13 ottobre

$\{u_n\}$ $\qquad u_n = u_{\varepsilon_n}$

$\forall \psi \in C^\infty_{c\sigma}([0,+\infty) \times \mathbb{R}^d, \mathbb{R}^d)$

$\langle u_n(t), \psi(t)\rangle =$

$$= \int_0^t \Big(\langle u_n, \Delta\psi(t)\rangle + \langle u_n, \partial_t \psi\rangle -$$

$$- \langle \mathbb{P}(\rho_{\varepsilon_n} * u_n \cdot \nabla u_n), \psi \rangle \Big) dt'$$

$$+ \langle \underbrace{\rho_{\varepsilon_n} * u_0}_{\downarrow\, u_0}, \psi(0)\rangle$$

$u_n \rightharpoonup u$ in $L^{\frac{10}{3}}(\mathbb{R}_+ \times \mathbb{R}^3, \mathbb{R}^3)$

$$\lim_{n\to+\infty} \int_0^t \left(\langle u_n, \Delta\psi\rangle + \langle u_n, \partial_t \psi\rangle \right) dt'$$

$$= \int_0^t \left(\langle u, \Delta\psi\rangle + \langle u, \partial_t\psi\rangle \right) dt'$$

Prop We have $u \in L^\infty(\mathbb{R}_+, L^2) \cap L^2(\mathbb{R}_+, \dot{H}^1)$ $\operatorname{div} u = 0$ and for any $T > 0$ and $K \subset\subset \mathbb{R}^3$ we have

$$u_n \longrightarrow u \quad \text{in} \quad L^2([0,T]\times K, \mathbb{R}^3)$$

Moreover $\forall \; \psi \in C^0([0,+\infty), H^1(\mathbb{R}^3, \mathbb{R}^3))$

we have $\quad u_n \in C^0([0,+\infty), L^2)$

$$\ast \quad \langle u_n, \psi \rangle_{L^2_x} \longrightarrow \langle u, \psi \rangle_{L^2_x} \quad \text{in } L^\infty_{loc}([0,+\infty))$$

So in particular $\langle u, \psi \rangle_{L^2_x} \in C^0([0,+\infty))$ $\square$

Before the proof, notice that ~~th~~ Prop implies

$$\langle u_n(t), \psi(t) \rangle \longrightarrow \langle u(t), \psi(t) \rangle$$

Notice that $\quad \operatorname{supp} \psi \subseteq [0,T]\times K$

and by prop $\quad u \in L^2([0,T]\times K, \mathbb{R}^3)$

$$\Rightarrow \quad \underset{[0,T]}{\overset{\cap}{t}} \longrightarrow u(t) \in L^2(K) \quad \text{defined a.e.}$$

and the uniform convergence guarantees

$$u \in C^0_w([0,T], L^2(K))$$

Proof of the Prop

Recall that we want to prove

$$u_n \to u \quad \text{in } L^2([0,T]\times K, \mathbb{R}^3)$$

$\forall\, T>0$ and any $K \subset\subset \mathbb{R}^3$.

<u>Claim 1</u> $\{u_n\}$ is relatively compact in $L^2([0,T]\times K, \mathbb{R}^3)$

<u>claim 2</u> $\forall \varepsilon > 0$ $\exists$ a finite cover of $\{u_n\}$ with balls in $L^2([0,T]\times K)$ of radius ε.

Proof. of claim 2

Instead of $\{u_n\}$ I consider

$$\{P_{n_0} u_n\} \qquad \widehat{P_{n_0} u_n}(\xi) = \chi_{D(0,n_0)}(\xi)\, \hat{u}_n(\xi)$$

$$\widehat{(1-P_{n_0})u_n}(\xi) = \chi_{|\xi|\geq n_0}\, \hat{u}_n(\xi)$$

$$|u_n - P_{n_0} u_n|^2_{L^2([0,T]\times K)}$$

$$\leq \int_0^T |\widehat{(1-P_{n_0})u_n}|^2_{L^2(\mathbb{R}^3)}\, dt$$

$$\le \frac{1}{m_0^2}\int_0^T \left| \xi \cancel{\chi_{|\xi|\ge m_0}}\, \hat{u}_n \right|^2_{L^2(\mathbb{R}^3)} dt$$

$$\lesssim \frac{1}{m_0^2}\int_0^T |\nabla u_n|^2_{L^2(\mathbb{R}^3)} dt$$

$$\le \frac{1}{m_0} \, |\rho_{\varepsilon_n} * u_0|^2_{L^2(\mathbb{R}^3)} \le \frac{1}{m_0^2} |u_0|^2_{L^2(\mathbb{R}^3)}$$

$$|u_n - P_{m_0} u_n|_{L^2([0,T]\times K)} \le \frac{1}{m_0} |u_0|_{L^2(\mathbb{R}}$$

$$\frac{1}{m_0} |u_0|_{L^2} < \frac{\varepsilon}{2}$$

Claim 2, then is equivalente to

Claim 3 $\{P_{m_0} u_n\}$ $H^1_x(\mathbb{R}^3) \lesssim 1$ is relatively compact in $L^2([0,T]\times K, \mathbb{R}^3)$

Claim 3 follows from

Claim 4 $\{P_{m_0} u_n\}$ is relatively compact in $C^0([0,T], L^2(K)) \subseteq L^\infty([0,T], L^2(K$

<u>Pf</u> We apply Ascoli - Arzelà

(a) $\{ P_{n_0} u(t) \}$ is relatively compact in $L^2(K)$

(b) $\{ P_{n_0} u_n \}$ is equicontinuous.

We start with (a)

<u>Claim</u> $H^1(\mathbb{R}^3) \longrightarrow L^2(K)$ is compact

$$f \longmapsto f|_K$$

This implies (a) because

$$| P_{n_0} u_n(t) |_{H^1(\mathbb{R}^3)} = | \langle \xi \rangle \chi_{|\xi| \le n_0} \hat{u}_n(t, \xi) |_{L^2(\mathbb{R}^3)}$$

$$\le \langle n_0 \rangle \, | u_n(t) |_{L^2(\mathbb{R}^3)} \le \langle n_0 \rangle \, | u_0 |_{L^2(\mathbb{R}^3)}$$

<u>Proof of Claim</u>

$$\begin{tikzcd} H^1(\mathbb{R}^3) \arrow[r] & L^2(K) \\ L^2(\mathbb{R}^3, \langle \xi \rangle^2 d\xi) \arrow[u, "\wr"', "\mathcal{F}^*"] \arrow[ur] \arrow[ur, bend right, "\sim"'] & \end{tikzcd}$$

$$\uparrow \qquad \frac{f}{\langle \xi \rangle}$$

$$\uparrow$$

$$L^2(\mathbb{R}^3, d\xi) \qquad f$$

$$Tf \doteq \chi_K \, \mathcal{F}^* \left(\frac{f}{\langle \xi \rangle} \right)$$

$$T : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3) \longrightarrow L^2(K)$$

$$Tf(x) = \int_{\mathbb{R}^3} K(x,\xi)\, f(\xi)\, d\xi$$

$$K(x,\xi) = \chi_K(x)\, e^{-i x\cdot\xi} \frac{1}{\langle \xi \rangle}$$

$\exists$ a sequence $\quad T_n : L^2(\mathbb{R}^3) \circlearrowleft$

$T_n \longrightarrow T$ in norm

$$T_n f(x) = \int_{\mathbb{R}^3} K_n(x,\xi)\, f(\xi)\, d\xi$$

$$K_n(x,\xi) = \chi_K(x)\, e^{-i x \xi} \frac{1}{\langle \xi \rangle} \chi_{B(0,n)}(\xi)$$

$$\left| (T_n - T) f \right|_{L^2(\mathbb{R}^3)} =$$

$$\leq \left\| \cancel{\chi_K(x)} \int_{\mathbb{R}^3} e^{-ix\xi} \frac{1}{\langle \xi \rangle} \chi_{|\xi| \geq n} f(\xi)\, d\xi \right\|_{L^2}$$

$$= \left\| \frac{1}{\langle \xi \rangle} \chi_{|\xi| \geq n} f \right\|_{L^2}$$

$$\leq \frac{1}{n} \| f \|_{L^2} \Rightarrow \| T_n - T \| \leq \frac{1}{n}$$

$$T_n \to T$$

If $\{T_n\}$ are compact operators, also $\{T\}$ is compact.

The T_n are Hilbert - Schmidt operators

$$K_n(x, \xi) = \chi_K(x)\, e^{-ix\xi} \frac{1}{\langle \xi \rangle} \chi_{B(0,n)}(\xi)$$

$$K_n \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$$

Notice that

$$\| T_n \| \leq \| K_n \|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}$$

$$\| T_n f \|_{L^2(\mathbb{R}^3)} = \left\| \int_{\mathbb{R}^3} K_n(x, \xi)\, f(\xi)\, d\xi \right\|_{L^2(\mathbb{R}^3)}$$

$$\leq \left| \left(\int_{\mathbb{R}^3} K_n^2(x,\xi)\, d\xi \right)^{\frac{1}{2}} \right|_{L^2(\mathbb{R}^3)} |f|_{L^2}$$

$$= |K_n|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \, |f|_{L^2}$$

Each T_n is compact

$K_n \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$

In $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ the space

$L^2(\mathbb{R}^3_x) \otimes_{\mathbb{R}} L^2(\mathbb{R}^3_\xi)$ is dense

$f(x) \quad g(\xi)$

This means that $\forall \varepsilon > 0$

$\exists \ \lambda_1 f_1 \otimes g_1 + \dots + \lambda_N f_N \otimes g_N$

so that

$$\Big| \underbrace{\lambda_1 f_1 \otimes g_1 + \dots + \lambda_N f_N \otimes g_N}_{S_N} - K_n \Big|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} < \varepsilon$$

$|S_N - K_n| \leq$

$$\| \lambda_1 f_1 \otimes g_1 + \cdots + \lambda_N f_N \otimes g_N - K_n \|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} < \varepsilon$$

$$S_N f(x) = \int_{\mathbb{R}^3} \left(\lambda_1 f_1(x) g_1(\xi) + \cdots + \lambda_N f_N(x) g_1(\xi) \right) f(\xi) d\xi$$

$$= \lambda f_1(x) \langle g_1, f \rangle + \cdots + \lambda f_N(x) \langle g_N, f \rangle$$

$$\in Sp\{ f_1, \ldots, f_N \}$$

$\Rightarrow$ S_N is compact