18 ottobre

Claim $\{P_{n_0} u_n\}_{n \in \mathbb{N}}$ is relatively compact inside $C^0([0,T], L^2(K))$ $K \subset\subset \mathbb{R}^d$.

We already discussed that $\{P_{n_0} u_n(t)\}$ is rel comp. in $L^2(K)$

Now we prove that $\{P_{n_0} u_n\}$ is equicontinuous

Claim $\exists\ C = C_T(n_0)$ s.t.

$$| \partial_t P_{n_0} u_n |_{L^{\frac{4}{d}}([0,T], L^2(\mathbb{R}^d))} \leq C \quad \forall n$$

$$| P_{n_0} u_n(t) - P_{n_0} u_n(t_0) |_{L^2(\mathbb{R}^d)} \leq \qquad t > t_0$$

$$\leq \int_{t_0}^{t} | \partial_s P_{n_0} u_n(s) |_{L^2(\mathbb{R}^d)} ds$$

$$\leq \quad | \partial_s P_{n_0} u_n |_{L^{\frac{4}{d}}([0,T], L^2_x)} \ | 1 |_{L^{(\frac{4}{d})'}[t_0,t)}$$

$$\leq (t-t_0)^\alpha \quad C \qquad P_{n_0}$$

$$\partial_t u_n = \Delta u_n - \mathbb{P}(\rho_{\varepsilon_n} * u_n \cdot \nabla u_n)$$

$$P_{n_0} \partial_t u_n = P_{n_0} \Delta u_n - P_{n_0}(\rho_{\varepsilon_n} * u_n \cdot \nabla u_n)$$

$$\partial_t P_{n_0} u_n = \quad " \qquad "$$

$$P_{n_0}\Delta : L^2_x \circlearrowleft$$

$$n_0^2$$

$$|P_{n_0}\Delta u_n|_{L^{\frac{4}{d}}([0,T], L^2_x)}$$

$$\leq n_0^2 \quad |u_n|_{L^{\frac{4}{d}}([0,T], L^2_x)}$$

$$\leq n_0^2 \quad |1|_{L^{\frac{4}{d}}([0,T])} \underbrace{|u_n|_{L^\infty([0,T], L^2_x)}}_{\leq |u_0|_{L^2_x}}$$

$$\leq C_{T n_0} \quad |u_0|_{L^2_x}$$

$$\leq |u_0|^2_{L^2(\mathbb{R}^d)}$$

$$|P_{n_0} \cancel{\mathbb{P}} (\rho_{\varepsilon_n} * u_n \cdot \nabla u_n)|_{L^{\frac{4}{d}}([0,T], L^2(\mathbb{R}^d))}$$

$$= \| P_{n_0} \operatorname{div} (\rho_{\varepsilon_n} * u_n \otimes u_n) \|_{L^{\frac{4}{d}}([0,T], L^2_x)}$$

$$\lesssim n_0 \quad | \rho_{\varepsilon_n} * u_n \otimes u_n |_{L^{4/d}([0,T], L^2_x)}$$

$$= n_0 \quad \Big| \quad | \rho_{\varepsilon_n} * u_n \otimes u_n |_{L^2_x} \Big|_{L^{\frac{4}{d}}([0,T])}$$

$$\leq n_0 \quad \Big| \quad \underbrace{| \cancel{\rho_{\varepsilon_n} *} u_n |_{L^4_x} \quad | u_n |_{L^4_x}}_{| u_n |^2_{L^4_x}} \Big|_{L^{\frac{4}{d}}[0,T]}$$

$$| u_n |_{L^4_x(\mathbb{R}^d)} \lesssim | u_n |^{1-\frac{d}{4}}_{L^2} \quad | \nabla u_n |^{\frac{d}{4}}_{L^2}$$

$$\lesssim \quad \Big| \quad | u_n |^{\frac{4-d}{2}}_{L^2} \quad | \nabla u_n |^{\frac{d}{2}}_{L^2} \Big|_{L^{\frac{4}{d}}[0,T]}$$

$$\lesssim \quad | u_n |^{\frac{4-d}{2}}_{L^\infty([0,T], L^2_x)} \quad \Big| \quad | \nabla u_n |^{\frac{d}{2}}_{L^2_x} \Big|_{L^{\frac{4}{d}}[0,T]}$$

$$\leq \quad | u_0 |^{\frac{4-d}{2}}_{L^2_x(\mathbb{R}^d)} \quad \Big| \quad | \nabla u_n |_{L^2_x} \Big|^{\frac{d}{2}}_{L^2([0,T])}$$

$$\leq \quad \| u_0 \|^{\frac{4-d}{2} + \frac{d}{2} = 2}_{L^2_{L^2(\mathbb{R}^d)}}$$

The first claim is proved.

there exists a subsequence, for simplicity the whole sequence,

$\{ P_{n_0} u_n \}$

which converges in $C^0([0,T], L^2(K))$

By taking a sequence $T_n \nearrow +\infty$

and a sequence of compact sets

$K_n \qquad K_n \subset \mathring{K}_{n+1}$

and with $\bigcup_n K_n = \mathbb{R}^d$

There is a sequence $\{u_n\}$ $\qquad \varepsilon_n \to 0^+$

$P_{n_0} u_n$ is convergent in $C^0([0,T], L^2(K))$

$\forall \; T > 0$ and any $K \subset\subset \mathbb{R}^d$

We know already that

$$u_n \longrightarrow u \quad \text{in } L^{\frac{10}{3}}(\cancel{[0,T]}\,\mathbb{R}_+ \times \mathbb{R}^3)$$

We want to emphasize that

$$P_{n_0} u_n \longrightarrow P_{n_0} u$$

First of all

$$u \in L^2([0,T] \times \mathbb{R}^3)$$

$$L^{\frac{10}{3}}([0,T] \times K) \subset L^2([0,T] \times K)$$

$$u_n \to u \implies u_n \rightharpoonup u$$

$$|u|_{L^2([0,T]\times K)} \leq \liminf_{n \to +\infty} |u_n|_{L^2([0,T]\times K)}$$

$$\leq C_T |u_0|_{L^2_x}$$

$$\Downarrow$$

$$|u|_{L^2([0,T]\times \mathbb{R}^3)} \leq C_T |u_0|_{L^2_x}$$

$$L^2([0,T] \times \mathbb{R}^3) \circlearrowleft P_{n_0}$$

is continuous also for the weak topology on it, since $u_n \to u$ in

$P_{n_0} u_n \to P_{n_0} u$

$P_{n_0} u_n \to v$ in $C^0([0,T], L^2(K))$

$P_{n_0} u_n \to P_{n_0} u$

$\Rightarrow \quad v = P_{n_0} u$

Let $\psi \in C^0([0,+\infty), H^1(\mathbb{R}^3,\mathbb{R}^3))$ $L^2(\mathbb{R}^3,\mathbb{R}^3)$

We want to show

$$\langle u_n(t), \psi(t)\rangle_{L^2(\mathbb{R}^3)} \xrightarrow{n\to\infty} \langle u(t), \psi(t)\rangle_{L^2(\mathbb{R}^3)}$$

uniformly on any interval $[0,T]$.

<u>Claim</u> For any $T>0$ and any $\varepsilon>0$ $\exists\ K \subset\subset \mathbb{R}^3$ s.t.

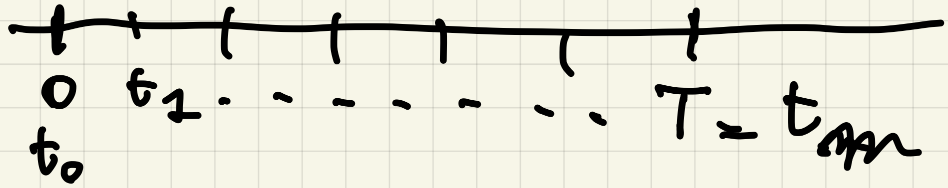
$$|\psi|_{L^\infty([0,T], L^2(\mathbb{R}^3\setminus K))} < \varepsilon$$

<u>Proof</u> We exploit $\psi \in C^0([0,T], L^2(\mathbb{R}^4))$

ψ is also uniformly continuous $[0,T] \to L^2(\mathbb{R}^4)$

So $\exists\ \delta > 0$ s.t. $|t-s| \leq \delta \Rightarrow |\psi(t)-\psi(s)|_{L^2_x} < \frac{\varepsilon}{2}$

So we decompose $[0,T]$ in m



$$\big| [t_j, t_{j+1}] \big| < \delta$$

I know that for any t_j $j=0,\dots,m$

$\exists\ K_j \subset\subset \mathbb{R}^3$ s.t.

$$|\psi(t_j)|_{L^2(\mathbb{R}^3 \setminus K_j)} < \frac{\varepsilon}{2}$$

$$K \supseteq \bigcup_{j=0}^{m} K_j$$

$t \in [0,T]$ $\exists\ t_j$ s.t. $|t-t_j| \leq \delta$

$$|\psi(t)|_{L^2(\mathbb{R}^3 \setminus K)} \leq$$

$$\leq \underbrace{|\psi(t)-\psi(t_j)|_{L^2(\mathbb{R}^3 \cancel{\setminus K})}}_{< \frac{\varepsilon}{2}} + \underbrace{|\psi(t_j)|_{L^2(\mathbb{R}^3 \setminus K)}}_{< \frac{\varepsilon}{2}}$$

$< \varepsilon$

Anagolously

Clen $\forall\ \varepsilon > 0$ and $T > 0$ $\exists\ n_0$ st

$$\|(P_{n_0} - 1)\psi(t)\|_{L^\infty([0,T], L^2(\mathbb{R}^3))} < \varepsilon$$

$$\langle u_n(t), \psi(t)\rangle_{L^2(\mathbb{R}^3)} \xrightarrow{n\to+\infty} \langle u(t), \psi(t)\rangle_{L^2(\mathbb{R}^3)}$$

uniformly in $[0,T]$

$$\langle u_n(t), \psi(t)\rangle_{L^2(\mathbb{R}^3)} - \langle u(t), \psi(t)\rangle_{L^2(\mathbb{R}^3)}$$

$$= \langle u_n(t) - u(t), (P_{n_0} + (1 - P_{n_0}))\psi(t)\rangle_{L^2(\mathbb{R}^3)}$$

$$= \langle u_n(t) - u(t), P_{n_0}\psi(t)\rangle_{L^2(\mathbb{R}^3)}$$

$$+ \langle u_n(t) - u(t), \underbrace{(1 - P_{n_0})\psi(t)}_{L^2_x \text{ norm} < \varepsilon}\rangle_{L^2_x(\mathbb{R}^3)}$$

$$\le (|u_n(t)|_{L^2_x} + |u(t)|_{L^2_x})\ \varepsilon$$

$$\le 2\,|u_0|_{L^2_x(\mathbb{R}^3)}\ \varepsilon$$

$$\langle P_{n_0}\big(u_n(t)-u(t)\big) \mid \psi(t)\rangle_{L^2(\mathbb{R}^3)} =$$

$$= \langle P\big(u_n(t)-u(t)\big)\quad \psi(t)\rangle_{L^2(K)}$$

$$+ \underbrace{\langle P_n\big(u_n(t)-u(t)\big), \cancel{P_{n_0}}\,\psi(t)\rangle}_{\leq 2|u_0|_{L^2}}\,\underbrace{{}_{L^2(\mathbb{R}^3\setminus K)}}_{\varepsilon}$$

$$P_{n_0}(u_n-u) \xrightarrow[n\to+\infty]{} 0 \quad \text{in } C^0([0,T], L^2(K))$$

$$\operatorname{div} u(t) = 0 \qquad \forall\, t.$$

Now we prove that u satisfies energy inequality

for a.e t

$$|u(t)|^2_{L^2_x} + 2\int_0^t |\nabla u(t')|^2_{L^2_x}\,dt' \leq |u_0|^2_{L^2_x}$$

$$|u_n(t)|^2_{L^2_x} + 2\int_0^t |\nabla u_n(t')|^2_{L^2_x}\,dt' = |u_0|^2_{L^2_x}$$

$\underline{L^2([0,t]\times\mathbb{R}^3)}$ $\{\nabla u_n\}$

$$u_n \rightharpoonup u \qquad L^2([0,+\infty) \times \mathbb{R}^3) \qquad \begin{array}{c} u_n \\ \Downarrow \\ u \end{array}$$

$$\Rightarrow \quad u_n(t) \rightharpoonup u(t) \quad \text{in } L^2(\mathbb{R}^3)$$

for a.e. $t \geq 0$.

$$\Rightarrow \quad |u(t)|_{L^2} \leq \liminf_{n\to+\infty} |u_n(t)|_{L^2}$$

$$\lim_{n\to+\infty} \int_0^t \langle \rho_{\varepsilon_n} * u_n \otimes u_n , \nabla\psi \rangle_{L^2(\mathbb{R}^3)} dt'$$

$$= \int_0^t \langle u \otimes u , \nabla\psi \rangle_{L^2(\mathbb{R}^d)} dt$$

$\forall \varepsilon > 0 \quad \cancel{\exists n} \quad 0 \leq t \leq T$

$\exists\, K \subset\subset \mathbb{R}^3$ s.t.

$$|\nabla\psi(t)|_{L^2(\mathbb{R}^3 \setminus K)} < \varepsilon$$

$$\lim_{n\to+\infty} \int_0^t \langle \rho_{\varepsilon_n} * u_n \otimes u_n , \nabla\psi \rangle_{L^2(K)} dt'$$

$\times$

$$= \int_0^t \langle u \otimes u , \nabla\psi \rangle_{L^2(K)} dt$$

$$\nabla\psi \in L^\infty([0,T], L^2(K))$$

$*$ will follow from

$$\rho_{\varepsilon_n} * u_n \otimes u_n \longrightarrow u \otimes u \quad \text{in } L^1([0,T], L^2(K))$$

The latter one will be a consequence

$$u_n \longrightarrow u \quad \text{in } L^2([0,T], L^4(K))$$

$\longrightarrow 0$ in $L^1([0,T], L^2(K))$

$$\rho_{\varepsilon_n} * u_n \; u_n - u \; u =$$

$$= (\rho_{\varepsilon_n} * u_n \; u_n - u \; u_n) + u \; u_n - u \; u$$

$$= (\rho_{\varepsilon_n} * u_n \; u_n - \rho_{\varepsilon_n} * u \; u_n) + (\rho_{\varepsilon_n} * u \; u_n - u \; u_n)$$

$$+ \; u \; u_n - u \; u$$

$|u(u_n - u)|_{L^2([0,T], L^4)}$

$\leq |u|_{L^\infty([0,T], L^2(\mathbb{R}^3))} |u_n - u|_{L^2([0,T])}$

$$|\rho_{\varepsilon_n} * (u_n - u) \; u_n|_{L^1([0,T], L^2(K))}$$

$$\leq |\rho_{\varepsilon_n} * (u_n - u)|_{L^2([0,T], L^4(K))} |u_n|_{L^2([0,T], L^4(\mathbb{R}^3))}$$