$f(x) = e^{-x^2}$ $\qquad f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$

PER VERIFICARE LA MISURABILITÀ, BASTA VERIFICARE CHE

$$\underbrace{f^{-1}((-\infty, a]))}_{\{x \in \mathbb{R} \mid f(x) \in (-\infty, a]\} = \{x \in \mathbb{R} \mid f(x) \leq a\}} \in \mathcal{B} \quad \text{PER OGNI } a \in \mathbb{R}$$

$= \{ x \in \mathbb{R} \mid e^{-x^2} \leq a \}$

$\underbrace{e^{-x^2}}_{>0} \leq a \iff$ IMPOSSIBILE SE $a \leq 0$

QUINDI

$f^{-1}((-\infty, a]) = \emptyset \in \mathcal{B}$

$e^{-x^2} \leq a \iff -x^2 \leq \log a$ SE $a > 0$

$\iff x^2 \geq -\log a$

$\underline{-\log a \le 0} \iff \log a \ge 0 \iff \underline{a \ge 1}$

$\underline{a \ge 1}$

$\underbrace{x^2}_{\ge 0} \ge \underbrace{-\log a}_{\le 0} \iff$ PER OGNI $x \in \mathbb{R}$

$f^{-1}((-\infty, a]) = \mathbb{R}$

$\cap$

UB

$\underline{a < 1}$ $(0 < a < 1)$

$\sqrt{\cdot}$

$x^2 \ge -\log a \iff |x| \ge \sqrt{-\log a}$

$\iff x \leq -\sqrt{-\log a}$ o $x \geq \sqrt{-\log a}$

quindi

$$f^{-1}((-\infty, a]) = \left(-\infty, -\sqrt{-\log a}\right] \cup$$

$$\cup \left[+\sqrt{-\log a}, +\infty\right)$$

$\in \mathcal{B}$

$$f(x,y) = (x+y,\ x,\ y) \qquad f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\mathcal{B}^2 \qquad\qquad \mathcal{B}^3$$

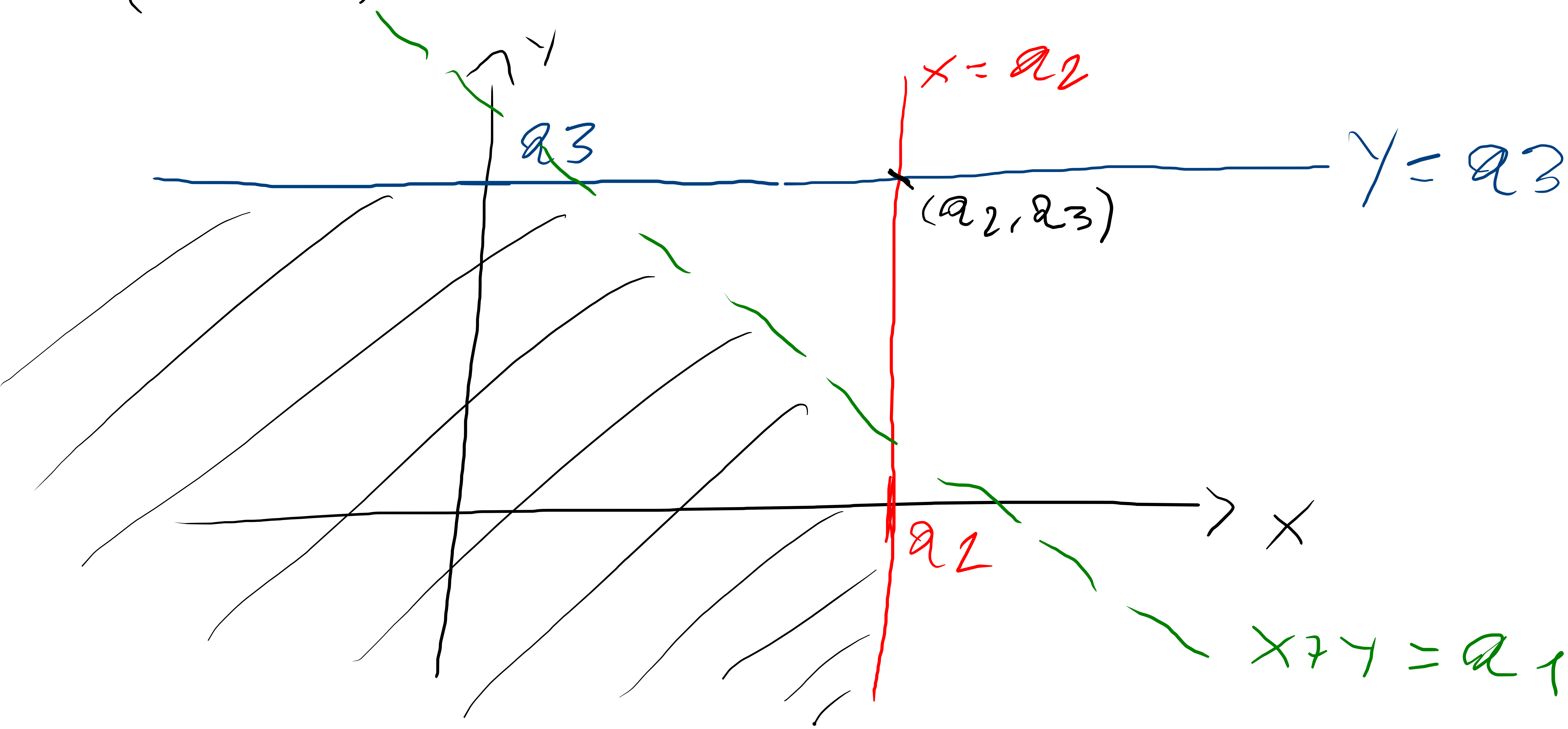
$$f^{-1}\Big(\underbrace{(-\infty,a_1] \times (-\infty,a_2] \times (-\infty,a_3]}_{I_{a_1,a_2,a_3}}\Big) = \qquad a_1, a_2, a_3 \in \mathbb{R}$$

$$\in \mathcal{B}^2$$

$$= \{(x,y) \mid f(x,y) \in I_{a_1,a_2,a_3}\} = \{(x,y) \mid (x+y, x, y) \in I_{a_1,a_2,a_3}\}$$

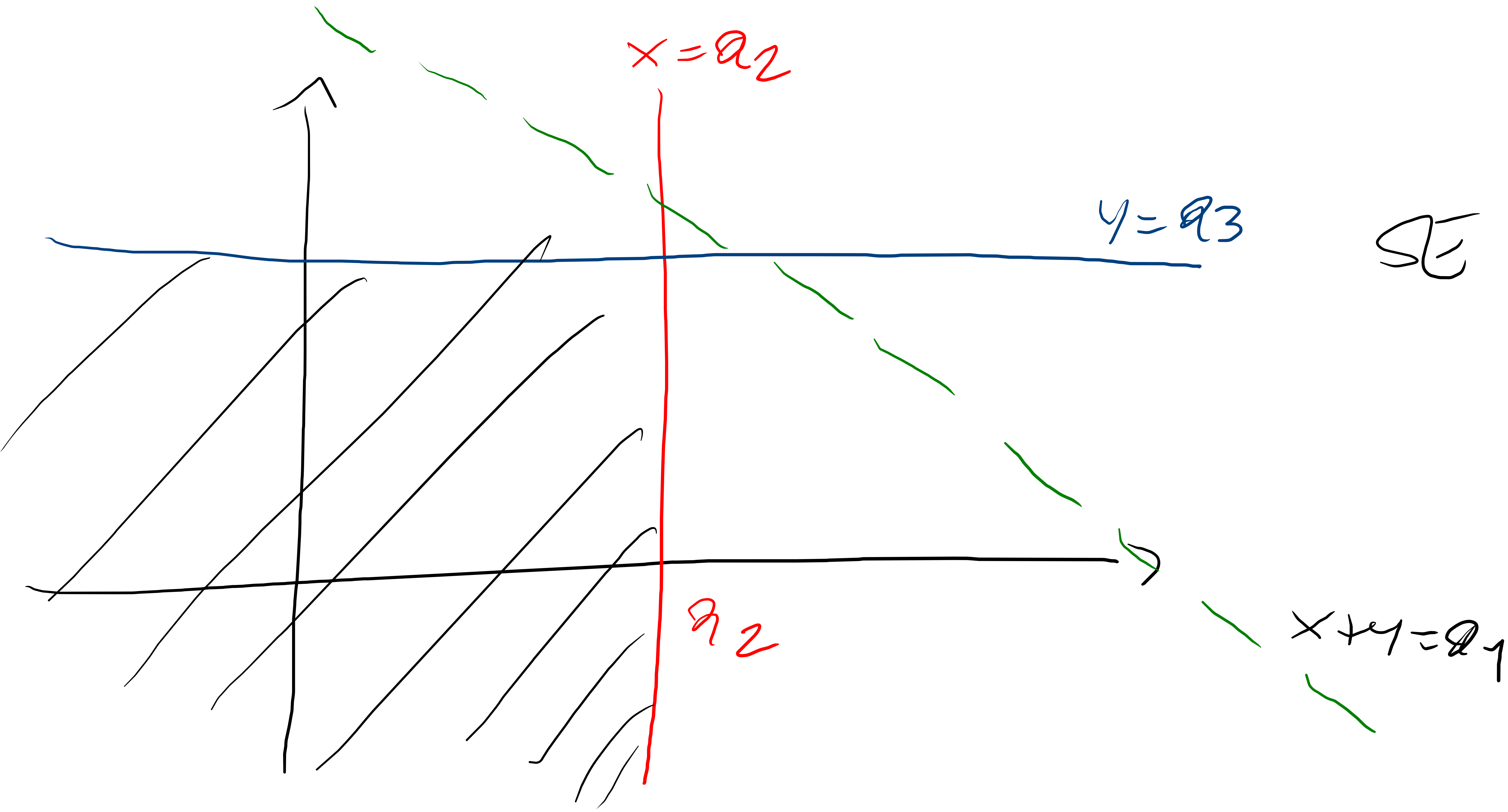
$$= \{ (x,y) \mid (x+y, x, y) \in (-\infty, a_1] \times (-\infty, a_2] \times (-\infty, a_3] \}$$

$$= \{ (x,y) \mid x+y \leq a_1, \ x \leq a_2, \ y \leq a_3 \}$$

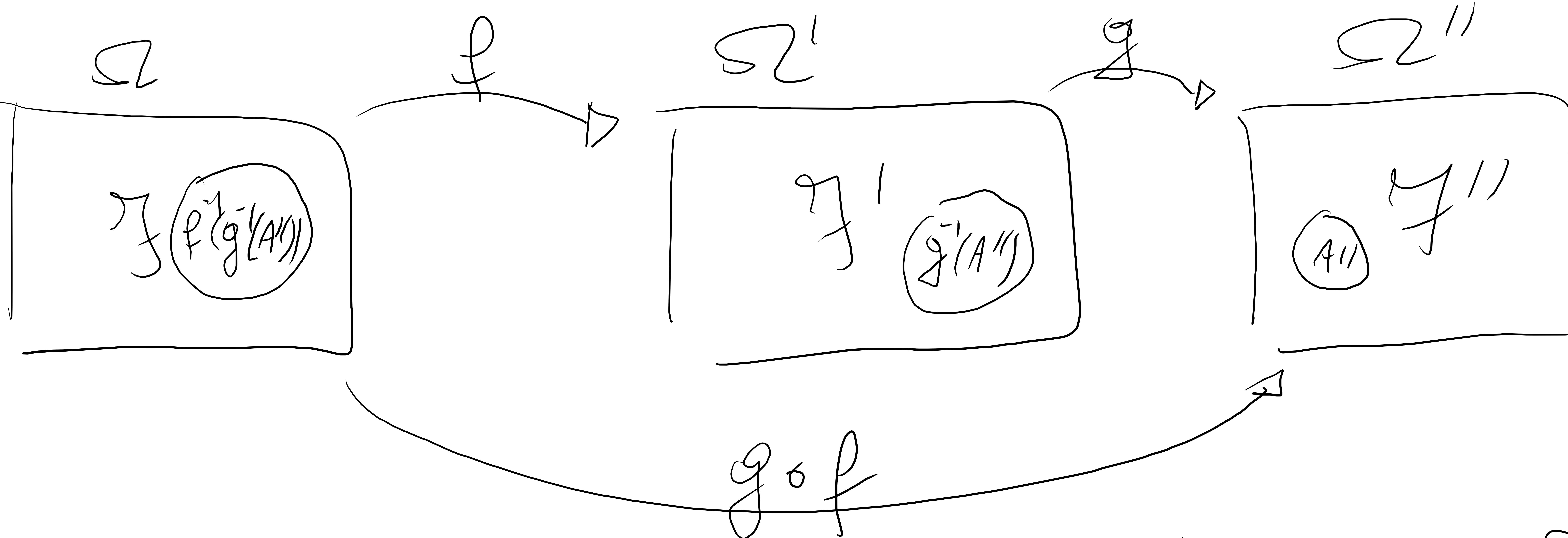


SE

$a_1 < a_2 + a_3$



SE $a_1 \geq a_2 + a_3$



PER OGNI $A'' \in \mathcal{F}''$, $\underbrace{(g \circ f)^{-1}(A'')}_{f^{-1}(\underbrace{g^{-1}(A'')}_{\in \mathcal{F}'}) \in \mathcal{F}} \in \mathcal{F}$?

$$\Omega = \{ L_1 L_2 L_3 \ldots L_n \ldots \mid L_i = T \circ L_i = C \}$$

$$W(CCCTCTCTT\ldots) = 4$$

$$W(CCCC\ldots C\ldots) = +\infty$$

$$f_n(x) = \begin{cases} 0 & \text{se } x \leq 0 \\ x^n & \text{se } x > 0 \end{cases} \qquad f_n : \mathbb{R} \to \mathbb{R} \quad n \geq 1$$

$$\left(\sup_n f_n\right)(x) = \sup_n f_n(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x = 1 \\ +\infty & x > 1 \end{cases}$$

$$= \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ +\infty & x > 1 \end{cases}$$

$$\sup_n f_n : \mathbb{R} \to \overline{\mathbb{R}}$$

$$\lim_{n\to+\infty} f_n(x) = \begin{cases} 0 & \text{SE} \quad x \leq 0 \\ 0 & \quad\quad 0 < x < 1 \\ 1 & \quad\quad x = 1 \\ +\infty & \quad\quad x > 1 \end{cases}$$

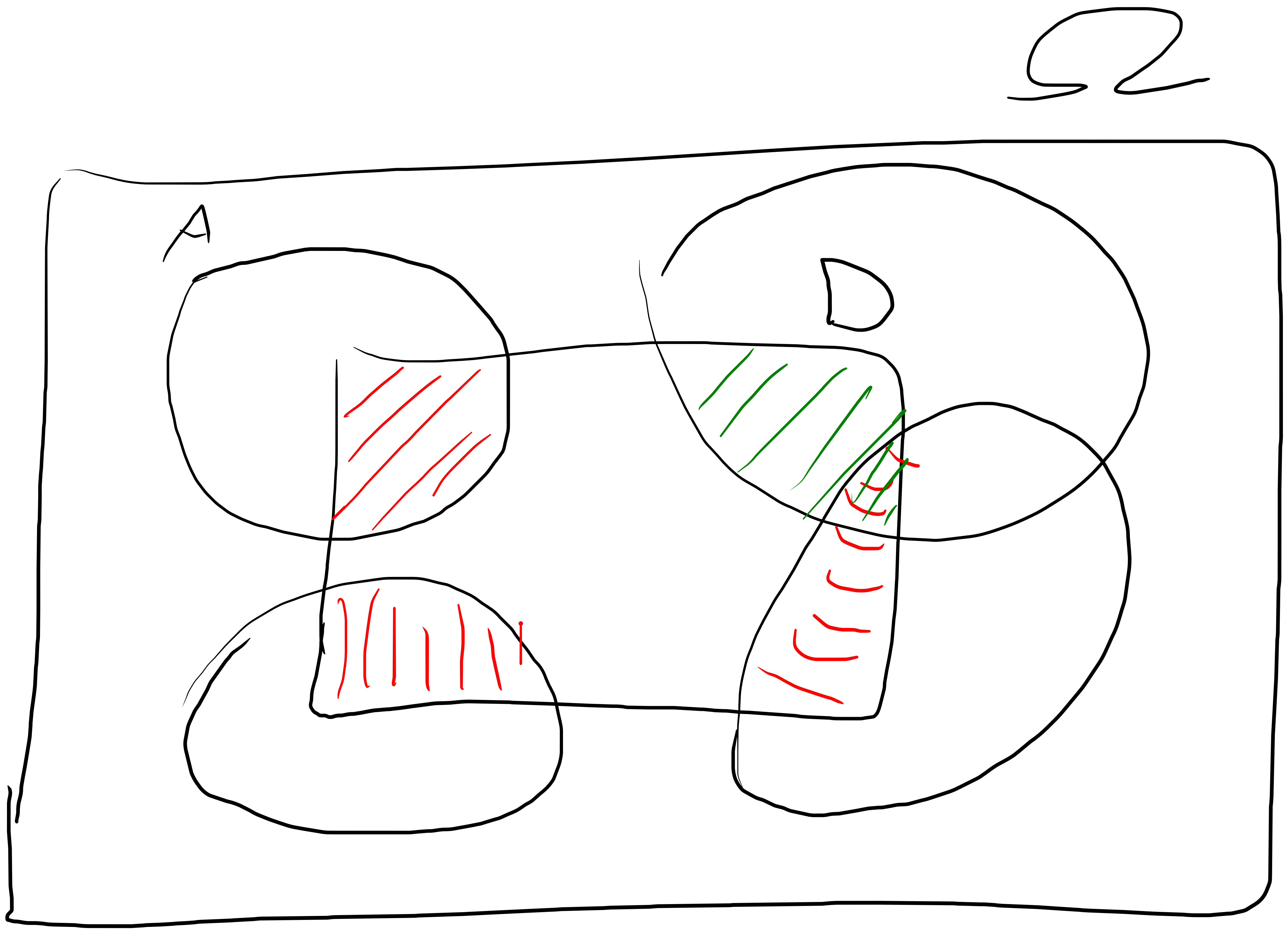
$$= \begin{cases} 0 & \quad\quad x < 1 \\ 1 & \text{SE} \quad x = 1 \\ +\infty & \quad\quad x > 1 \end{cases}$$

$$\lim_n f_n : \mathbb{R} \to \overline{\mathbb{R}} \quad \text{MISURABILE}$$

$$E_2 = \Omega = \mathbb{R} \qquad E_1 = (-\infty, 1]$$

σ-ALGEBRA TRACCIA

DEF



$\mathcal{F}$

$\mathcal{F}_D = \{ A \cap D \mid A \in \mathcal{F} \}$

È UNA σ-ALGEBRA SU D

$\Omega = \mathbb{R}$ $\qquad D = [0,1]$

$\mathcal{B}_D = \{ B \cap D \mid B \in \mathcal{B} \} = \sigma(\{ I \cap D \mid \text{CON } I \text{ INTERVALLO} \})$

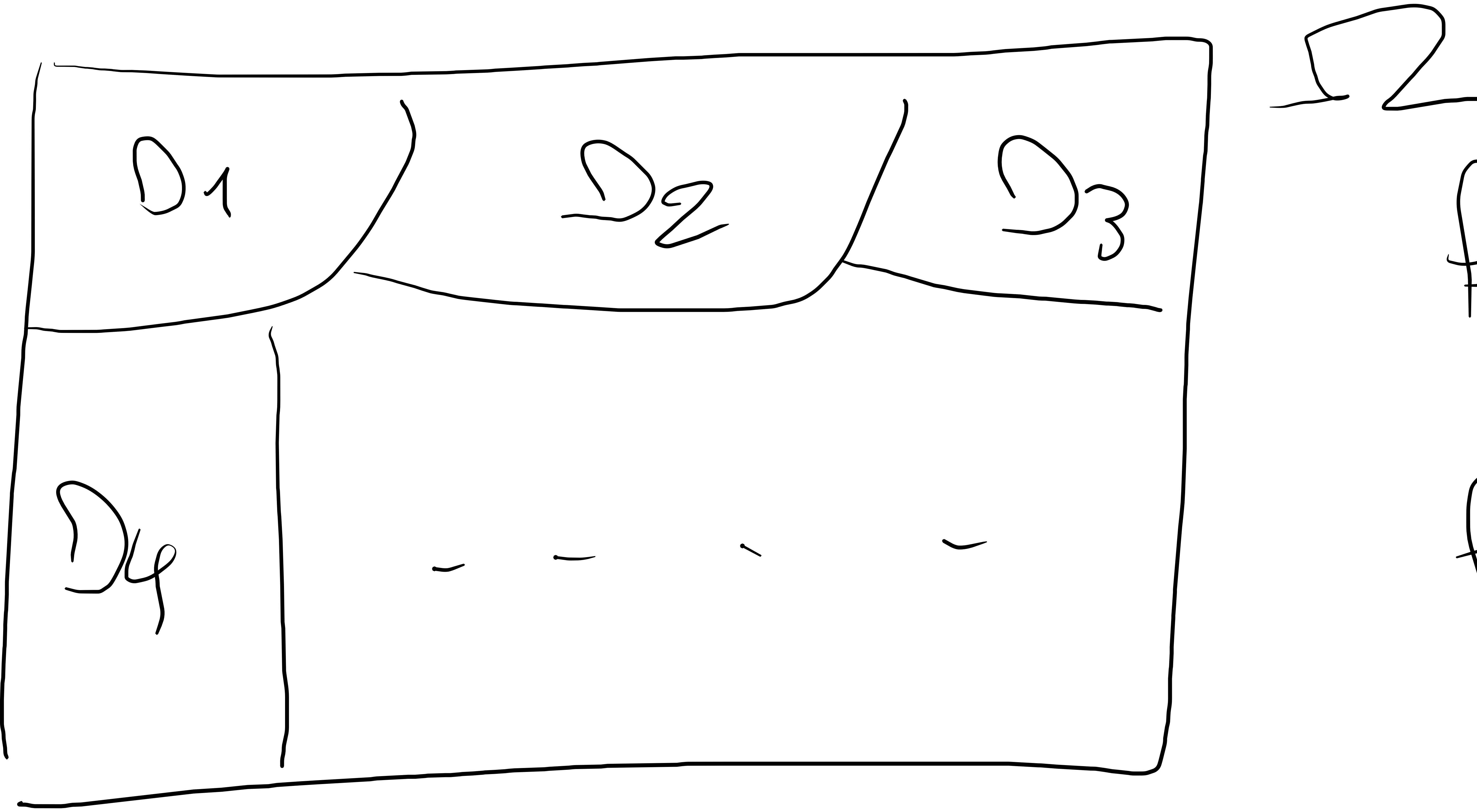
$\mathcal{B} = \sigma(\text{ INTERVALLI })$

$$f_D : \underset{\mathcal{F}_D}{D} \longrightarrow \underset{\mathcal{F}'}{\Omega'} \qquad f_D(\omega) = f(\omega) \quad \omega \in D$$

$$A' \in \mathcal{F}' \qquad \underbrace{f_D^{-1}(A')}_{} \in \mathcal{F}_D \ ?$$

$$\underbrace{\{\omega \in D \mid \underbrace{f_D(\omega)}_{= f(\omega)} \in A'\}}_{}$$

$$\{\omega \in D \mid f(\omega) \in A'\} = \underbrace{f^{-1}(A')}_{\in \mathcal{F}} \cap D$$



$$f_1 : \begin{matrix} D_1 \\ \mathcal{F}_{D_1} \end{matrix} \longrightarrow \begin{matrix} \Omega' \\ \mathcal{F}' \end{matrix}$$

$$f_2 : \begin{matrix} D_2 \\ \mathcal{F}_{D_2} \end{matrix} \longrightarrow \begin{matrix} \Omega' \\ \mathcal{F}' \end{matrix}$$

$$\vdots \qquad \qquad \vdots$$

ALLORA $f(\omega) = \begin{cases} f_1(\omega) & \omega \in D_1 \\ f_2(\omega) & \omega \in D_2 \\ \dots & \end{cases}$ È $\mathcal{F}/\mathcal{F}'$ MISURABILE

$A' \in \mathcal{F}' \qquad f^{-1}(A') \in \mathcal{F} \ ?$

$$f^{-1}(A') = f^{-1}(A') \cap \Omega = f^{-1}(A') \cap \left(\bigcup_n D_n\right)$$

$$= \bigcup_n \big(\underbrace{f^{-1}(A') \cap D_n}_{\{\omega \in D_n \mid \underbrace{f(\omega)}_{f_n(\omega)} \in A'\}}\big) \in \mathcal{F}$$

$\{\omega \in D_n \mid f_n(\omega) \in A'\}$

$= f_n^{-1}(A') \in \mathcal{F}_{D_n} = \underbrace{\underset{\mathcal{F}}{A_n} \cap \underset{\mathcal{F}}{D_n}}_{\in \mathcal{F}}$

$$f(x,y) = \begin{cases} x+y & x+y>0 \\ \sqrt{-x-y} & x+y \le 0 \end{cases} : \mathbb{R}^2 \to \mathbb{R}$$

$$\mathcal{B}^2 \qquad \mathcal{B}$$

$$= \begin{cases} f_1(x,y) & (x,y) \in D_1 = \{(x,y) \mid x+y>0\} \\ f_2(x,y) & D_2 \qquad \le \end{cases}$$

BASTA MOSTRARE CHE

$$f_1 : D_1 \to \mathbb{R} \quad , \quad f_2 : D_2 \to \mathbb{R}$$

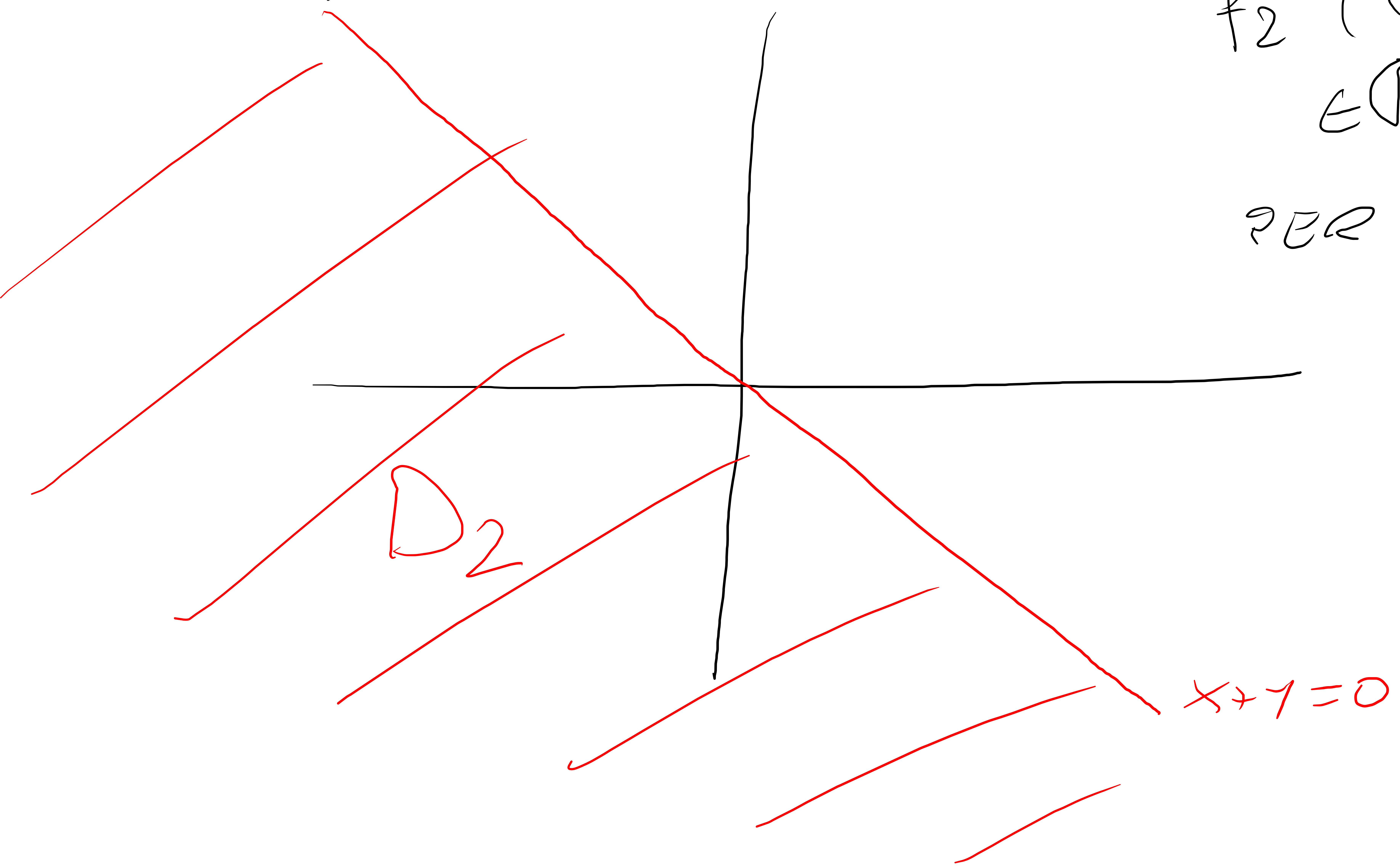
$$\mathcal{B}^2_{D_1} \qquad \mathcal{B} \qquad\qquad \mathcal{B}^2_{D_2} \qquad \mathcal{B}$$

SONO MISURABILI

$f_2 : D_2 \to \mathbb{R}$

$f_2^{-1}((-\infty, a]) \in \mathcal{B}^2_{D_2}$

PER OGNI $a \in \mathbb{R}$



$$f_2^{-1}((-\infty, a]) = \{(x,y) \in D_2 \mid \underbrace{f_2(x,y)}_{\sqrt{-x-y}} \leq a\}$$

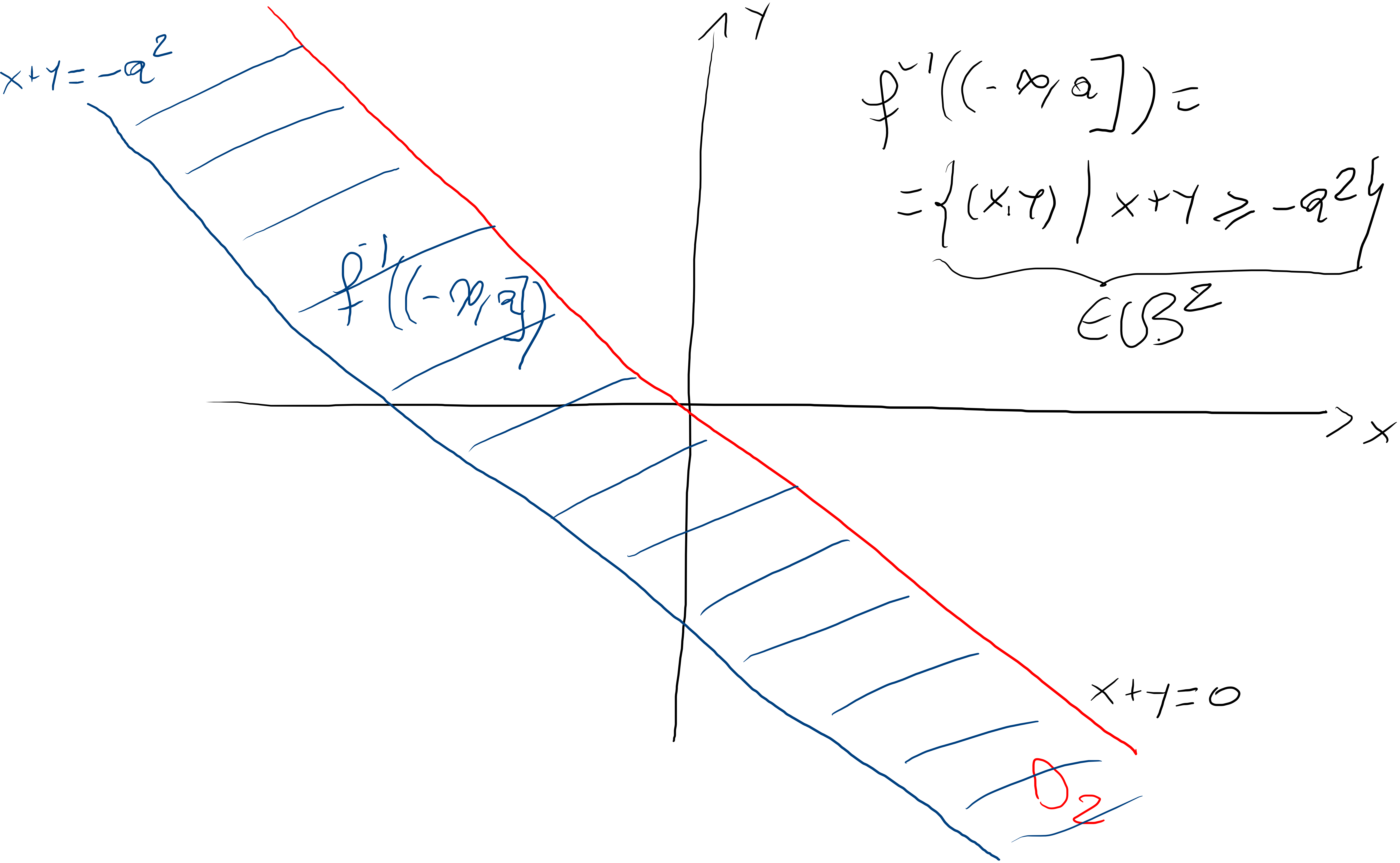
$\sqrt{-x-y} \leq a \iff$ IMPOSSIBILE SE $a < 0$

$f_2^{-1}((-\infty, a]) = \emptyset \in \mathcal{B}^2_{D_2}$

$a \geq 0$ ELEVO AL QUADRATO

$$-x-y \leq a^2 \iff x+y \geq -a^2$$

$$f^{-1}((-\infty, a]) = \underbrace{\{(x,y) \mid x+y \geq -a^2\}}_{\in \mathcal{B}^2} \cap D_2$$



$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} : \mathbb{R}^n \to \mathbb{R}^m$$

MISURABILE SE E SOLO SE

$$f^{-1}\left((-\infty, a_1] \times (-\infty, a_2] \times \ldots \times (-\infty, a_m] \right) =$$

$$= \{ (x_1 \ldots x_n) \in \mathbb{R}^n \mid \underbrace{f(x_1 \ldots x_n)}_{\begin{bmatrix} f_1(\quad) \\ \vdots \\ f_m(\ldots) \end{bmatrix}} \in (-\infty, a_1] \times \ldots \times (-\infty, a_m] \}$$

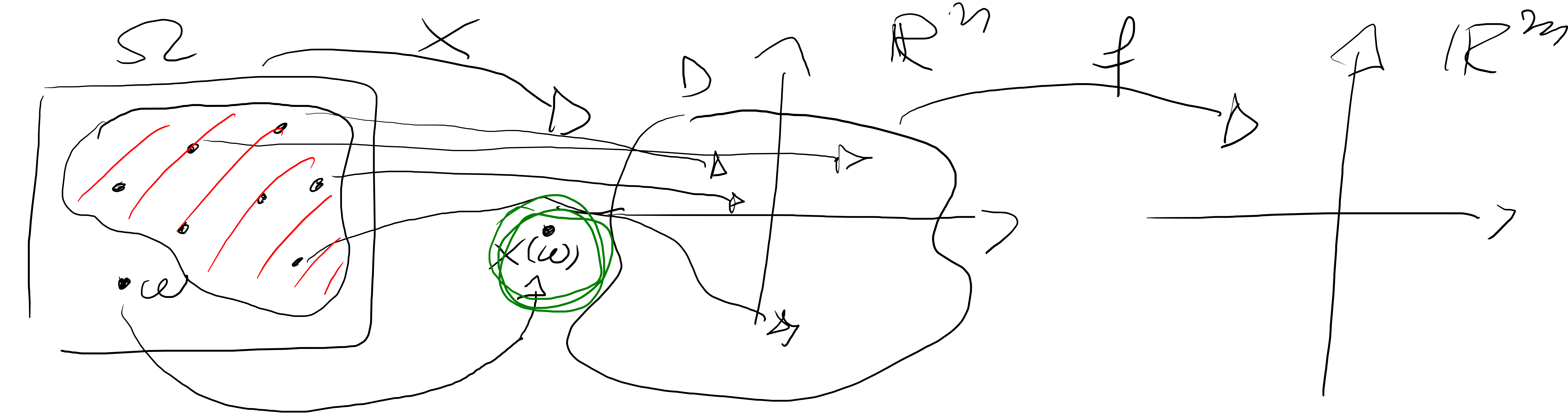
$$= \{(x_1 \ldots x_n) \in \mathbb{R}^n \mid f_1(x_1 \ldots x_n) \leq a_1, \ldots$$

$$\ldots, f_m(x_1 \ldots x_n) \leq a_m\}$$

$$= \underbrace{f_1^{-1}((-\infty, a_1])}_{\in \mathcal{F}} \cap \ldots\ldots \cap \underbrace{f_m^{-1}((-\infty, a_m])}_{\in \mathcal{F}} \in \mathcal{F}$$

SE $f_1, f_2, \ldots f_m$ SONO MISURABILI

VICEVERSA $\quad f_1 = g \circ f \quad \underbrace{g(y_1, \ldots, y_m) = y_1}_{\text{MISURABILE}}$



$$P(X \in D) = 1$$

$$f(X)(\omega) = \begin{cases} f(X(\omega)) & \text{se } X(\omega) \in D \\ 0 & X(\omega) \notin D \end{cases}$$

$\Omega \qquad X : \Omega \longrightarrow \mathbb{R}^n \quad \mathcal{B}^n$

$\qquad\qquad Y : \Omega \longrightarrow \mathbb{R}^m \quad \mathcal{B}^m$

$$\sigma(X,Y) = \mathcal{F} = \sigma\left(\left\{ X^{-1}(B_1), Y^{-1}(B_2) \text{ con } B_1 \in \mathcal{B}^n, B_2 \in \mathcal{B}^m \right\} \right)$$

$\mathcal{F}$ È LA PIÙ PICCOLA σ-ALGEBRA SU Ω TALE CHE

$X : \Omega, \mathcal{F} \to \mathbb{R}^n, \mathcal{B}^n$ MISURABILI

$Y : \Omega, \mathcal{F} \to \mathbb{R}^m, \mathcal{B}^m$

$$f : \underset{\sigma(f)}{\Omega} \longrightarrow \underset{\mathcal{F}'}{\Omega'}$$

$$\sigma(f) = \{ f^{-1}(A') \mid A' \in \mathcal{F}' \}$$ È UNA σ-ALGEBRA SU Ω

$$\emptyset = f^{-1}(\underbrace{\emptyset}_{\in \mathcal{F}'}) \in \mathcal{F}$$

$$\Omega = f^{-1}(\underbrace{\Omega'}_{\in \mathcal{F}'}) \in \mathcal{F}$$

$\sigma(f)$ CHIUSO RISPETTO COMPLEMENTARE

$A \in \sigma(f) \Rightarrow \overline{A} \in \sigma(f)$?

$A = f^{-1}(A')$ $\quad \overline{A} = \overline{f^{-1}(A')} = f^{-1}(\underbrace{\overline{A'}}_{\in \mathcal{F}'}) \in \mathcal{F}$

CON $A' \in \mathcal{F}'$

$\sigma(f)$ CHIUSO RISPETTO UNIONI NUMERABILI

$(A_n) \in \sigma(f) \Rightarrow \bigcup A_n \in \sigma(f)$?

$A_n = f^{-1}(A'_n)$ CON $A'_n \in \mathcal{F}'$

$$\bigcup_n A_n = \bigcup_n f^{-1}(A'_n) = f^{-1}(\underbrace{\bigcup_n A'_n}_{\in \mathcal{F}'}) \in \mathcal{F}$$