25 october

$$Q(u,v) = -\frac{1}{2}\mathbb{P}\left(\operatorname{div}(u\otimes v) + \operatorname{div}(v\otimes u)\right)$$

$$\begin{cases} \partial_t u - \Delta u = Q(u,u) \\ \nabla \cdot u = 0 \\ u_{|t=0} = u_0 \in \dot{H}^{\overset{1}{\boxed{\frac{d}{2}-1}}}(\mathbb{R}^d) \end{cases} \quad d=2,3$$

$$\begin{cases} \partial_t B(u,v) - \Delta B(u,v) = Q(u,v) \\ B(u,v)_{|t=0} = 0 \end{cases}$$

$$\partial_t u - \Delta u = f \in L^2([0,T], \dot{H}^{1-1}(\mathbb{R}^d))$$
$$u_{|t=0} = u_0 \in \dot{H}^1(\mathbb{R}^d)$$

For $u, v \in L^4([0,T], \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d))$

$d = 2,3$

we will need $Q(u,v) \in L^2([0,T], \dot{H}^{\frac{d}{2}-2})$

$$u = e^{t\Delta} u_0 + B(u,u) \quad *$$

Theo $\forall\, u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d)$ $\exists\, T$ and a solution u of $*$ with $u \in L^4([0,T], \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d, \mathbb{R}^d))$.

Furthermore we have

$u \in C^0([0,T), \dot{H}^{\frac{d}{2}-1})$

$\nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}-1})$

Furthermore, let T_{u_0} be the lifespan

1) $\exists$ a $\varepsilon_0 > 0$ s.t.

$$|u_0|_{\dot{H}^{\frac{d}{2}-1}} < \varepsilon_0 \Rightarrow T_{u_0} = +\infty$$

2) If $T_{u_0} < +\infty$ then

$$\int_0^{T_{u_0}} |u(t)|^4_{\dot{H}^{\frac{d-1}{2}}} \, dt = +\infty$$

$$\left(u(t) \in L^4([0,T], \dot{H}^{\frac{d-1}{2}}) \quad \forall \; 0 < T < T_{u_0} \right)$$

3) If $T_{u_0} < +\infty$ then

$$\int_0^{T_{u_0}} |\nabla u(t)|^2_{\dot{H}^{\frac{d}{2}-1}} \, dt = +\infty$$

$$\left(|u(t)|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t |\nabla u(t')|^2_{\dot{H}^{\frac{d}{2}-1}} \, dt' = |u_0|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \langle u, Q(u,u) \rangle_{\dot{H}^{\frac{d}{2}-1}} \, dt \right)$$

(it is also true that $\lim_{t \to T_{u_0}^-} |u(t)|_{\dot{H}^{\frac{d}{2}-1}} = +\infty$, but it is hard to prove)

If u and v are solutions then

$$|u(t)-v(t)|^2_{\dot{H}^{\frac{d}{2}-1}} + \int_0^t |\nabla(u-v)|^2_{\dot{H}^{\frac{d}{2}-1}} dt'$$

$$\leq |u_0 - v_0|^2_{\dot{H}^{\frac{d}{2}-1}} \, e^{C \, |(u,v)|^4_{L^4([0,t], \dot{H}^{\frac{d-1}{2}})}}$$

for $C = C_d$.

<u>Lemma</u> $d = 2, 3$. $\exists$ $C_d > 0$ st

$$|Q(u,v)|_{\dot{H}^{\frac{d}{2}-2}} \leq C_d \, |u|_{\dot{H}^{\frac{d-1}{2}}} \, |v|_{\dot{H}^{\frac{d-1}{2}}}$$

<u>Pf</u> For $d = 2$

$$|Q(u,v)|_{\dot{H}^{-1}} = \sum_{j,k=1}^{2} \left(|\partial_k(u_k v_j)|_{\dot{H}^{-1}} + |\partial_k(u_j v_k)|_{\dot{H}^{-1}} \right)$$

$$|\partial_k(u_k v_j)|_{\dot{H}^{-1}} \leq |u_k v_j|_{L^2} \leq$$

$$\leq \| u_k \|_{L^4(\mathbb{R}^2)} \| v_j \|_{L^4(\mathbb{R}^2)}$$

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$$

$$\frac{1}{4} = \frac{1}{2} - \frac{\frac{1}{2}}{2} \quad \checkmark$$

$$\leq C_2^2 \| u_k \|_{\dot{H}^{\frac{1}{2}}} \| v_j \|_{\dot{H}^{\frac{1}{2}}}$$

$d = 3$

$$\| Q(u,v) \|_{\dot{H}^{-\frac{1}{2}}} = \sum_{j,k=1}^{3} \left(\| \partial_k (u_k v_j) \|_{\dot{H}^{-\frac{1}{2}}} + \dots \right)$$

$$\| \partial_k (u_k v_j) \|_{\dot{H}^{-\frac{1}{2}}} \leq \| \nabla u_k \, v_j \|_{\dot{H}^{-\frac{1}{2}}} + \| u_k \nabla v_j \|_{\dot{H}^{-\frac{1}{2}}}$$

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$$

$$L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$$

$$\lesssim \left(\| \nabla u_k \, v_j \|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \right) + \dots$$

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6} = \frac{3+1}{6} = \frac{4}{6}$$

$$|\nabla u \; v|_{L^{\frac{3}{2}}} \leq |\nabla u|_{L^2} \; |v|_{L^6} \leq$$

$$\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$$

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$$

$$\leq |\nabla u|_{L^2} \; |\nabla v|_{L^2}$$

$$\approx |u|_{\dot{H}^1} \; |v|_{\dot{H}^1}$$

$\underline{\text{Lemma}}$ $\quad d = 2, 3 \qquad u, v \in L^4([0,T], \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d))$

$$|Q(u,v)|_{L^2([0,T], \dot{H}^{\frac{d}{2}-2})} \leq$$

$$\leq C_d \; |u|_{\underline{L^4([0,T], \dot{H}^{\frac{d-1}{2}})}} \; |v|_{\underline{L^4([0,T], \dot{H}^{\frac{d-1}{2}})}}$$

Pf

$$\| \, |Q(u,v)|_{\dot H^{\frac{d}{2}-2}} \|_{L^2([0,T])} \leq$$

$$\leq C_d \, \| \, |u|_{\dot H^{\frac{d-1}{2}}} \, |v|_{\dot H^{\frac{d-1}{2}}} \|_{L^2(0,T]}$$

$$\leq C_d \, \underline{\| \, |u|_{\dot H^{\frac{d-1}{2}}} \|_{L^4(0,T)}} \, \| |v|_{\dot H^{\frac{d-1}{2}}} \|_{L^4(0,T)}$$

$B(u,v)$

$$L^4([0,T], \dot H^{\frac{d-1}{2}}) \times L^4([0,T], \dot H^{\frac{d-1}{2}})$$

$$\downarrow$$

$$L^4([0,T], \dot H^{\frac{d-1}{2}})$$

$$\begin{cases} \partial_t B(u,v) - \Delta B(u,v) = Q(u,v) \\ B(u,v)|_{t=0} = 0 \end{cases}$$

$$|B(u,v)|_{L^4([0,T], \dot H^{\frac{d-1}{2}})} \quad \cdots$$

$$= |B(u,v)|_{L^p([0,T],\dot H^{s+\frac{2}{p}})}$$

$$\boxed{s = \frac{d}{2} - 1 \qquad p = 4}$$

$2 \le p \le +\infty$

$$s + \frac{2}{p} = \frac{d}{2} - 1 + \frac{1}{2} = \frac{d-1}{2}$$

$$\le |Q(u,v)|_{L^2([0,T],\dot H^{\frac{d}{2}-2})}$$

$$\le C_d \underbrace{|u|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}}_{X} |v|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}$$

$$B : X \times X \to X$$

$$\|B\| \le C_d .$$

$$u_0 \in \dot H^{\frac{d}{2}-1} \Rightarrow |e^{t\Delta} u_0|_{L^\infty([0,+\infty),\dot H^{\frac{d-1}{2}})} \le$$

$$\le |u_0|_{\dot H^{\frac{d}{2}-1}}$$

$$|e^{t\Delta} u_0|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t |\nabla e^{t'\Delta} u_0|^2_{\dot H^{\frac{d}{2}-1}} = |u_0|^2_{\dot H^{\frac{d}{2}-1}}$$

$(**)\ u = e^{t\Delta}u_0 + B(u,u)$ $\qquad x = x_0 + B(x,x)$

Lemma X $\quad B : X \times X \to X$

$\|B\| < \infty, \qquad \alpha < \frac{1}{4\|B\|}$

if $|e^{t\Delta}u_0|_X \leq \alpha \Rightarrow$

$\exists$ a solution u of $(**)$

with $|u|_X \leq 2\alpha$.

$X = L^4([0,T], \dot{H}^{\frac{d-1}{2}})$

$\|B\| \leq C_d$

We will seek

$$|e^{t\Delta}u_0|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} < \frac{1}{4C_d}$$

Two cases

1) If $|u_0|_{\dot{H}^{\frac{d-1}{2}}} \leq \varepsilon_0$ for ε_0 small

$$\| e^{t\Delta} u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} \leq \| u_0 \|_{\dot{H}^{\frac{d}{2}-1}} \leq \varepsilon_0 < \frac{1}{4C_d}$$

$\forall\ T > 0$, in fact for $T = +\infty$

$\Rightarrow \quad \exists\ u \in L^4([0,+\infty), \dot{H}^{\frac{d-1}{2}})$

with $\quad \| u \|_{L^4([0,+\infty), \dot{H}^{\frac{d-1}{2}})} \leq 2\varepsilon_0$

2) u_0 not small. $\qquad \rho > 0$

$$u_0 = \chi_{\sqrt{-\Delta} \leq \rho}\, u_0 + \chi_{\sqrt{-\Delta} \geq \rho}\, u_0$$

For ρ large enough

$$\| \chi_{\sqrt{-\Delta} \geq \rho}\, u_0 \|_{\dot{H}^{\frac{d}{2}-1}} < \frac{1}{8C_d}$$

$$\| e^{t\Delta} u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} \leq$$

$$\leq \| e^{t\Delta} \chi_{\sqrt{-\Delta} \leq \rho}\, u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})}$$

$$+ \underbrace{\| e^{t\Delta} \chi_{\sqrt{-\Delta} \geq \rho}\, u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})}}_{\leq \| \chi_{\sqrt{-\Delta} \geq \rho}\, u_0 \|_{\dot{H}^{\frac{d}{2}-1}} < \frac{1}{8C_d}}$$

$$\| e^{t\Delta} \chi_{\sqrt{-\Delta} \leq \rho} u_0 \|_{L^4([0,T], \dot H^{\frac{d-1}{2}})}$$

$$\leq \| e^{t\Delta} \chi_{\sqrt{-\Delta} \leq \rho} \sqrt{\rho} \underbrace{\cancel{\left(\frac{\sqrt[4]{-\Delta}}{\sqrt{\rho}}\right)}}_{\leq 1} u_0 \|_{L^4([0,T], \dot H^{\frac{d}{2}-1})}$$

$$= \sqrt{\rho} \, \| e^{t\Delta} \chi_{\sqrt{-\Delta} \leq \rho} u_0 \|_{L^4([0,T], \dot H^{\frac{d}{2}-1})}$$

$$\leq \sqrt{\rho} \, \| 1 \|_{L^4([0,T])} \| e^{t\Delta} \chi_{\sqrt{-\Delta} \leq \rho} u_0 \|_{L^\infty([0,T], \dot H^{\frac{d}{2}-1})}$$

$$\leq \sqrt{\rho} \sqrt[4]{T} \, \| \cancel{\chi_{\sqrt{-\Delta} \leq \rho}} u_0 \|_{\dot H^{\frac{d}{2}-1}}$$

$$\| e^{t\Delta} u_0 \|_{L^4([0,T], \dot H^{\frac{d-1}{2}})} \leq$$

$$\leq \sqrt{\rho} \sqrt[4]{T} \, \| u_0 \|_{\dot H^{\frac{d}{2}-1}} + \frac{1}{8 C_d} \leq \frac{1}{4 C_d}$$

$$\sqrt{\rho} \, T^{\frac{1}{4}} \| u_0 \|_{\dot H^{\frac{d}{2}-1}} \leq \frac{1}{8 C_d}$$

$$T \leq \left(\frac{1}{8 \rho^{\frac{1}{2}} C_d \| u_0 \|_{\dot H^{\frac{d}{2}-1}}} \right)^4$$

$$T = T(u_0)$$

$$u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1})$$

$$\nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}-1})$$

$$T = T_{u_0} < +\infty$$

$$\Rightarrow \int_0^T |u(t)|^4_{\dot{H}^{\frac{d-1}{2}}} \, dt = +\infty$$

By contradiction suppose

$$\int_0^T |u(t)|^4_{\dot{H}^{\frac{d-1}{2}}} \, dt < +\infty$$

$$= |Q(u,u)|_{L^2([0,T], \dot{H}^{\frac{d}{2}-2})} < +\infty$$

<u>Claim</u> Solution u extends to $L^4([0, T+\varepsilon), \dot{H}^{\frac{d-1}{2}})$