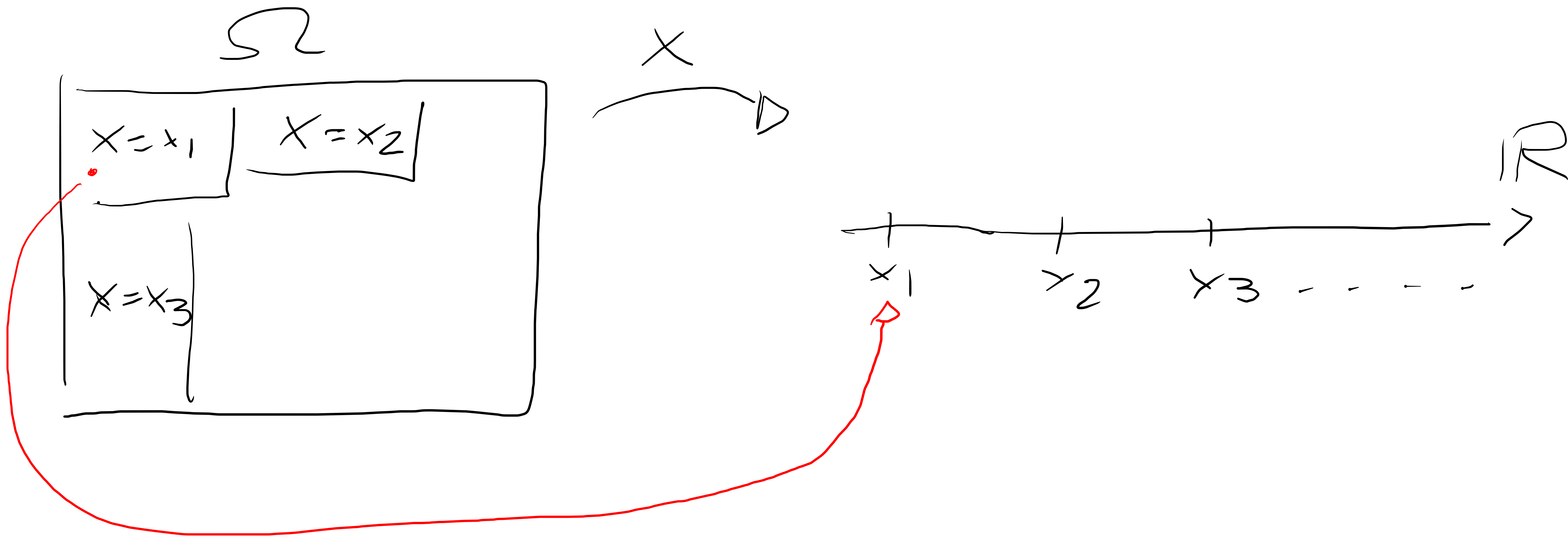


f è $\sigma(1_A)$-MISURABILE

f COSTANTE SU $A \longrightarrow x$

$\bar{A} \longrightarrow y$

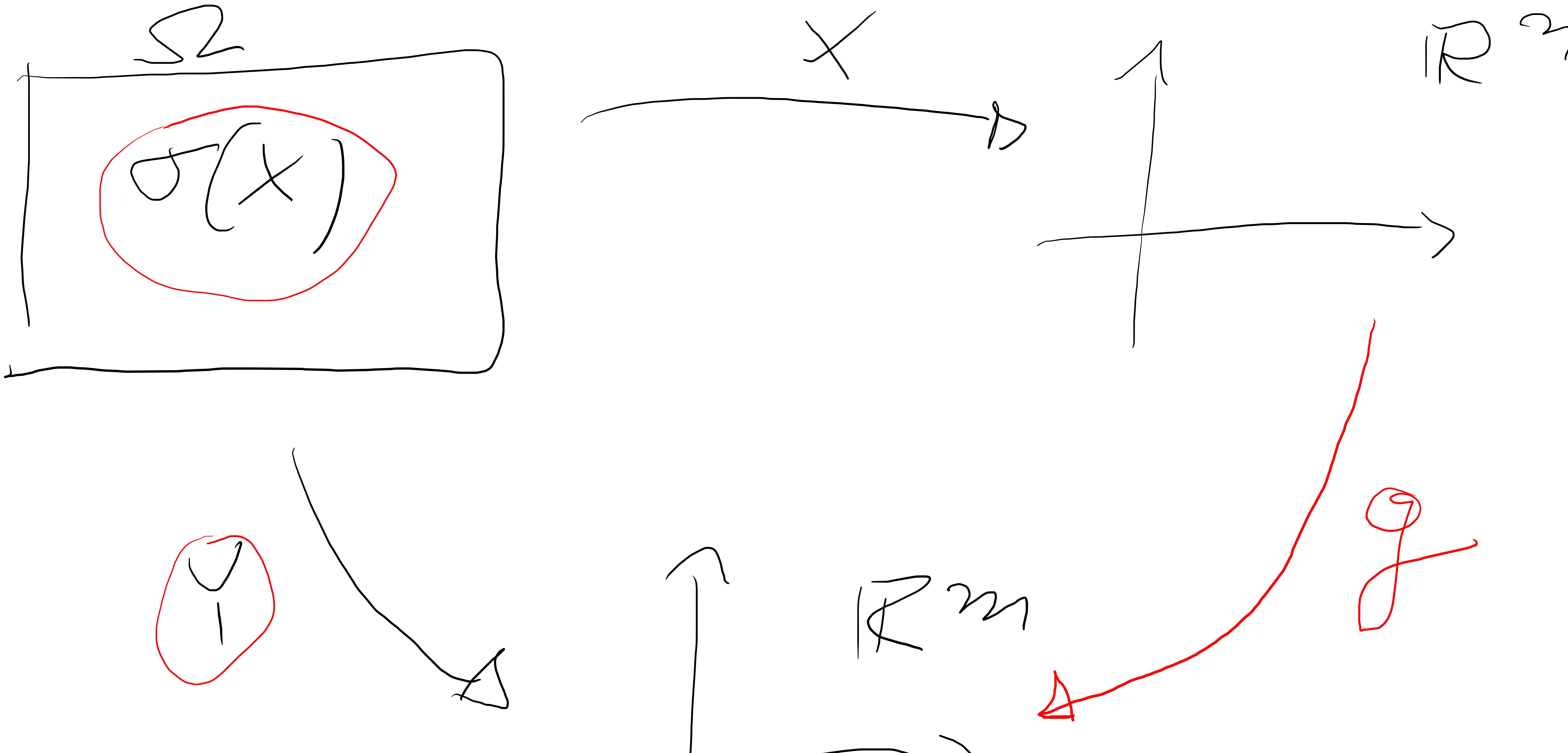
$$f(\omega) = \begin{cases} x & \text{SE } 1_A(\omega) = 1 \\ y & \text{SE } 1_A(\omega) = 0 \end{cases} = g(1_A(\omega))$$



Y È σ(X) - MISURABILE

<=> Y È COSTANTE SU $\{X = x_i\}$

<=> Y È FUNZIONE DI X



$$Y = g(X)$$

1) Y È $\sigma(X)$ MISURABILE

2) $\sigma(Y) \subset \sigma(X)$

3) $\exists g$ MISURABILE : $Y = g(X)$

$(1) \Leftrightarrow (2)$ SEGUE DALLA MINIMALITÀ DI $\sigma(Y)$

$(3) \Rightarrow (1)$ COMPOSIZIONE DI FUNZIONI MISURABILI

$Y_1 = X_1 \quad Y_2 = X_1 + X_2 \quad \ldots \quad Y_n = X_1 + \cdots + X_n$

$\Rightarrow \sigma(Y_1 \cdots Y_n) \subset \sigma(X_1 \cdots X_n)$

$X_1 = Y_1 \quad X_2 = Y_2 - Y_1 \quad X_3 = Y_3 - Y_2$

$\ldots \quad X_n = Y_n - Y_{n-1}$

$\Rightarrow \sigma(X_1 \cdots X_n) \subset \sigma(Y_1 \cdots Y_n)$

VARIANTE DELL' ESEMPIO

$Y_1 = X_1 + X_2$

$Y_2 = X_1 + X_3$

$Y_3 = X_1 + X_4$

$\sigma(Y_1, Y_2, Y_3)$

$\subset \sigma(X_1, \ldots, X_4)$

MA NON $\supset$

$$\{\emptyset, \Omega\} \subsetneq \sigma(X^2) = \sigma(|X|) \subsetneq \sigma(X) = \sigma(X+c) \subset$$

$$\subsetneq \sigma(X, X^2 - Y^2) \subsetneq \sigma(X, Y) = \sigma(X, X+Y) = \sigma(X+Y, X-Y) \subset \mathcal{F}$$

$$\sigma(X) \supset \sigma(|X|) = \sigma(X^2)$$

$$\sigma(X) = \sigma(X+c)$$

A MENO CHE NON SIA $X \geq 0$

LANCIO DI DUE DADI

$$\sigma(Z, U) = \sigma(Z, V) \qquad Z = U + V$$

$$W = U \cdot V$$

$$X_1 - X_2 = \begin{cases} V - U & \text{se } \mathbb{1}_{X_1 > X_2} = 1 \\ -(V - U) & \text{se } \mathbb{1}_{X_1 > X_2} = 0 \end{cases}$$

$$\mathbb{1}_{X_1 > X_2} = \mathbb{1}_{X_1 - X_2 > 0}$$

$\sigma(Z) \cap \sigma(W)$

$\{Z=3\} = \{W=2\} \in \sigma(Z) \cap \sigma(W)$

$\{Z=12\} = \{W=36\}$ //

$\{Z=11\} = \{W=30\}$ //

$\{Z=4\} \in \sigma(Z)$ MA $\notin \sigma(W)$

$$\begin{array}{ccc} \sigma(X, X+Y) & = & \sigma(X+Y, X-Y) \\ \| & & \\ \sigma(X,Y) & \supsetneq & \underbrace{\sigma(X, X^2-Y^2)}_{=\,\sigma(X,Y^2)} \end{array}$$