20 ottobre

$$\|u(u_n-u)\|_{L^1([0,T],L^2(K))} \xrightarrow{} 0 \leq$$

$$\leq \|u\|_{L^2([0,T],L^4(\mathbb{R}^3))} \|u_n-u\|_{L^2([0,T],L^4(K))} \to 0$$

$$\| \rho_{\varepsilon_n} * (u_n-u)\|_{L^2([0,T],L^4(K))}$$

$$= \left\| \int_{\mathbb{R}^3} \rho_{\varepsilon_n}(y) \, (u_n(x-y)-u(x-y))\,dy \right\|_{L^2([0,T],L^4_x(K))}$$

$$\leq \int_{\mathbb{R}^3} \underline{\rho_{\varepsilon_n}(y)} \quad \|u_n(x-y)-u(x-y)\|_{L^2([0,T],L^4_x(K))} \, dy$$

$$|y|\leq 1$$

$$\operatorname{supp} \rho \subseteq D_{\mathbb{R}^3}(0,1) \qquad \varepsilon_n < 1$$

$$\operatorname{supp} \rho_{\varepsilon_n} \subseteq D_{\mathbb{R}^3}(0,1) \qquad x-y \in K-y$$

$$x-y \in \bigcup_{|y|\leq 1} K-y \subset \tilde{K} \subset \mathbb{R}^3$$

$$\leq \int_{\mathbb{R}^3} \rho_{\varepsilon_n}(y) \, \|u_n-u\|_{L^2([0,T],L^4(\tilde{K}))} \, dy$$

$$= \|u_n - u\|_{L^2([0,T],\, L^4(\tilde K))} \xrightarrow{n \to \infty} 0$$

$$H^1(\mathbb{R}^d)$$

$$u_n \longrightarrow u \quad \text{in } L^2([0,T],\, L^4(K))$$

$$u_n - u \to 0 \qquad \forall\ K \subset\subset \mathbb{R}^3 \text{ and } T > 0.$$

$d = 2, 3$

$$\underset{\sim K}{\|f\|_{L^4(\mathbb{R}^d)}} \le C\, \underset{K}{\|f\|^{1-\frac{4}{d}}_{L^2(\mathbb{R}^d)}}\ \underset{K}{\|\nabla f\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}}$$

$$\chi \in C_c^\infty(\mathbb{R}^d, [0,1])$$

$$\chi \equiv 1 \text{ in } K, \quad \Omega \doteq \operatorname{supp} \chi$$

$$\Omega \subset\subset \mathbb{R}^d$$

$$\|\nabla \chi\|_{L^\infty(\mathbb{R}^d)} \le 1$$

$$\nabla u \in L^2([0,T] \times \mathbb{R}^d)$$

$$f \in H^1(\mathbb{R}^d)$$

$$\|\chi f\|_{L^4(\mathbb{R}^d)} \le C\, \|\chi f\|^{1-\frac{4}{d}}_{L^2(\mathbb{R}^d)} \cdot \|\nabla(\chi f)\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}$$

$$\|f\|_{L^4(K)} \le C\, \|f\|^{1-\frac{4}{d}}_{L^2(\Omega)} \cdot \left(\|(\nabla \cancel{\chi}) f\|_{L^2(\mathbb{R}^d)} + \|\cancel{\chi} \nabla f\|_{L^2(\mathbb{R}^d)} \right)^{\frac{4}{d}}$$

$$|f|_{L^4(K)} \leq C \, |f|_{L^2(\Omega)}^{1-\frac{d}{4}} \, |f|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}}$$

$$|u-u_n|_{L^2([0,T],L^4(K))} =$$

$$= \Big| \, |u-u_n|_{L^4(K)} \Big|_{L^2(0,T)}$$

$$\lesssim \Big| \, |u-u_n|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \, |u-u_n|_{L^2(\Omega)}^{\frac{4-d}{4}} \Big|_{L^2([0,T]}$$

$$\frac{1}{2} = \frac{4-d}{8} + \frac{d}{8} \qquad d = 2,3$$

$$\leq \Big| \, |u-u_n|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \Big|_{L^{\frac{8}{d}}([0,T])}$$

$$\Big| \, |u-u_n|_{L^2(\Omega)}^{\frac{4-d}{4}} \Big|_{L^{\frac{8}{4-d}}([0,T])}$$

$$= \Big| \, |u-u_n|_{H^1(\mathbb{R}^d)} \Big|_{L^2([0,T]}^{\frac{d}{4}}$$

$$\Big| \, |u-u_n|_{L^2(\Omega)} \Big|_{L^2([0,T])}^{\frac{4-d}{4}}$$

$$= |u-u_n|_{L^2([0,T],H^1(\mathbb{R}^d))}^{\frac{d}{4}} \cdot$$

$$|u-u_n|_{L^2([0,T]\times\Omega)}^{\frac{4-d}{4}} \xrightarrow{n\to+\infty} 0$$

$$|u|_{L^2([0,T],H^1(\mathbb{R}^d))} \leq$$

$$\underbrace{|u|_{L^2([0,T],L^2(\mathbb{R}^d))}}_{\leq \sqrt{T}\,\underbrace{|u|_{L^\infty([0,T],L^2(\mathbb{R}^d))}}_{\leq |u_0|_{L^2}}} + \underbrace{|\nabla u|_{L^2([0,T]\times\mathbb{R}^d)}}_{\leq |u_0|_{L^2}}$$

$$\int_0^t \langle \operatorname{div}(\rho_{\varepsilon_n} * u_n \otimes u_n), \Psi \rangle_{L^2_x}\, dt$$

$$\xrightarrow{n\to+\infty} \int_0^t \langle \operatorname{div}(u \otimes u), \Psi \rangle_{L^2_x}\, ds$$

Remarks The above u satisfies the energy inequality not only with respect to 0, that is

$$|u(t)|^2_{L^2(\mathbb{R}^3)} + 2\int_s^t |\nabla u(t')|^2_{L^2(\mathbb{R}^3)}\, dt' \leq |u(s)|^2_{L^2(\mathbb{R}^3)}$$

$$\forall\ t > s \geq 0$$

Weak solutions u of this type are called Leray-Hopf solutions.

Kato theory

$$\begin{cases}\partial_t u - \Delta u = -\mathbb{P}\,\mathrm{div}(u\otimes u) \quad Q(u,u)\\ u_{|t=0} = u_0\end{cases}$$

$Q(u,v)$ is a symmetrization of $-\mathbb{P}\,\mathrm{div}(u\otimes v)$

$$Q(u,v) = \frac{1}{2}\,\mathbb{P}\left(u\otimes v + v\otimes u\right)$$

Suppose u and v are defined in spacetime such that $Q(u,v)$ satisfies the hypothesis of the external force f in the linear heat equation $\partial_t w - \Delta w = f$

$$\begin{cases}\partial_t B(u,v) - \Delta B(u,v) = Q(u,v)\\ B(u,v)_{|t=0} = 0\end{cases}$$

$$B(u,v) = \int_0^t e^{(t-t')\Delta}\, Q(u,v)(t')\,dt'$$

Then NS

$$\begin{cases}\partial_t u - \Delta u = Q(u,u)\\ u_{|t=0} = u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)\end{cases}$$

Duhamel formula

$$u = e^{t\Delta}u_0 + B(u,u)$$

$X_T = L^4([0,T], \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d))$

Recall

<u>Lemma</u> $B : X \times X \to X$ bilinear bounded, X Banach space.

$\alpha < \frac{1}{4\|B\|}$. Then $\forall\, x_0 \in D_X(0,\alpha)$

$\exists!\ x \in \overline{D_X(0,2\alpha)}$ st

$$x = x_0 + B(x,x).$$

Remark NS is "invariant" by the scaling $\lambda > 0$

$$u \longrightarrow u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

$\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$

$u(0,x) = u_0(x)$

$u_\lambda(0,x) = \lambda u_0(\lambda x)$

$$|\lambda u_0(\lambda \cdot)|_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)} = |u_0|_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)}$$

Ex $d=2$

$$|\lambda u_0(\lambda \cdot)|_{L^2(\mathbb{R}^2)} = \lambda\, |u_0(\lambda \cdot)|_{L^2(\mathbb{R}^2)}$$

$$= \lambda\, \lambda^{-\frac{d}{2}} |u_0|_{L^2(\mathbb{R}^2)} = |u_0|_{L^2(\mathbb{R}^2)}$$

$\frac{d}{2}$ " 1

$d = 3$

$$\|\lambda u_0(\lambda \cdot)\|_{\dot H^{\frac12}(\mathbb{R}^3)} = \|u_0\|_{\dot H^{\frac12}(\mathbb{R}^3)}$$

$$\dot H^{\frac12}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$$

$$\frac12 - \frac{\frac12}{3} = \frac12 - \frac16 = \frac13$$

$$\|\lambda u_0(\lambda \cdot)\|_{L^3(\mathbb{R}^3)} = \|u_0\|_{L^3(\mathbb{R}^3)}$$

$$= \lambda \quad \|u_0(\lambda \cdot)\|_{L^3(\mathbb{R}^d)} = \cancel{\lambda} \quad \cancel{\lambda^{-\frac{d}{3}}} \quad \|u_0(\lambda \cdot)\|_{L^3(\mathbb{R})}$$

u_0

$T(u_0)$ $\quad u \quad [0, T)$

u_0

$T \quad [0, \frac{T}{\lambda^2}) \quad u_\lambda(t,x) = u(\lambda^2 t, \lambda x)\,\lambda$

$u_{0\lambda}$

$$T(\|u_0\|_{L^2}) = 1$$

$$t \to T$$

$$\|u(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)} \xrightarrow{t \to T} +\infty$$

$$\dot H^{\frac12}(\mathbb{R}^3)$$

$$\cap$$

$$L^3(\mathbb{R}^3) \subset \cdots \subseteq \dot B^{-1}_{\infty\infty}$$

$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

$\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^3)$ $\qquad d=2,3$

$$X = L^4([0,T], \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^3))$$

$$\begin{cases} \partial_t u - \Delta u = f \\ \nabla \cdot u = 0 \\ u_0 \end{cases}$$

$$\boxed{\|u\|_{L^p([0,T], \dot{H}^{s+\frac{2}{p}}(\mathbb{R}^d))} \lesssim \|u_0\|_{\dot{H}^s} + \|f\|_{L^2([0,T], \dot{H}^{s-1}(\mathbb{R}^d))}}$$

$s = \frac{d}{2} - 1 \qquad p = 4$

$$s + \frac{2}{p} = \frac{d}{2} - 1 + \frac{2}{4} = \frac{d-1}{2}$$

$$X_T = L^4([0,T], \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d))$$

$$\partial_t B(u,v) - \Delta B(u,v) = Q(u,v)$$

$$B(u,v)|_{t=0} = 0$$

We want $(u,v) \mapsto B(u,v) \qquad X_T \times X_T \to X_T$

step 1 is to show

$$Q : X_T \times X_T \longrightarrow L^2([0,T], \dot{H}^{\frac{d}{2}-2}(\mathbb{R}^d))$$