Image Processing for Physicists

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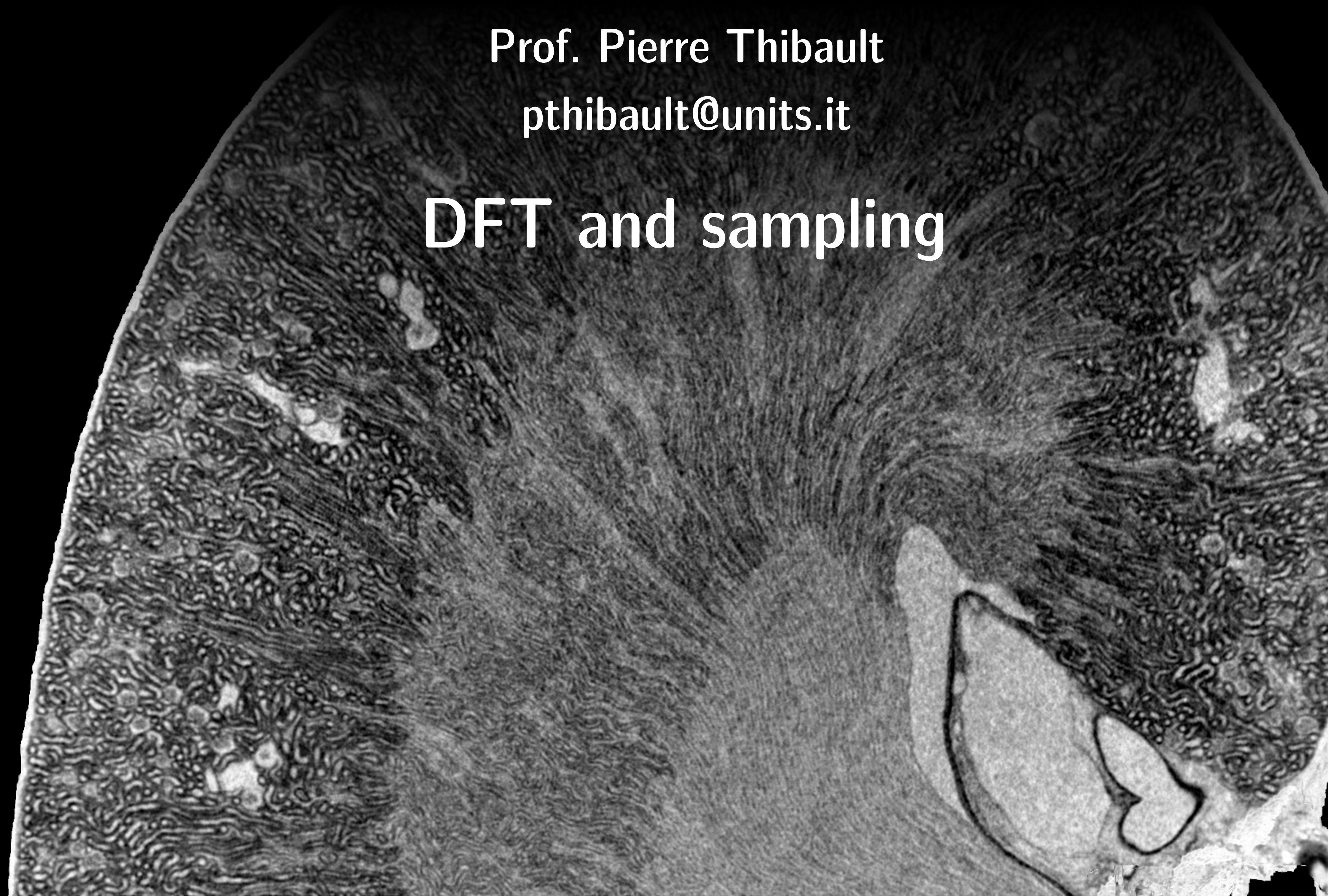
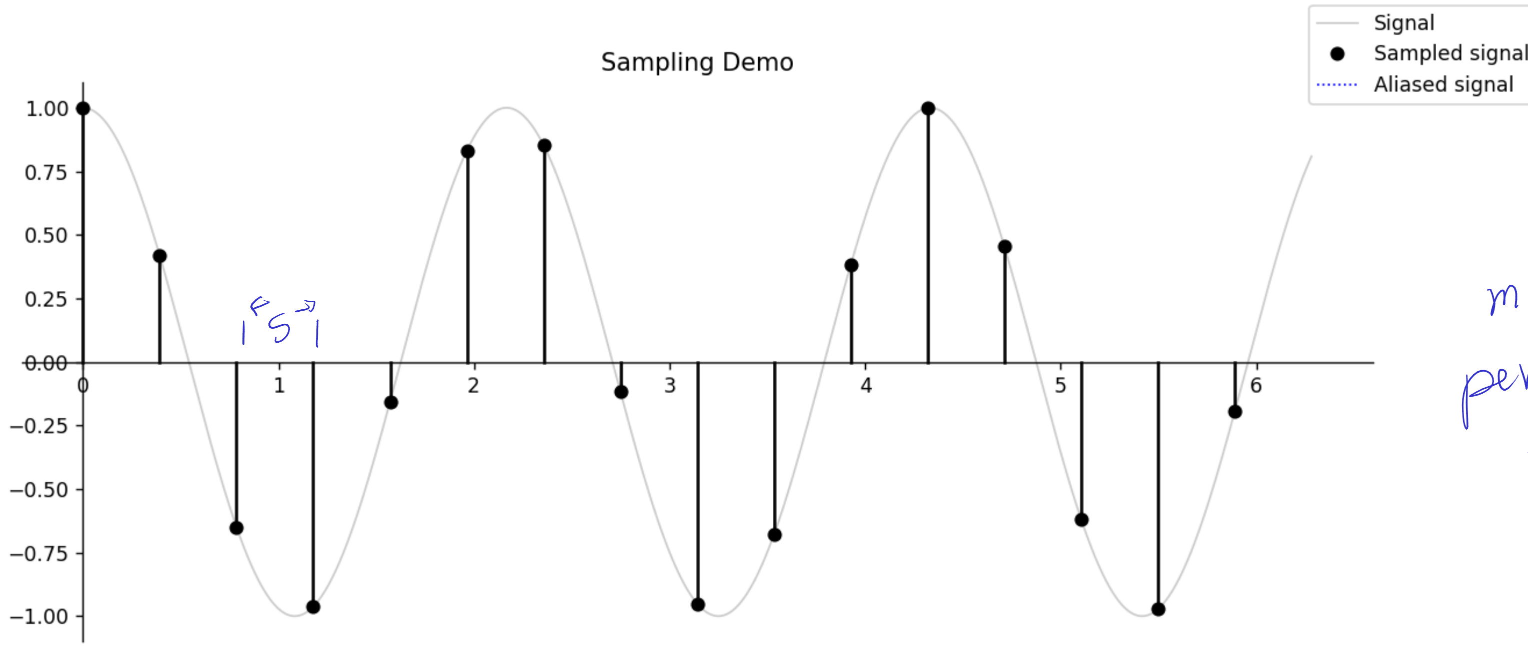
DFT and sampling

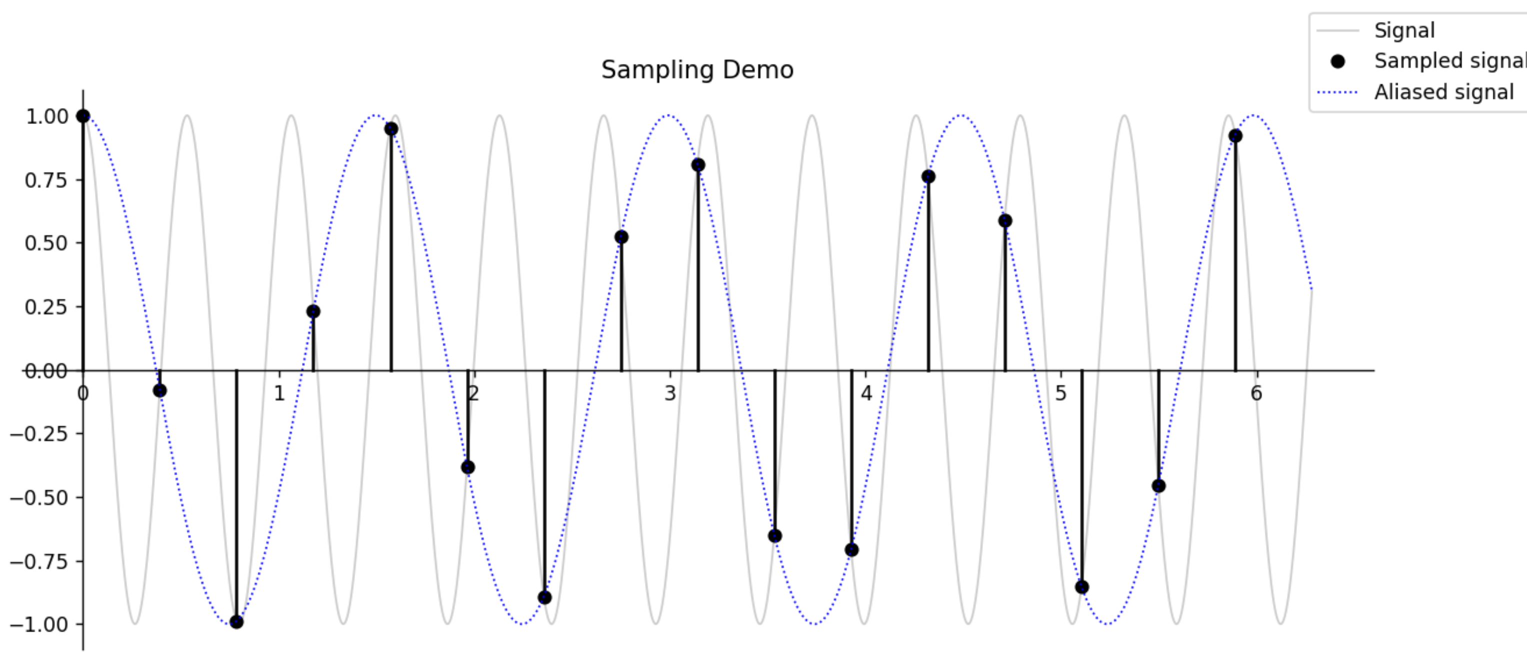


Overview

- Sampling
  - Nyquist theorem
- Discrete Fourier transform
  - Undersampling and Aliasing
- Interpolation (resampling)

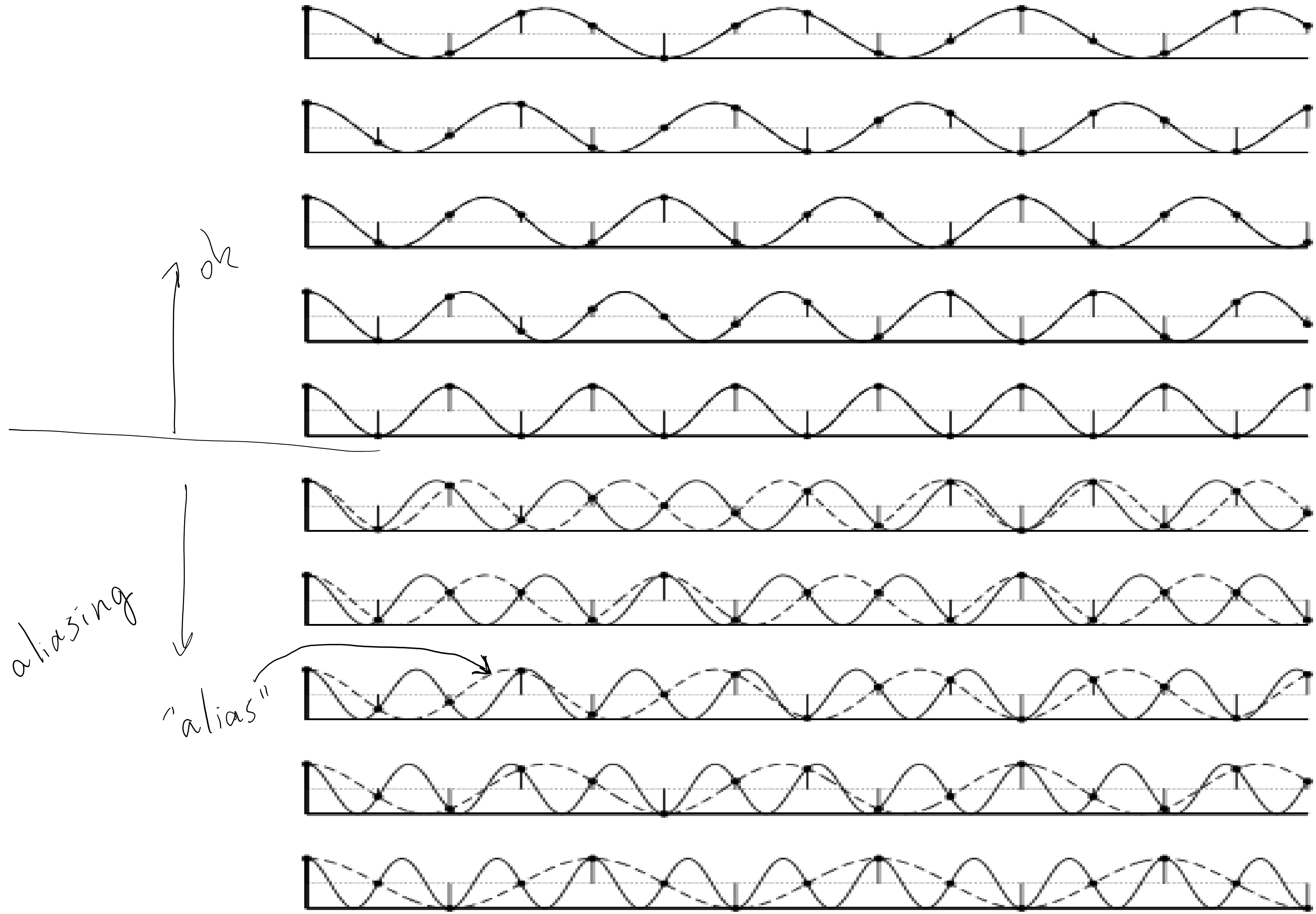
Sampling





minimum period: 2s
⇕
maximum frequency $\frac{1}{2s}$

Undersampling and aliasing



The Nyquist-Shannon sampling theorem

"The largest frequency that can be represented in a signal sampled at intervals s is 1/2s"

Periodic signals

$f(x)$: periodic function with period p

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i x k/p}$$

$$\langle k/p \mid u \rangle = \delta(u - k/p)$$

What is F.T. of f?

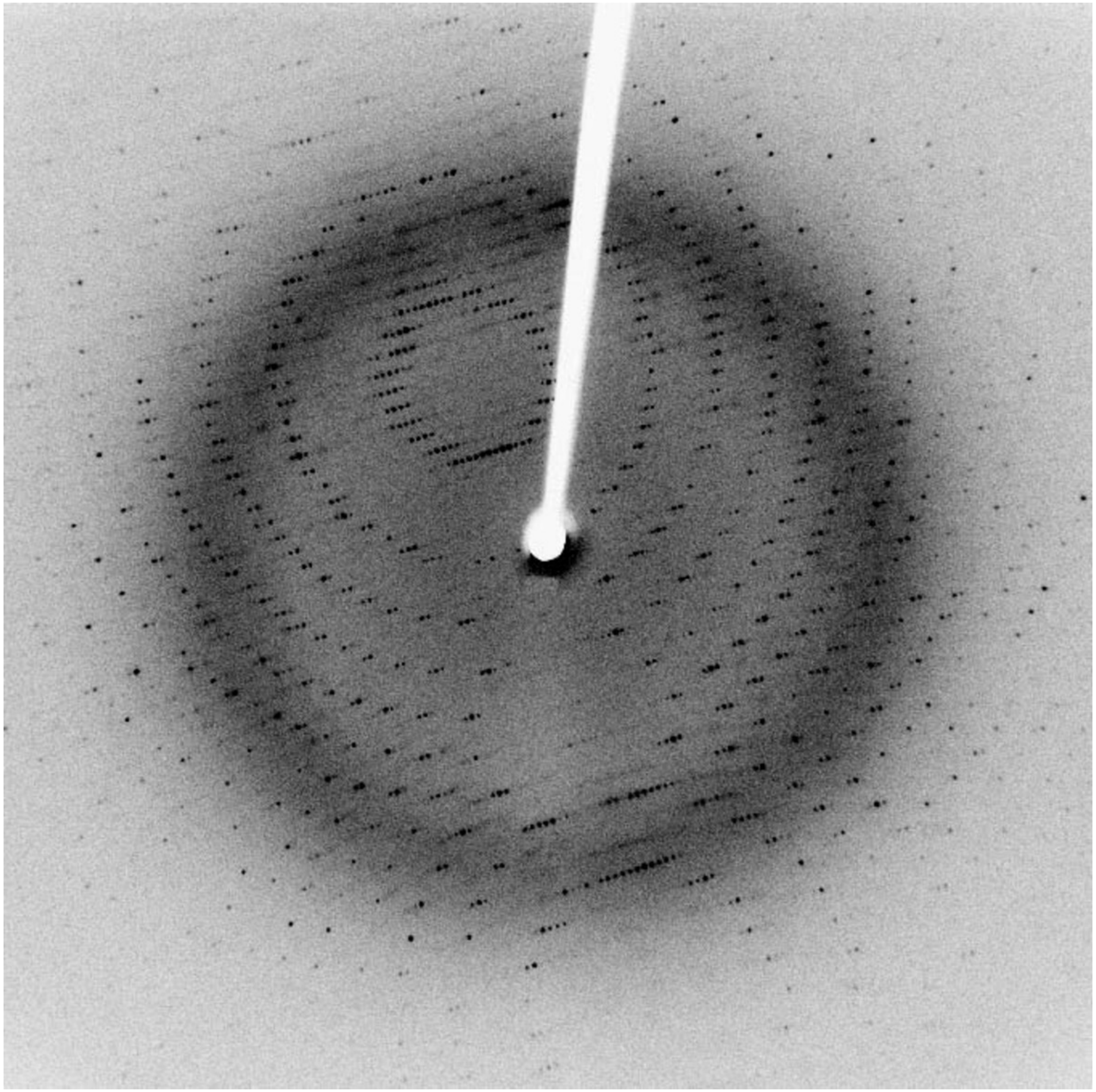
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x u} dx = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k e^{2\pi i x k/p} e^{-2\pi i x u} dx$$

$$F(u) = \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty} e^{2\pi i x (k/p - u)} dx = \sum_{k=-\infty}^{\infty} c_k \delta(u - k/p)$$

The Fourier transform of a periodic signal has a discrete spectrum located at multiples of $1/p$

Periodic signals

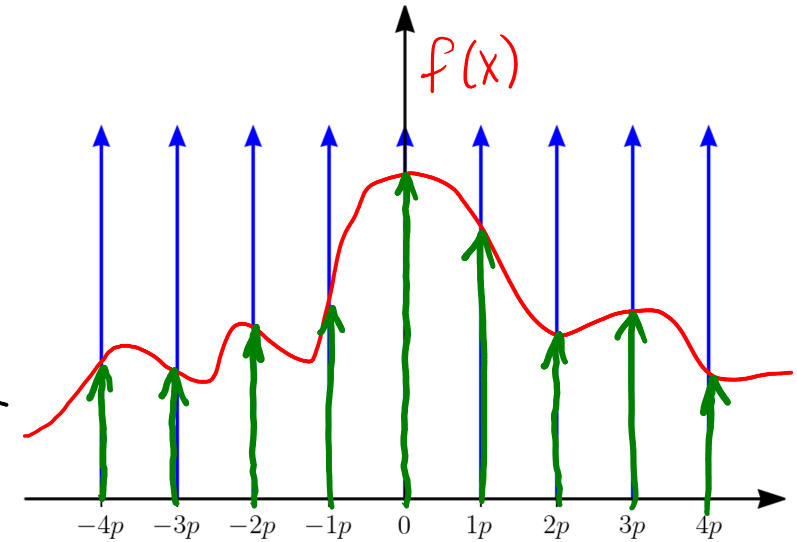
X-ray diffraction by a crystal



Sampling with the Dirac comb

A periodic function made of Dirac functions

$$\Delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - np)$$



$\Delta_p(x)$ can be used to represent sampling of a function $f(x)$:

$$f(x)\,\Delta_p(x) = \sum_{n=\infty}^{\infty} f(x)\,\delta(x - np) = \sum_{n=\infty}^{\infty} f(np)\,\delta(x - np)$$

Fourier transform of a Dirac comb

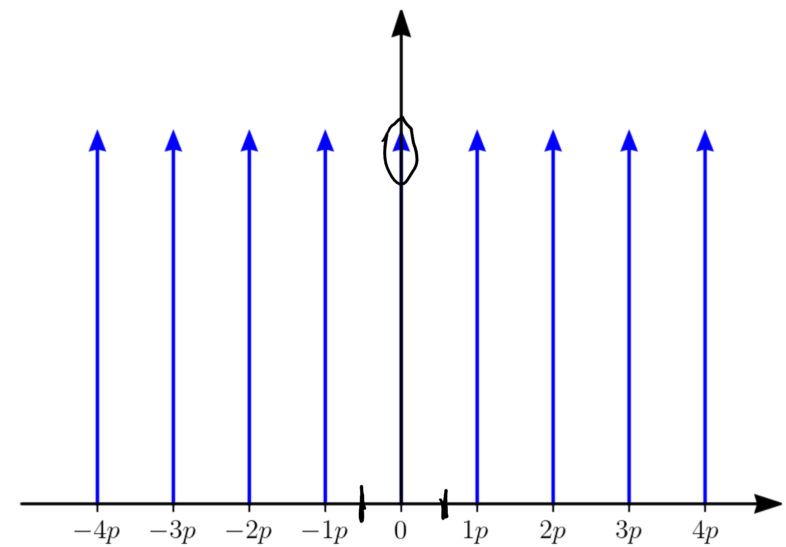
Fourier series coefficients that correspond to $\Delta_p(x)$

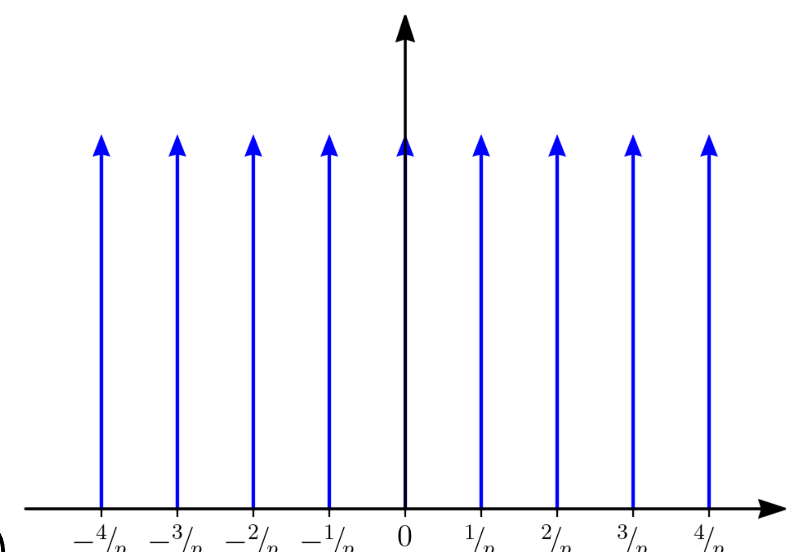
$$c_k = \int_{-p/2}^{p/2} \Delta_p(x)\, e^{-2\pi i k x/p}\, dx$$

$= 1$

$$\Rightarrow \mathcal{F}\{\Delta_p(x)\} = \sum_{k=-\infty}^{\infty} c_k\, \delta(u - k/p) \qquad (c_k = 1)$$

$$= \sum_{k=\infty}^{\infty} \delta(u - k/p)$$



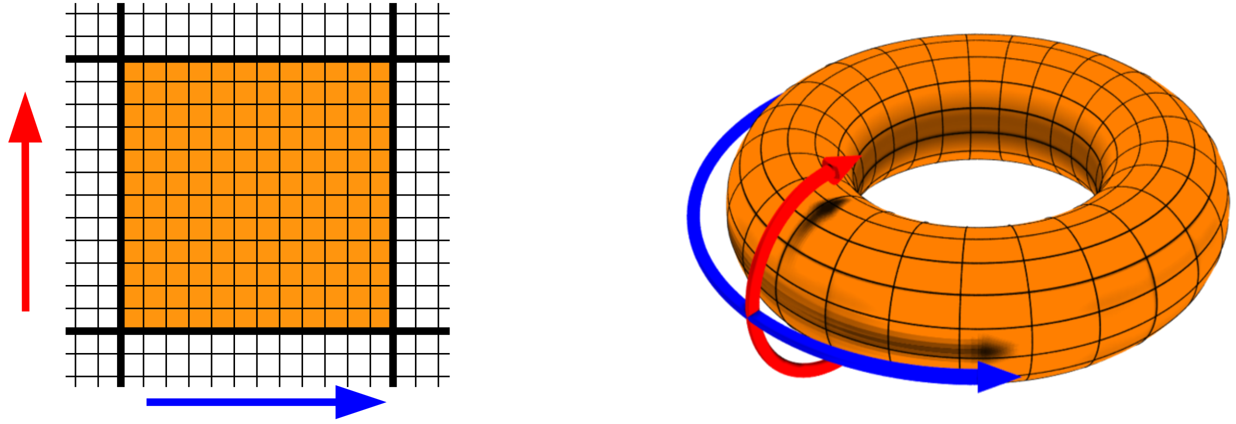


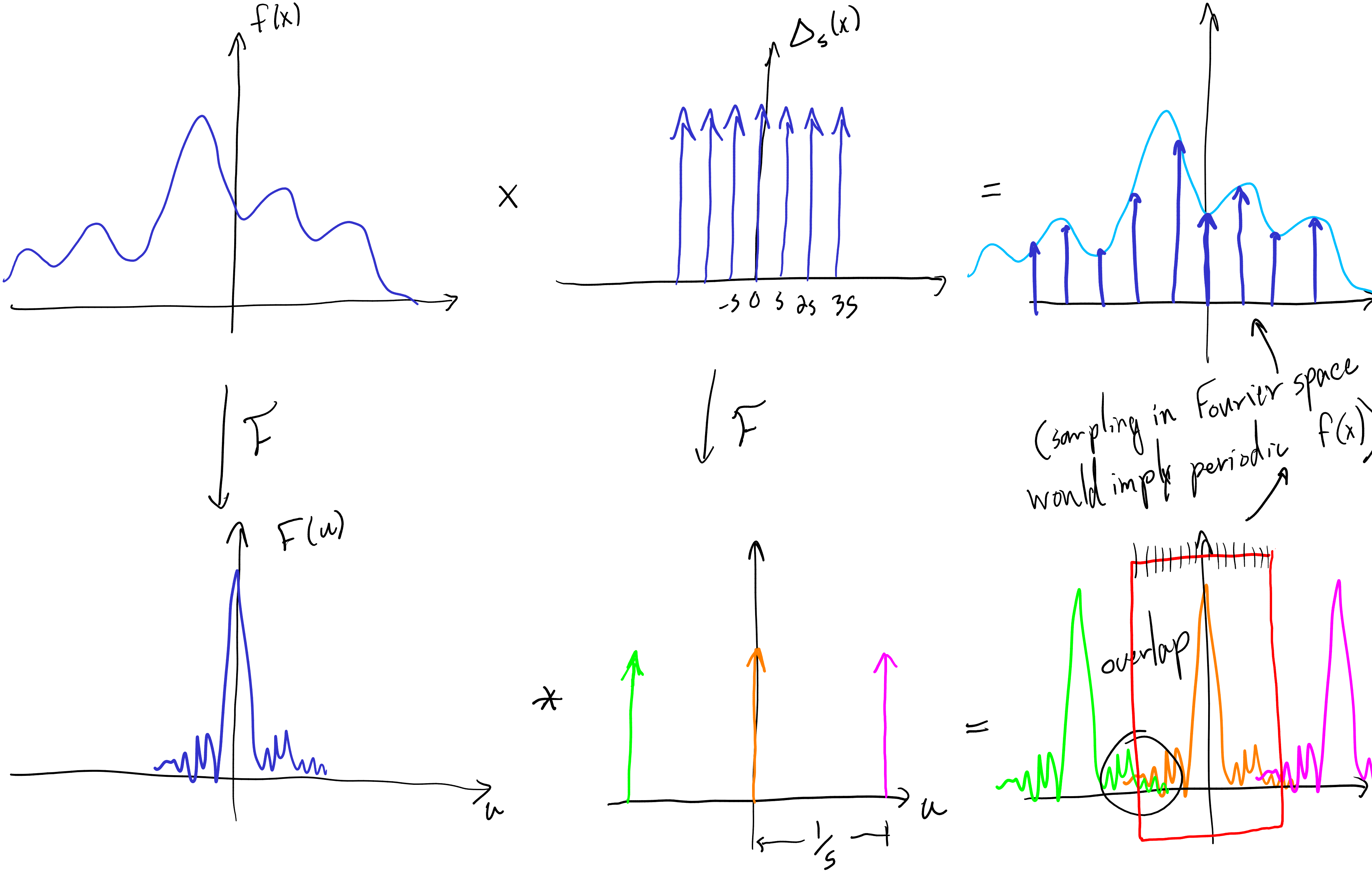
F.T. of a Dirac comb is another Dirac comb

Discrete Fourier Transform

- A **periodic** function has a **discrete** spectrum in the Fourier domain;
- A function with **discrete** values in the spatial domain is **periodic** in the Fourier domain;

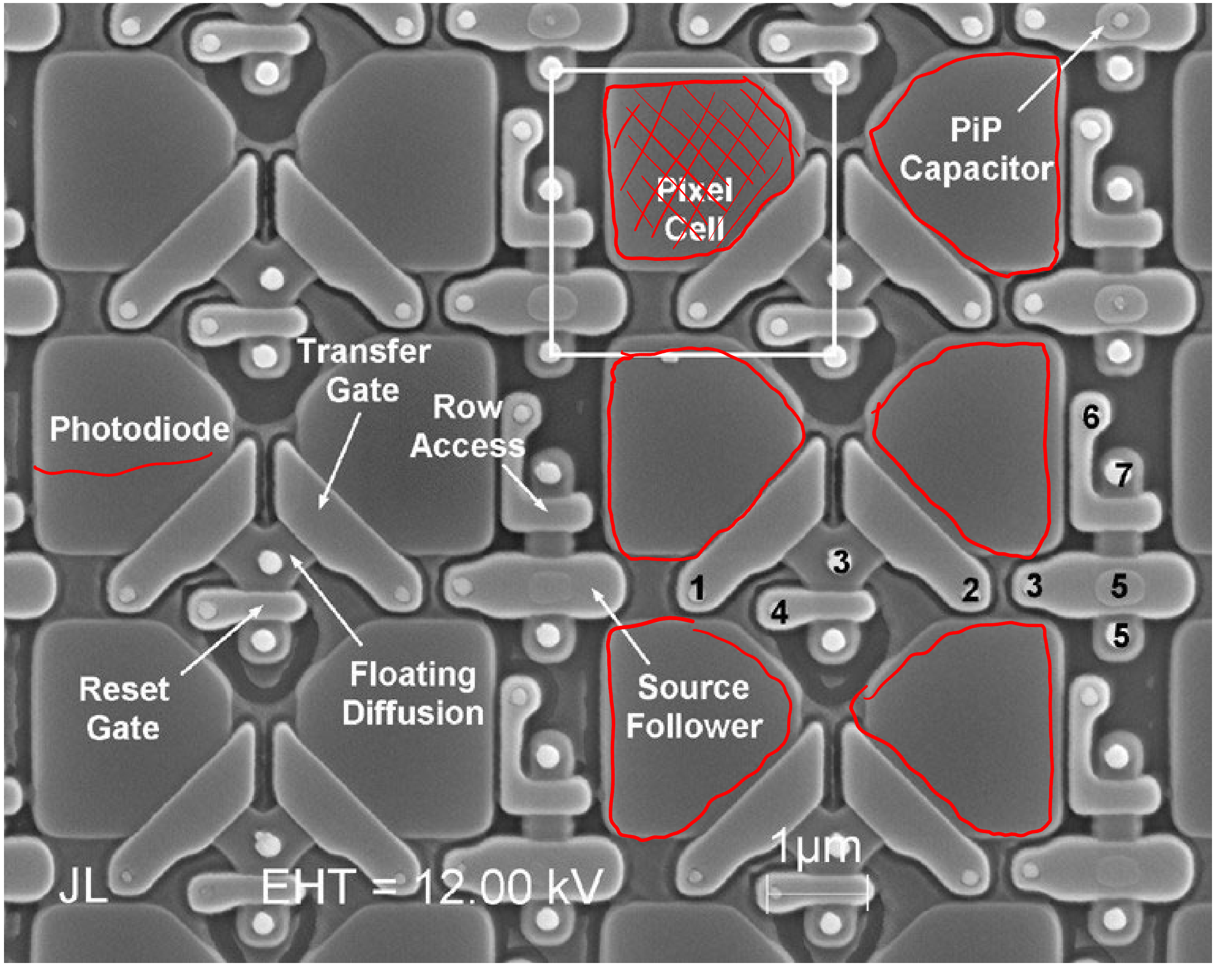
$\Rightarrow$ A periodic and discrete function has a periodic and discrete Fourier transform.

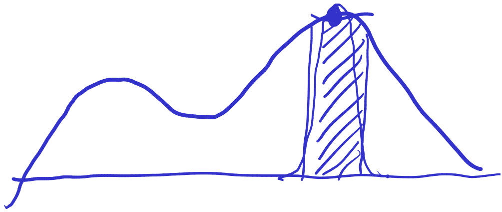



f(x)
X
Δs(x)
-s 0 s 2s 3s
=
(sampling in Fourier space
would imply periodic f(x))
F
F
F(u)
u
*
u
1/s
=
overlap

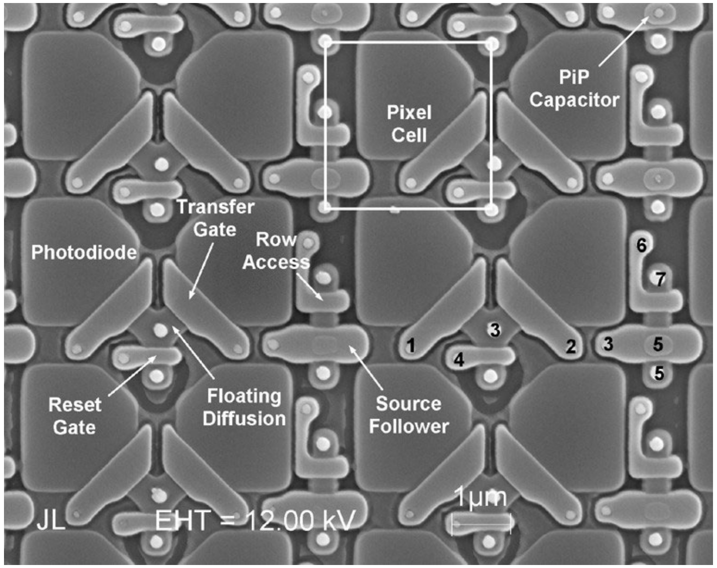
Sampling with a pixel-array detector

- A 2D light field is sampled with a 2D pixel-array detector.



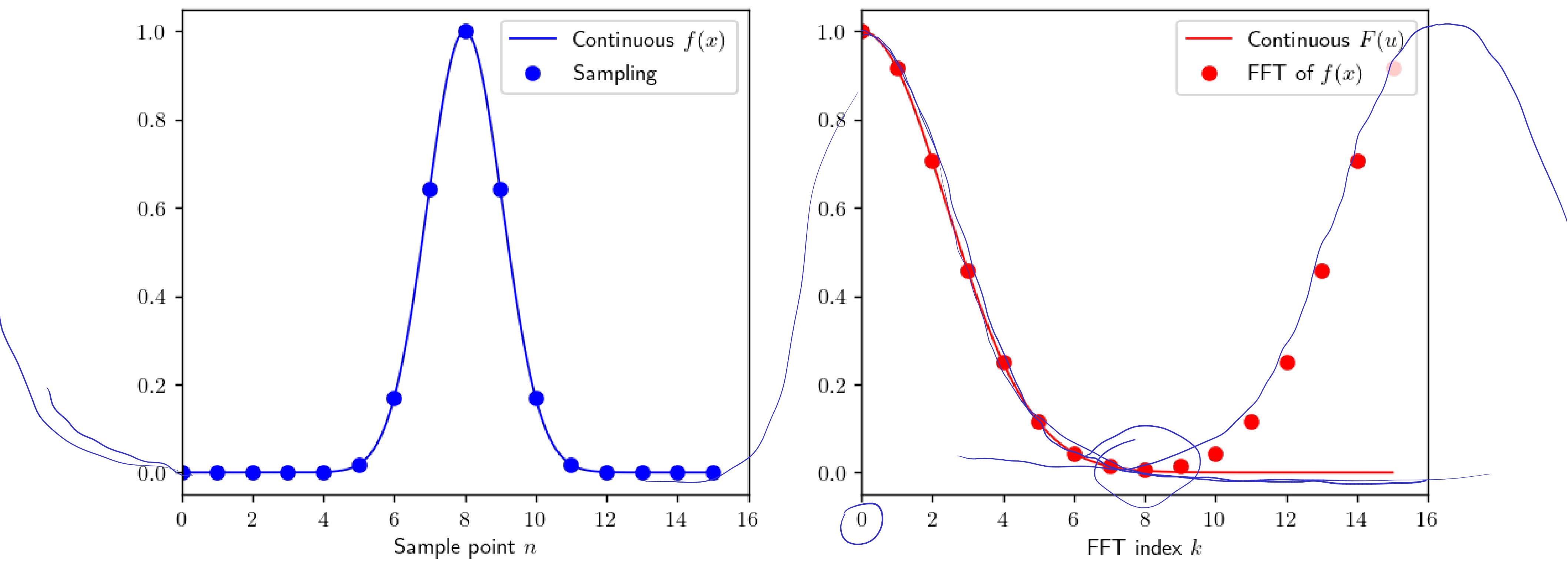


Sampling with a pixel-array detector



DFT example

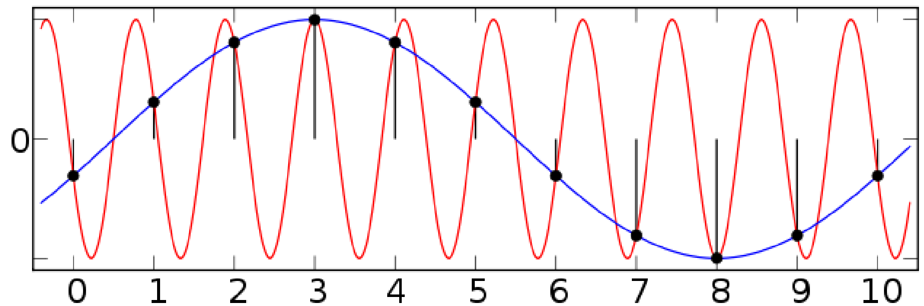
- Example: relation between space, sampling and frequency



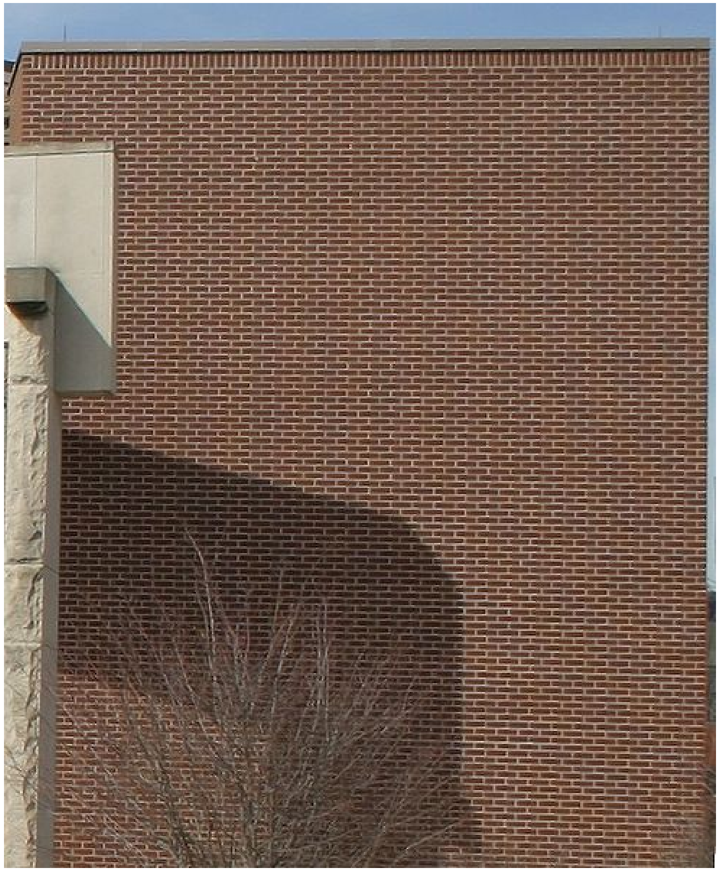
zero frequency component is in the top left corner output array.

Aliasing

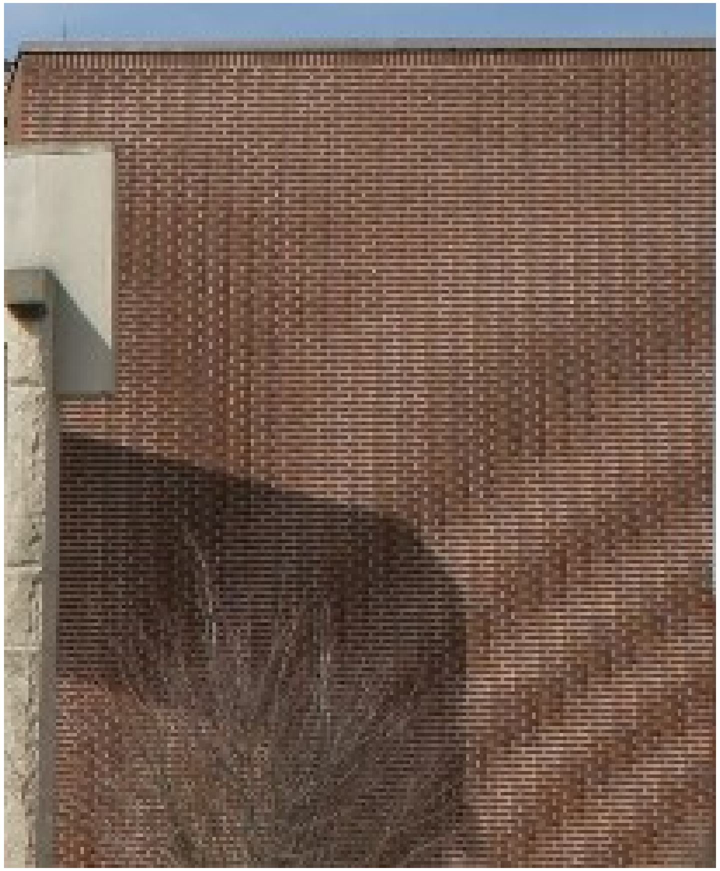
Moiré: after resampling, high spatial frequencies appear as low spatial frequencies



high resolution



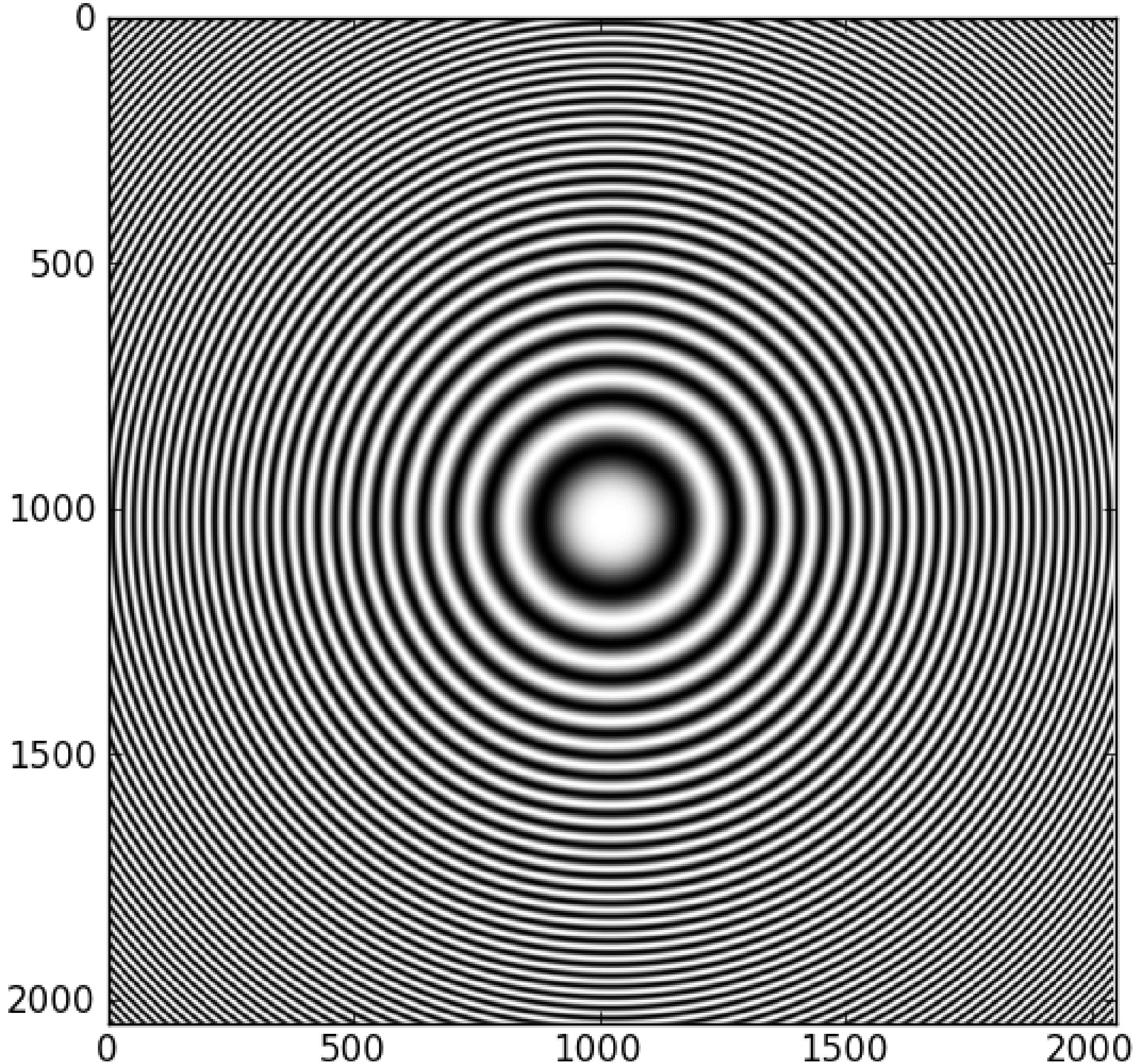
low resolution



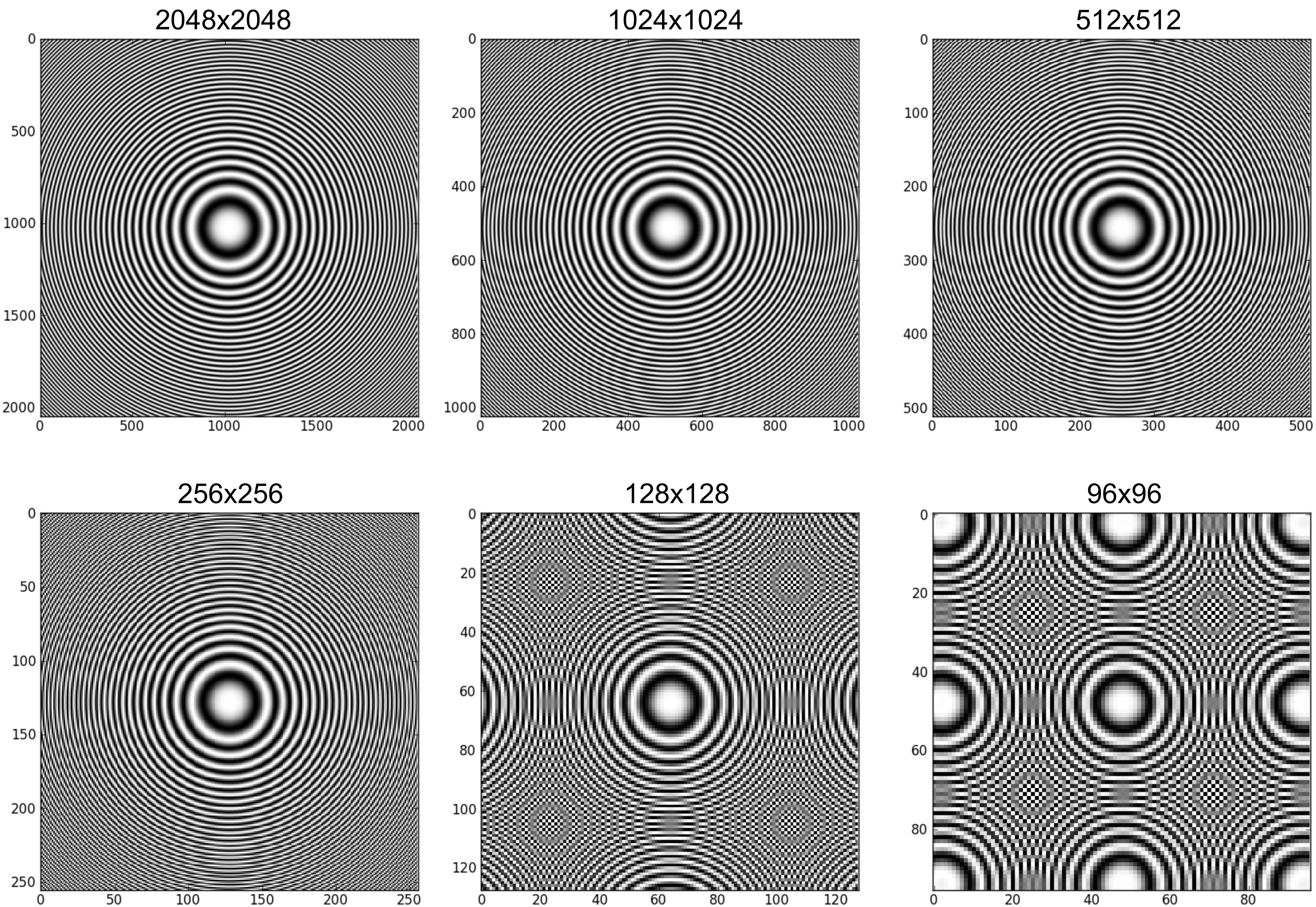
source: http://wikipedia.org

Undersampling

"Fresnel zone" test pattern: radial linear increase in spatial frequency



Undersampling & aliasing



Recall: Discrete Fourier transform:

$$F_k = \sum_{n=0}^{N-1} f_n e^{-2\pi i kn/N} \qquad f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{2\pi i kn/N}$$

On the computers: signal is both sampled and of finite extent

$\Rightarrow$ using DFT on this signal means that it is assumed to be periodic.

If this signal is a sufficiently sampled representation of an underlying continuous function then, the DFT can be interpreted as a sampled version of the continuous F.T.

Conversion is done looking at the exp argument

continuous: $e^{2\pi i u x}$

discrete: $e^{2\pi i n k / N}$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad ux = nk/N$

$f(x) \longrightarrow f_n$ sample step is s: $f_n = f(x = ns)$

$$x = ns$$

$$u \cancel{n} s = \cancel{n} k/N \qquad u = k/Ns$$

Observation: $$F_{k+N} = \sum_{n=0}^{N-1} f_n e^{-2\pi i n (k+N)/N}$$

$$= \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N} \underbrace{e^{-2\pi i n}}_{1}$$

$$= F_k \longrightarrow$$ of course! It's periodic!

$\Rightarrow$ the bounds of the sum can be shifted

$$f_n = \sum_{k=0}^{N-1} \cdots = \sum_{k=-N/2}^{(N-1)/2} \cdots$$

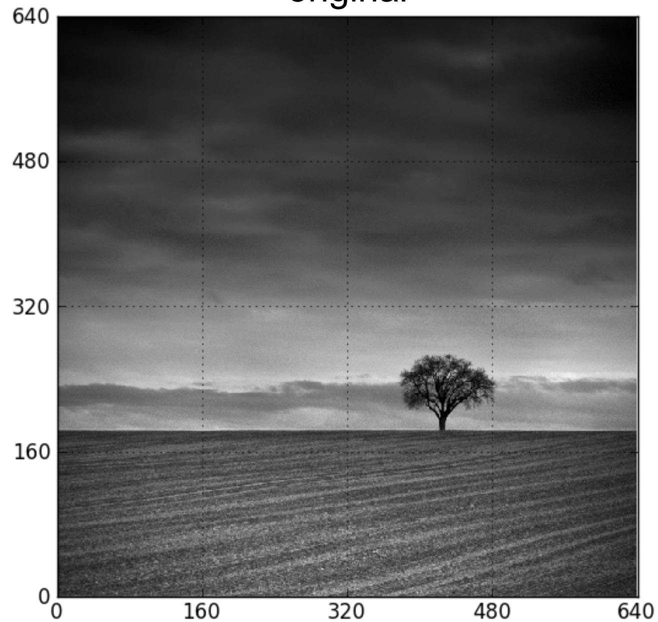
$u = k/NS$ if $k = -\frac{N}{2} \ldots \frac{N-1}{2}$ then

$$u = -\frac{1}{2S}, \ldots \frac{1}{2S}\left(1 - \frac{1}{N}\right)$$

numpy.fft.fftfreq gives you this sequence with $S=1$

Fourier space translation

original



amplitude of Fourier spectrum

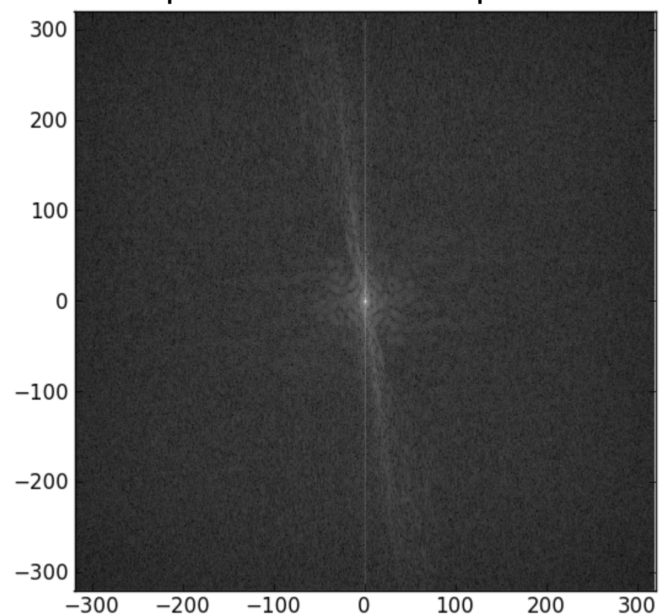
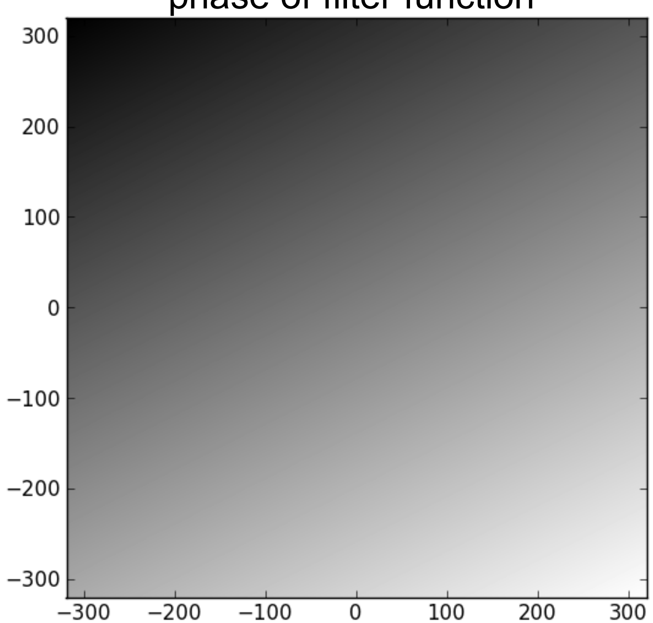


Image shifting using shifting property of FT

$$\mathcal{F}\{f(x-x_0)\} = \mathcal{F}\{f(x)\}\, e^{2\pi i u x_0}$$

phase of filter function



shifted image

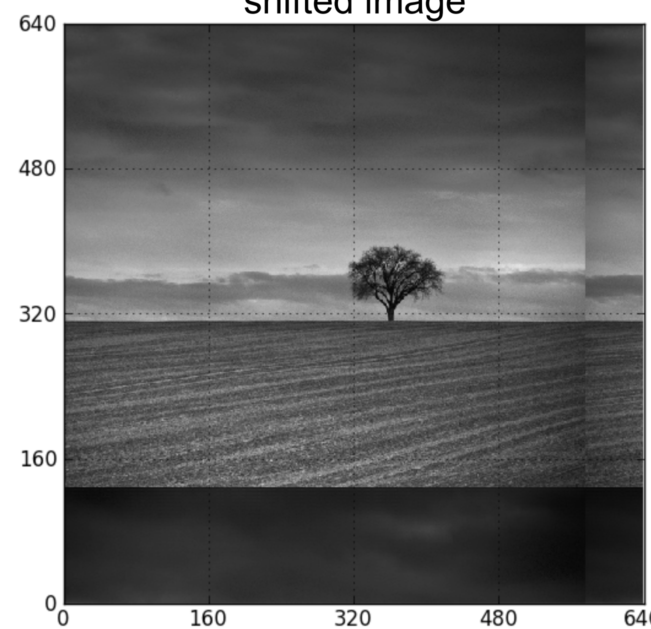
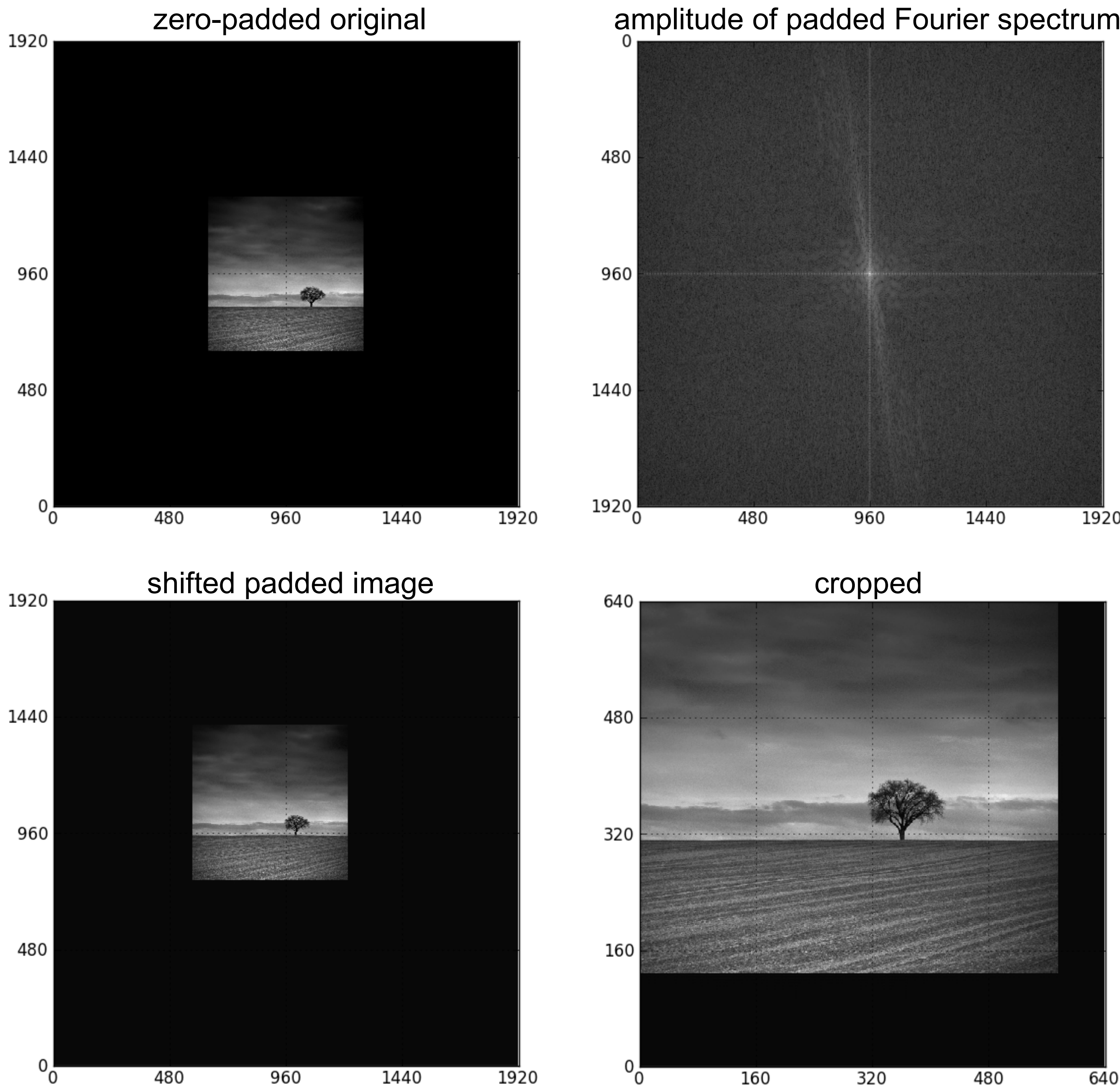


Image gets wrapped around

Zero-padding

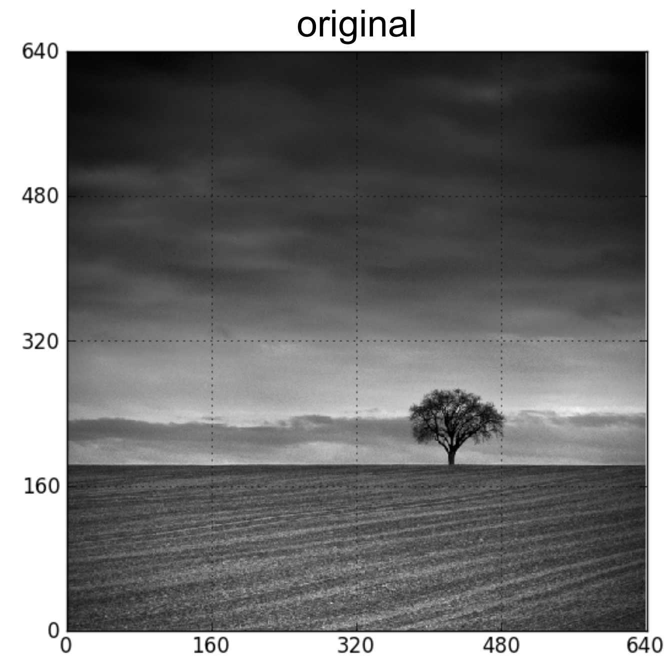


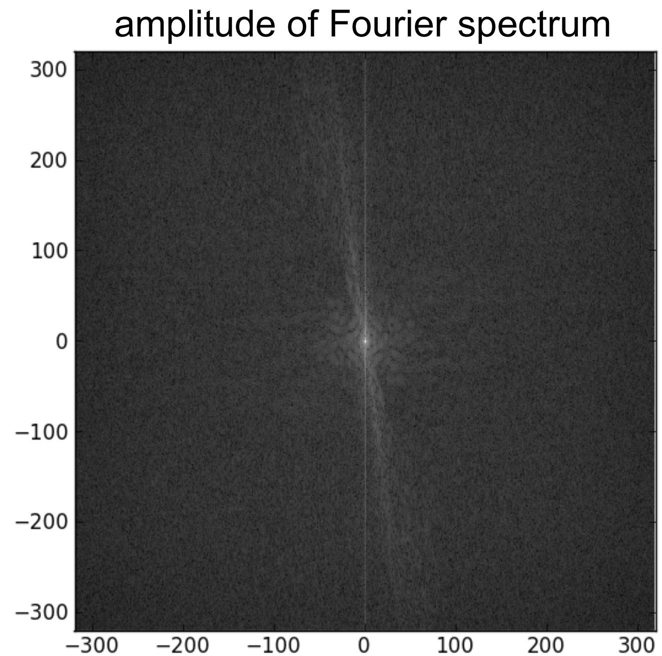
1. Add zeros around original image (*zero-padding*)

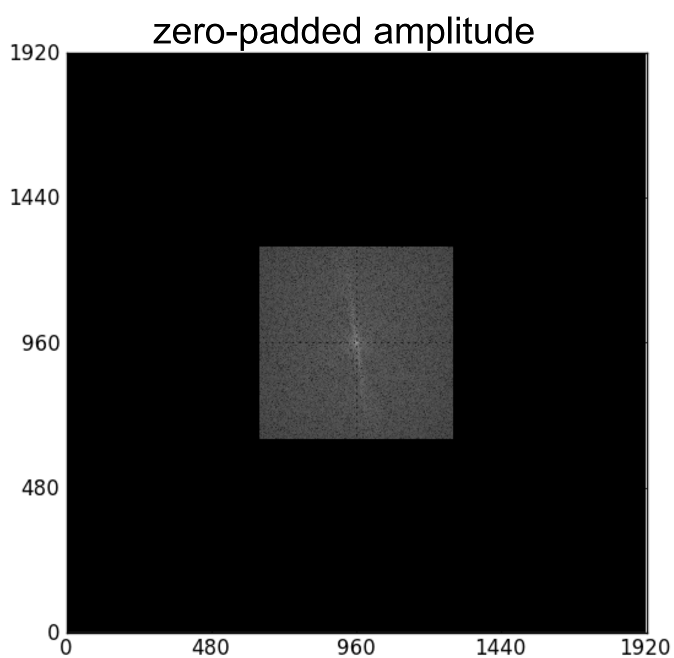
2. Shift using FT

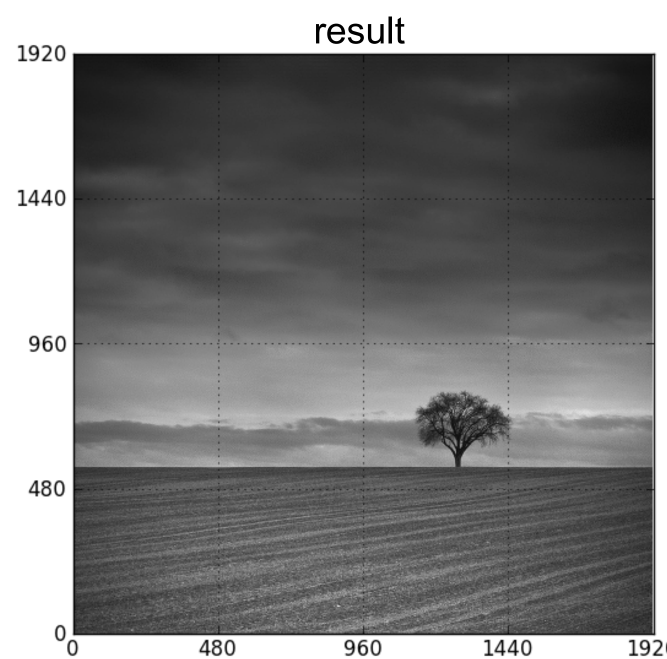
3. Crop result

Zero-padding in Fourier space









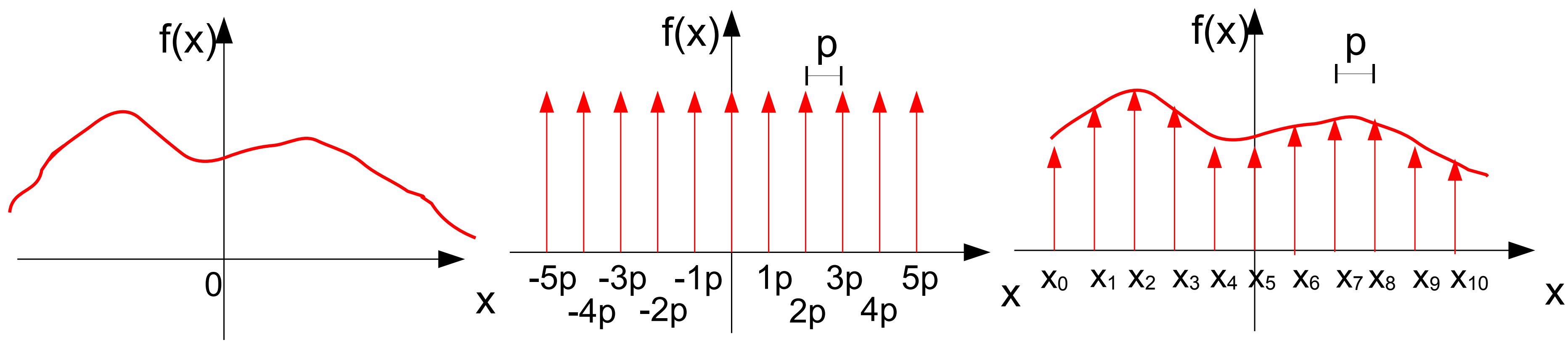
Result: increased sampling!

"upscaling"

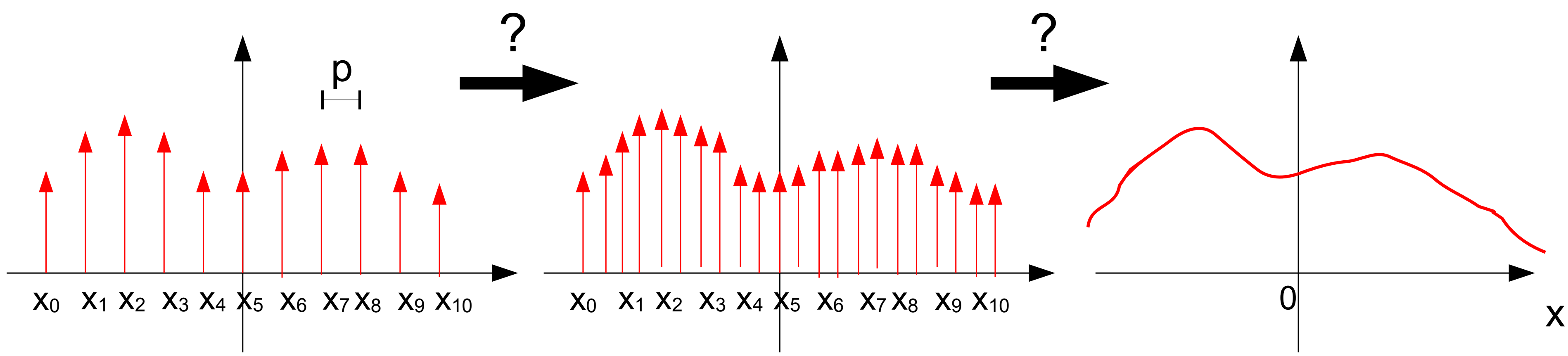
"interpolation"

Interpolation

- Discrete sampling of a continuous function



- Reconstruct original function from sampled data?

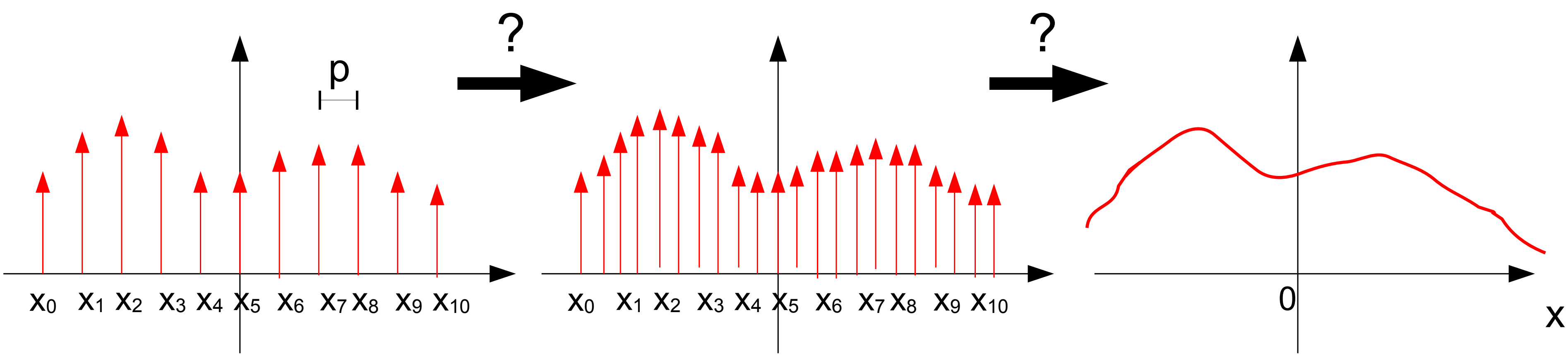


Interpolation

Finding unknown points between known ones

- wide field, many different approaches
- closely related to approximation theory and curve fitting

difference: interpolated curve has to pass through all known samples



Interpolation

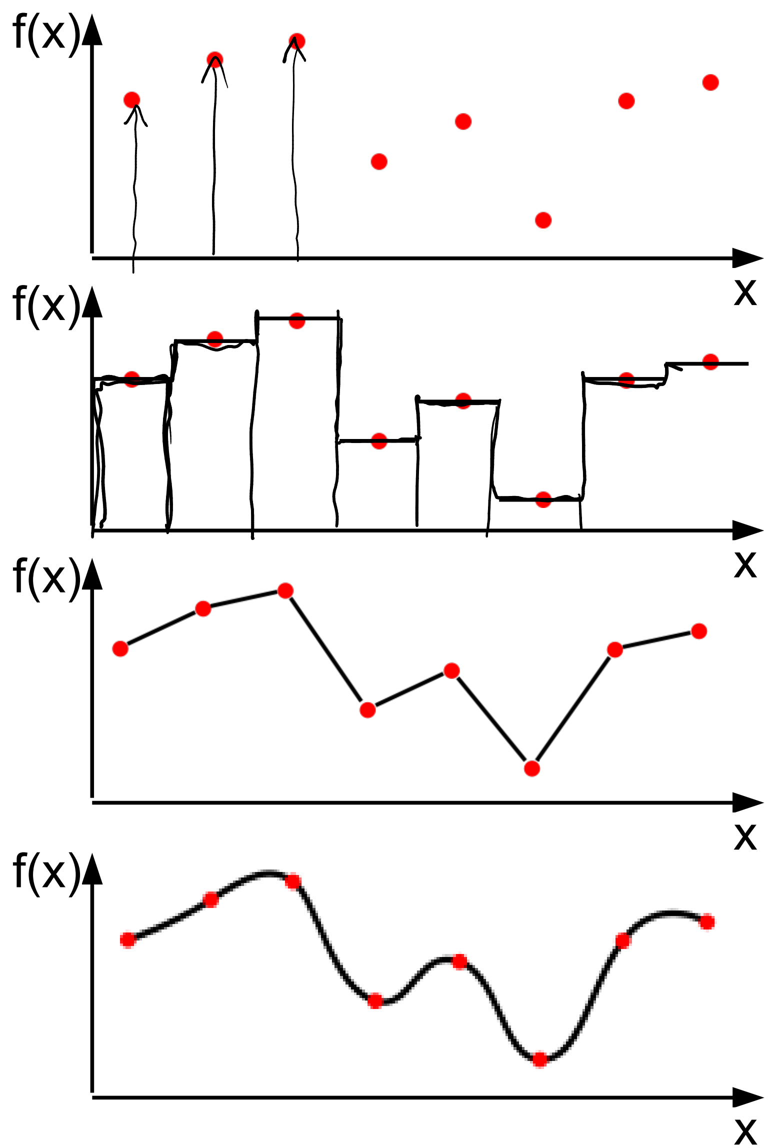
Various “classical” interpolation methods available

sampled points

nearest neighbor

linear

cubic spline



Linear interpolation

- Interpolation as an operator

$$f(x) = \mathcal{L}\{f_n\}$$

- Linear interpolation

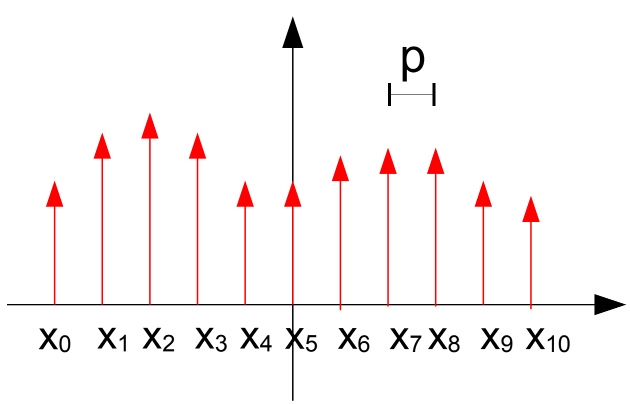
$$\mathcal{L}\{f_n + g_n\} = \mathcal{L}\{f_n\} + \mathcal{L}\{g_n\}$$

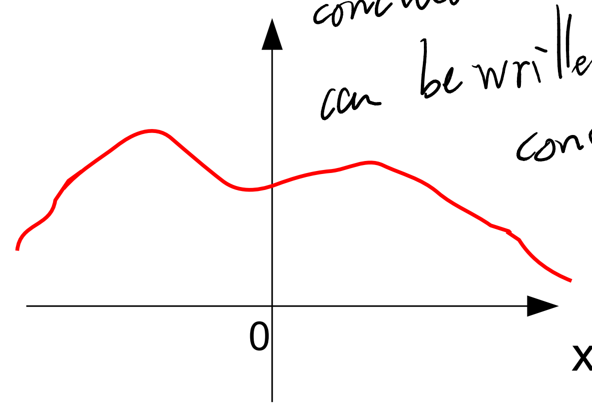
- Shift invariance

$$\mathcal{L}\{f_{n+n_0}\} = f(x + n_0 s)$$

sufficient to conclude that $\mathcal{L}$ can be written as a convolution ?

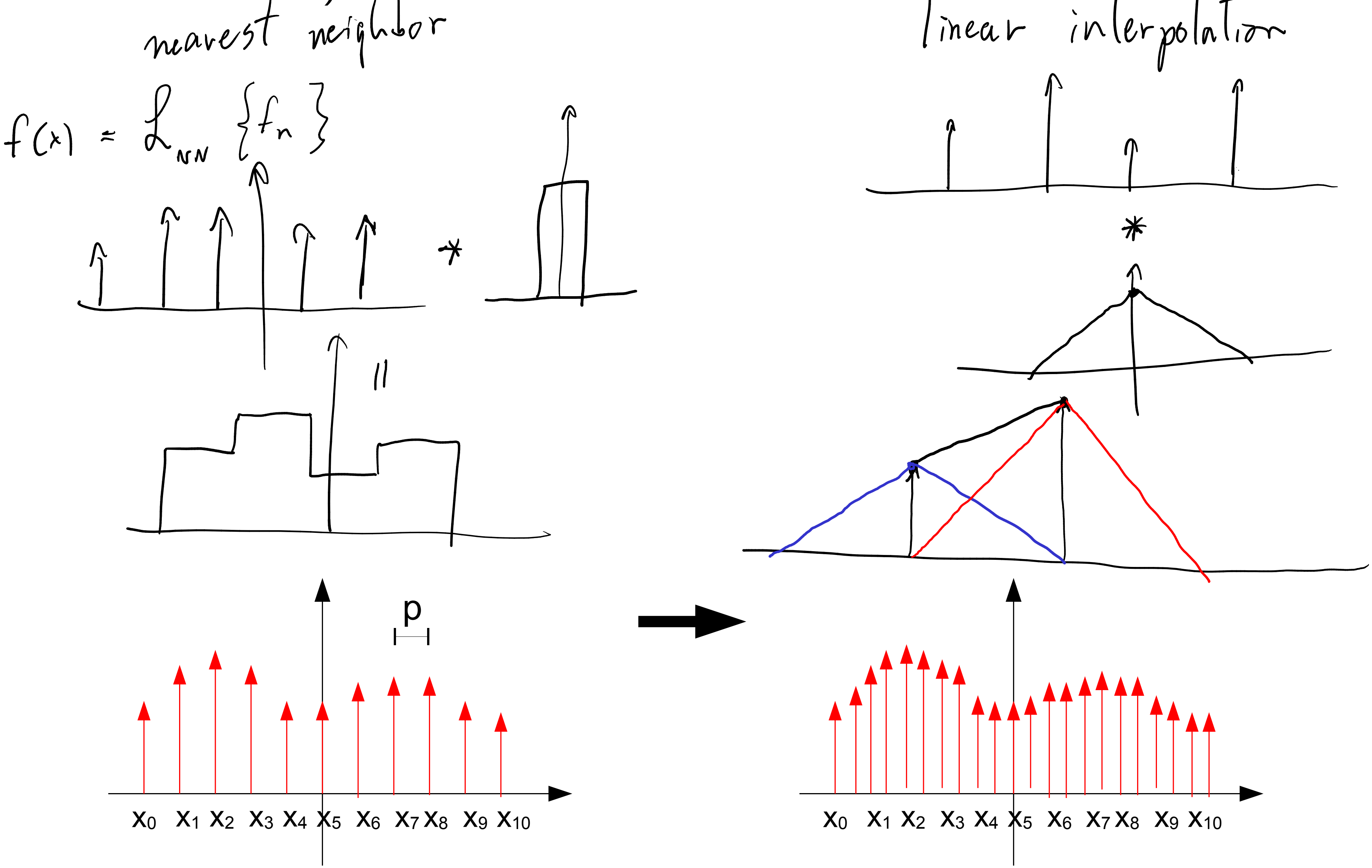
- Kernel





Linear interpolation

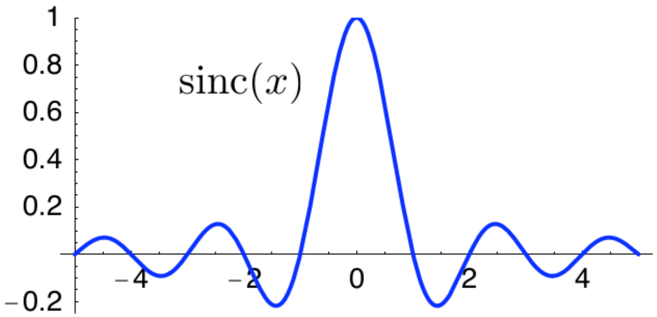
- Linear interpolation can be written as a convolution with a kernel (e.g. a basis function)



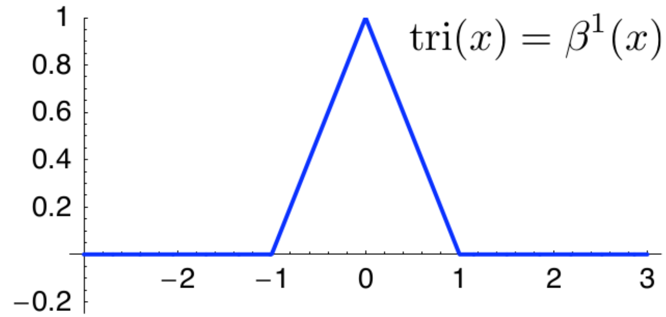
Linear interpolation

- **Bandlimited**

$\frac{\sin(x)}{x}$



- **Piecewise linear**

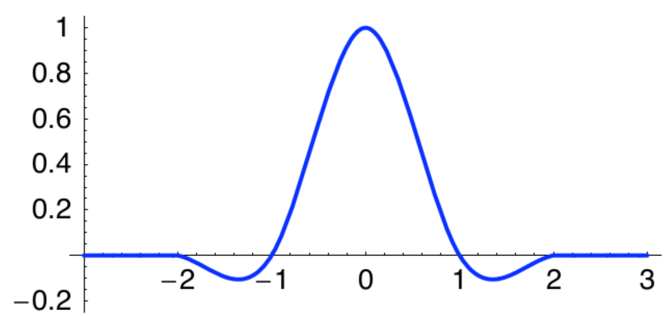


Interpolation condition:

$$\varphi_{\text{int}}(k) = \delta_k = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

↑ ensures that the interpolated curve passes through all the samples

- **Cubic convolution**



[Keys, 1981; Karup-King 1899]

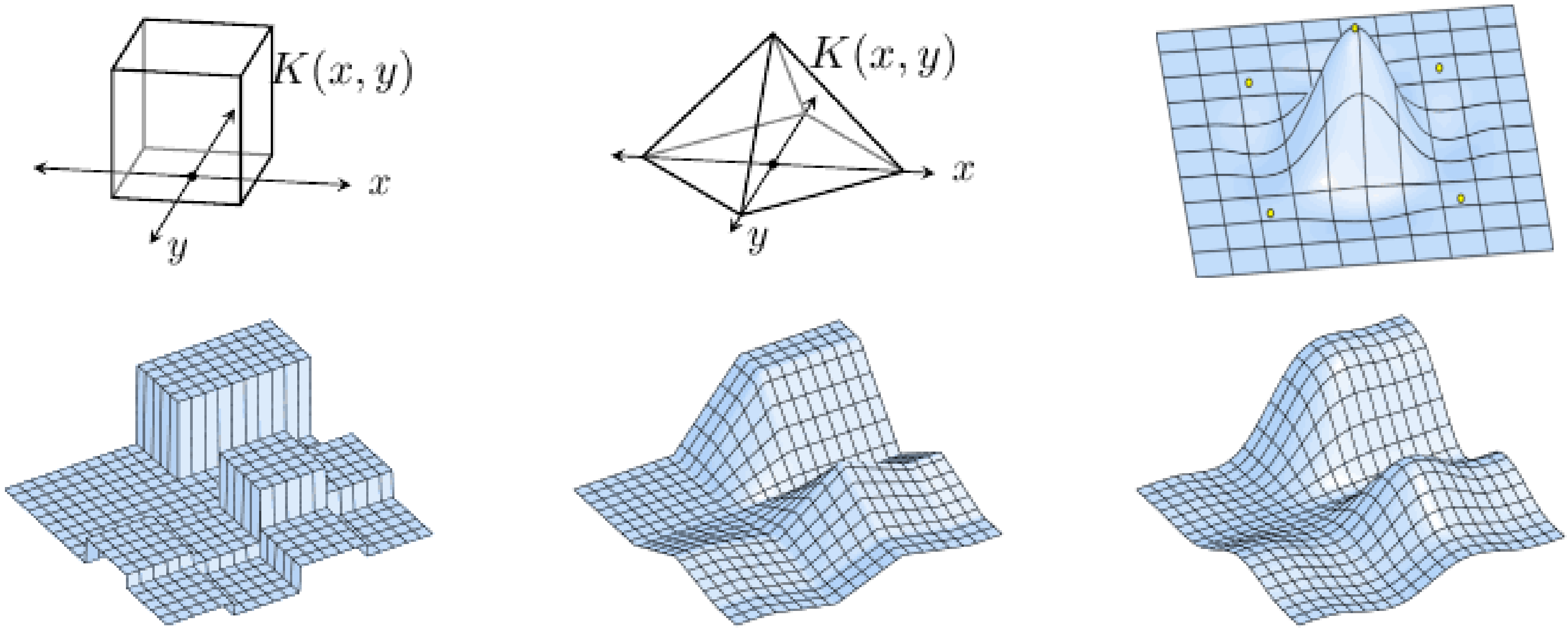
source: http://bigwww.epfl.ch/tutorials/unser_isbi_06_part1

Interpolation via convolution

2D interpolation

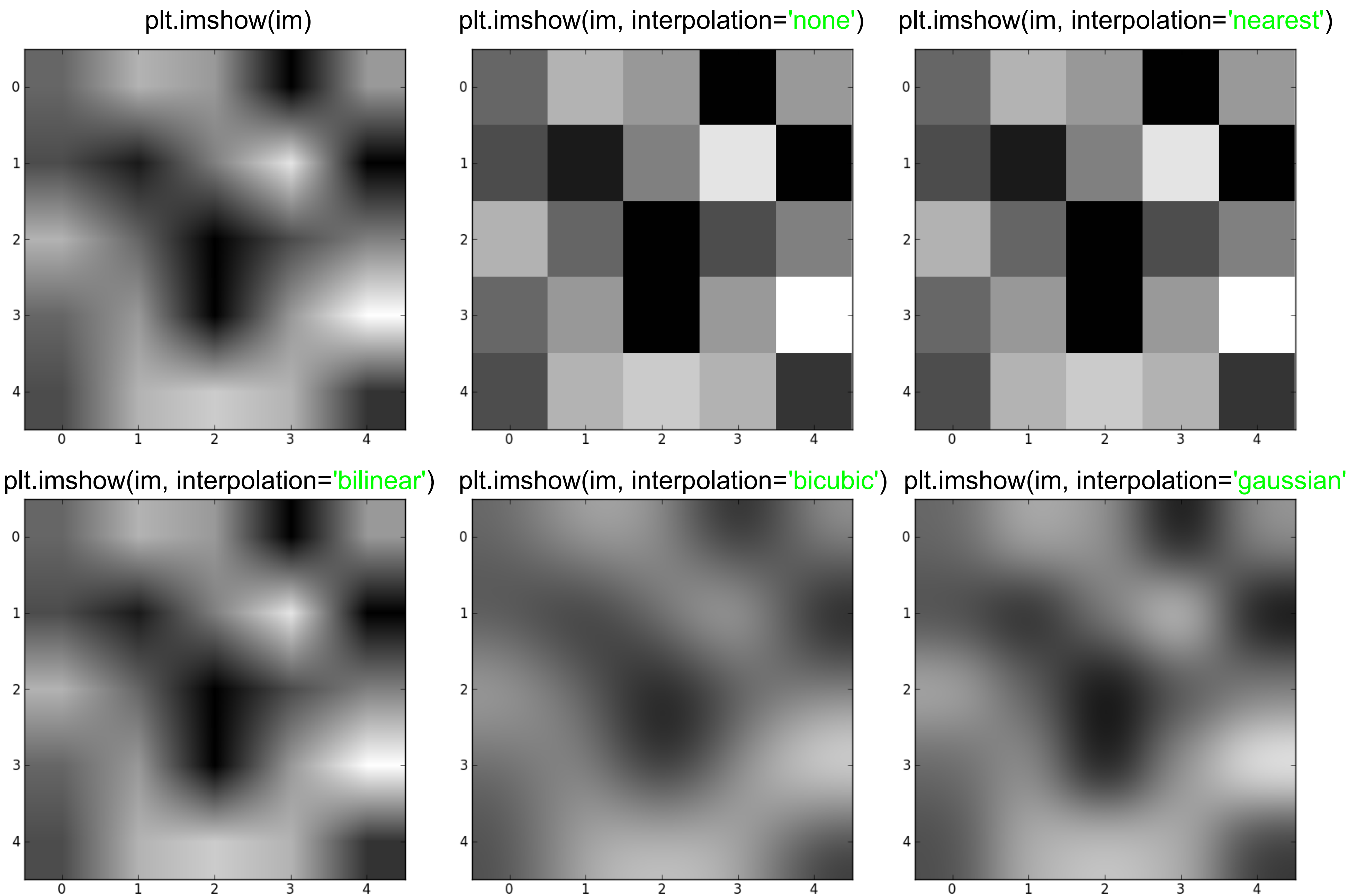
- Make 2D interpolation linear in each variable

nearest neighbor | bilinear | bicubic

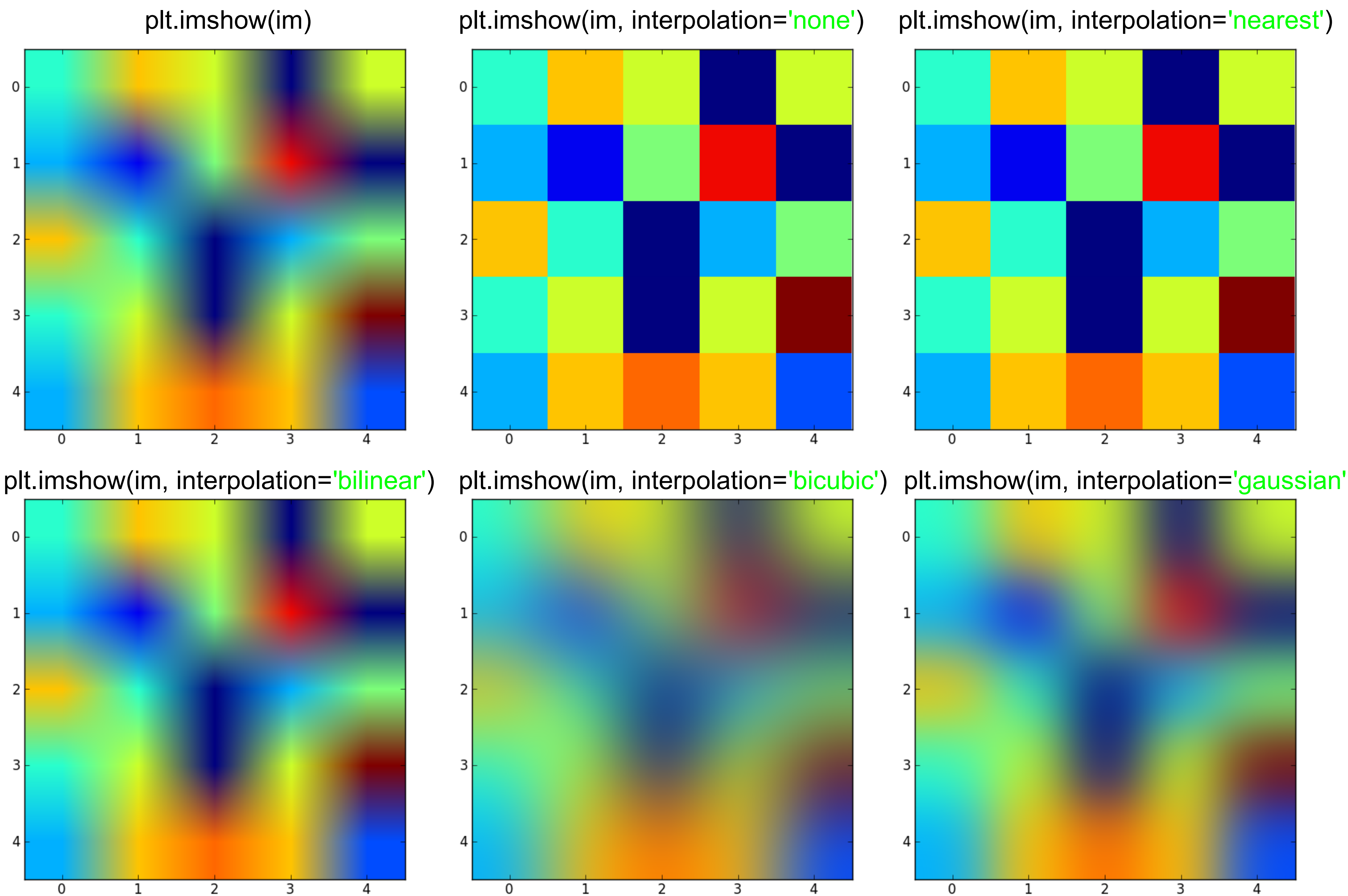


source: http://www.ipol.im/pub/art/2011/g_lmii/

Python plotting

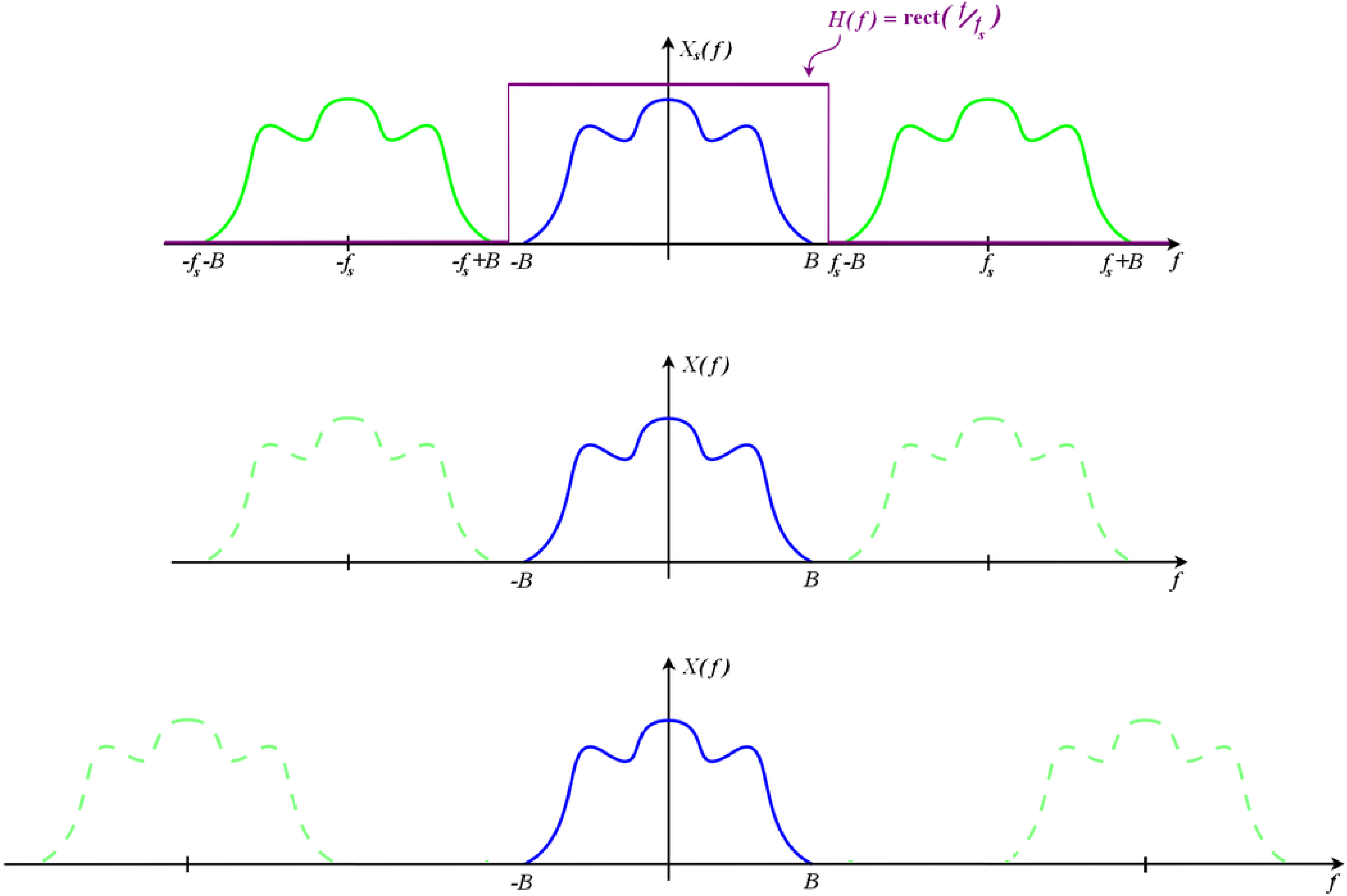


Python plotting



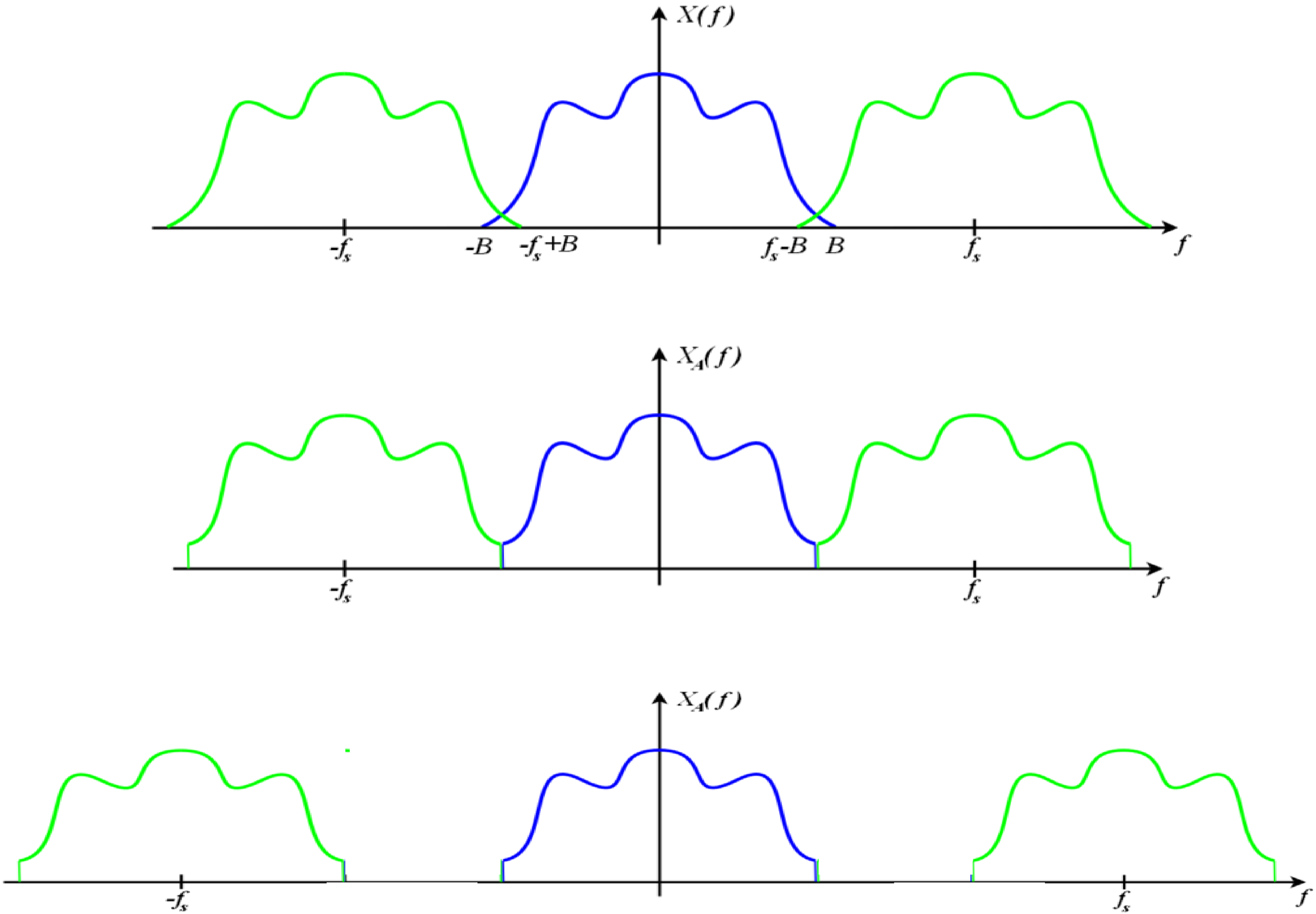
Sinc interpolation and zero-padding

Also known as "Whittaker–Shannon interpolation"



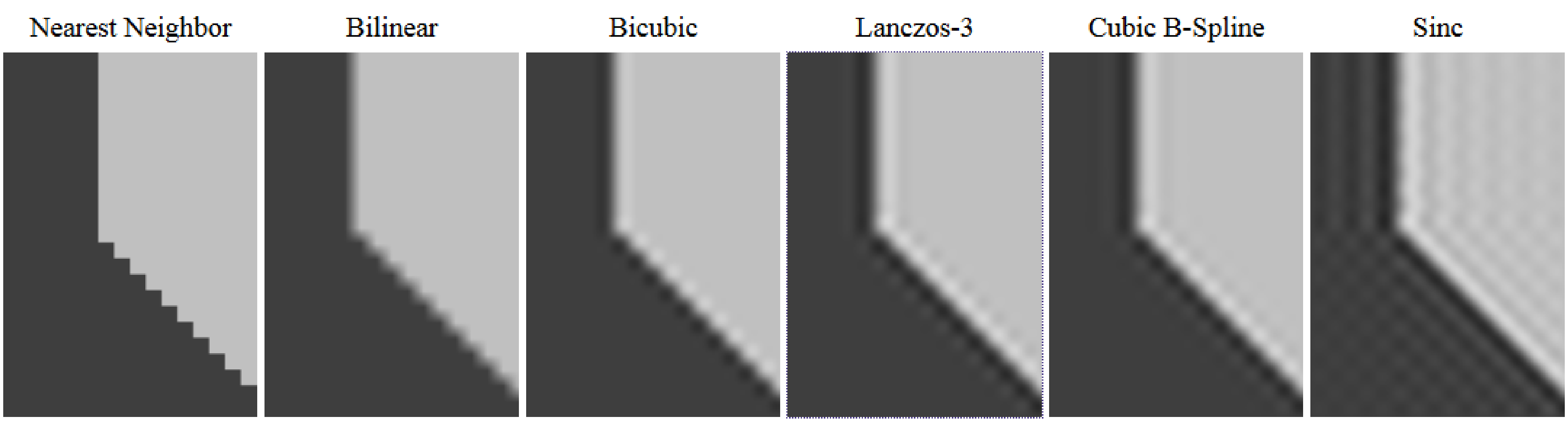
Sinc interpolation and zero-padding

Also known as “Whittaker–Shannon interpolation”



Reconstruction from samples

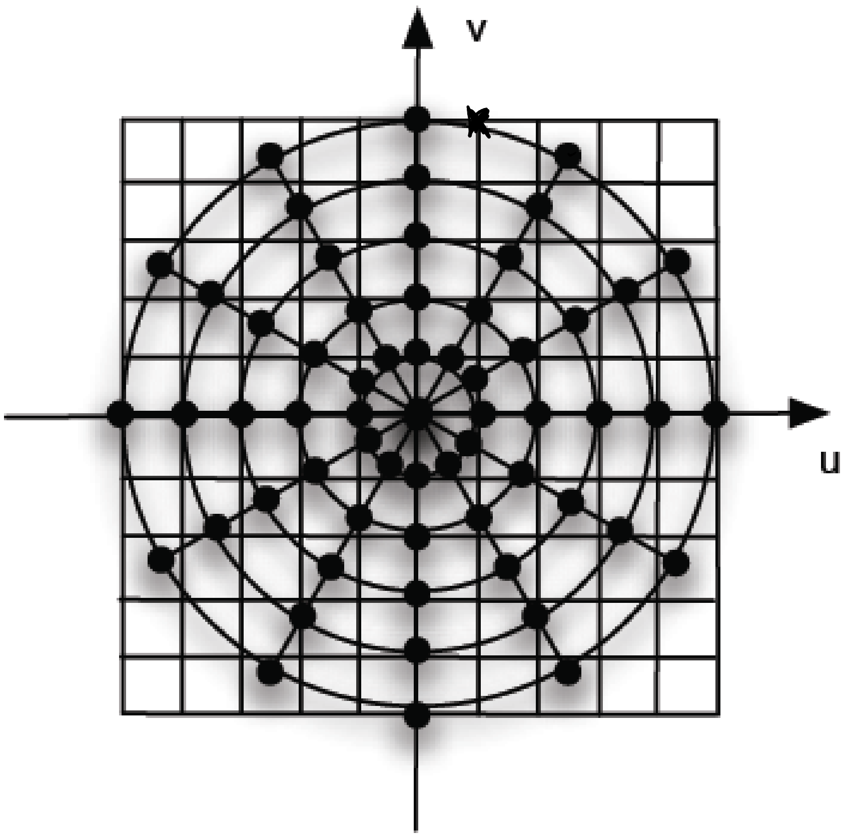
- Sinc interpolation can perfectly reconstruct a function from its samples if
  - sampled at a rate higher than Nyquist rate
  - bandlimited up to Nyquist frequency
  - no aliasing
- Sinc interpolation introduces ringing otherwise, due to leakage of aliased frequencies



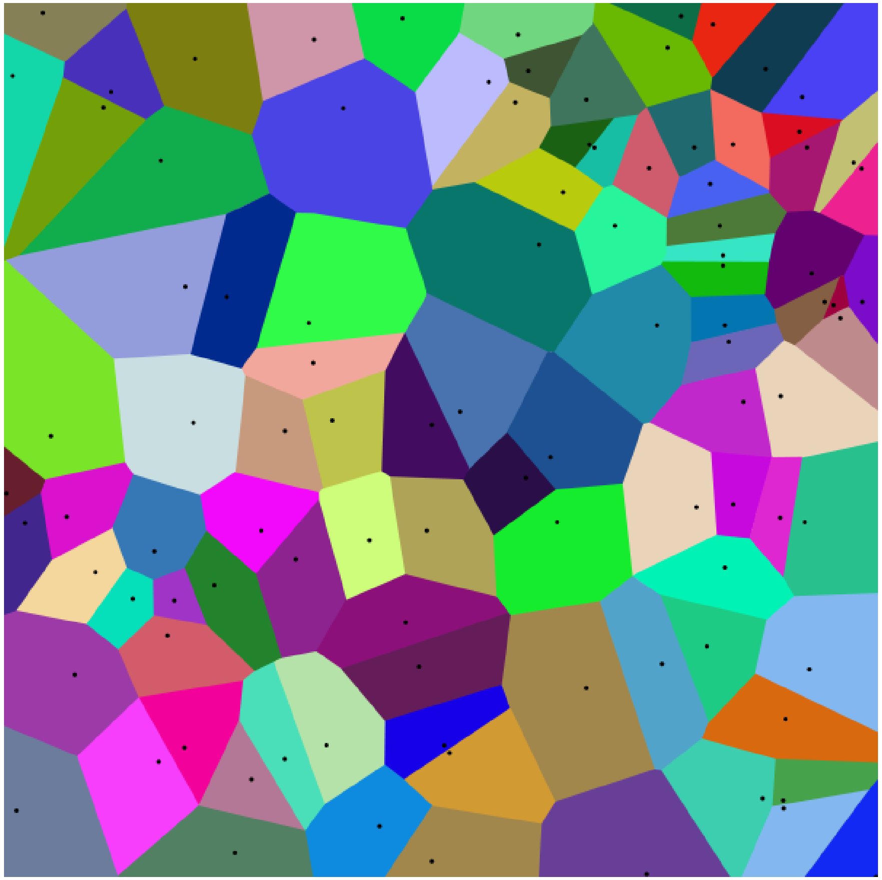
Linear interpolation of a step edge: a balance between staircase artifacts and ripples.

Other Interpolation

- Change from polar to cartesian grid
- Linear, but not translation invariant



polar vs. cartesian sampling



irregular sampling

Summary

- Images can be represented as a sampling grid and pixel basis functions
- Need for interpolation arises when changing the grid
- Linear and translation invariant interpolation can be written as a convolution with an interpolation kernel function
- Typical interpolation kernels include nearest neighbor, linear, cubic and higher B-spline interpolation
- Zero-padding in one domain equals sinc interpolation in the other
- "ideal" sinc interpolation may lead to ringing artifacts