27 october

$$(1)\begin{cases} \partial_t u - \Delta u = \overbrace{-\mathbb{P}\,\mathrm{div}(u\otimes u)}^{Q(u,u)} \\ \nabla\cdot u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

$\mathbb{P}$: Leray projector

$$\mathrm{div}(u\otimes u) = u\cdot\nabla u$$

$$\begin{cases} \partial_t B(u,v) - \Delta B(u,v) = Q(u,v) \\ B(u,v)|_{t=0} = 0 \end{cases}$$

We framed (1) as

$$(2)\quad u = e^{t\Delta}u_0 + B(u,u)$$

<u>Thm</u> $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)$ $\quad d=2,3$

$\exists\ T>0$ and a solution of 2

$u \in L^4([0,T],\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))$.

Furthermore

$u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1})$

$\nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}-1})$

and if T_{u_0} is the maximum time of existence

1) If $|u_0|_{\dot{H}^{\frac{d}{2}-1}} < \varepsilon_0(d)$

$\Rightarrow \quad T_{u_0} = +\infty$

2) If $T_{u_0} < +\infty$ then

$$\int_0^{T_{u_0}} |u|^4_{\dot{H}^{\frac{d-1}{2}}} \, dt = +\infty$$

3)

If u and v are solutions then

$$|u(t)-v(t)|^2_{\dot{H}^{\frac{d}{2}-1}} + \int_0^t |\nabla(u-v)|^2_{\dot{H}^{\frac{d}{2}-1}} dt'$$

$$\leq |u_0 - v_0|_{\dot{H}^{\frac{d}{2}-1}} \, e^{C_d \|(u,v)\|^4_{L^4([0,t], \dot{H}^{\frac{d-1}{2}})}}$$

Pf

Suppose u is a solution $T = T_{u_0}$ and that

$$\int_0^T \| u(t) \|^4_{\dot{H}^{\frac{d-1}{2}}} \, dt < +\infty$$

$$\Rightarrow \quad u \in L^4\left((0,T), \dot{H}^{\frac{d-1}{2}}\right)$$

$$\| Q(u,u) \|_{L^2\left([0,T), \dot{H}^{\frac{d}{2}-2}\right)}$$

$$\leq C_d \quad \| u \|^2_{L^4\left((0,T), \dot{H}^{\frac{d-1}{2}}\right)}$$

$$g(\xi) := \sup_{0 \leq t' \leq T} | \hat{u}(t', \xi) |$$

<u>Claim</u> $\quad |\xi|^{\frac{d}{2}-1} \, g \in L^2(\mathbb{R}^d)$

Pf

$$\| \, |\xi|^{\frac{d}{2}-1} g \|_{L^2} = \left(\int_{\mathbb{R}^d} |\xi|^{d-2} \, |g(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathbb{R}^d} |\xi|^{d-2} \left(\sup_{0 \leq t' < T} | \hat{u}(t', \xi) | \right)^2 d\xi \right)^{\frac{1}{2}}$$

$$\begin{cases} \partial_t u - \Delta u = Q(u,u) \in L^2((0,T), \dot{H}^{\frac{d}{2}-2}) \\ u|_{t=0} = u_0 \in \dot{H}^{\frac{d}{2}-1} \end{cases}$$

$$\begin{cases} \partial_t u - \Delta u = f \in L^2((0,T), \dot{H}^{s-1}) \\ u|_{t=0} = u_0 \in \dot{H}^{s} \end{cases}$$

$$\Rightarrow \left(\int_{\mathbb{R}^d} |\xi|^{2s} \left(\sup_{0 \le t < T} |\hat{u}(t',\xi)| \right)^2 d\xi \right)^{\frac{1}{2}}$$

$$\le |u_0|_{\dot{H}^s} + \frac{1}{\sqrt{2}} |f|_{L^2(0,T), \dot{H}^{s-1})}$$

$$s = \frac{d}{2} - 1$$

$$\left(\int_{\mathbb{R}^d} |\xi|^{d-2} \left(\sup_{0 \le t' < T} |\hat{u}(t',\xi)| \right)^2 d\xi \right)^{\frac{1}{2}}$$

$$\le |u_0|_{\dot{H}^{\frac{d}{2}-1}} + |Q(u,u)|_{L^2((0,T), \dot{H}^{\frac{d}{2}-2})}$$

$$< +\infty$$

Since $\quad |\xi|^{\frac{d}{2}-1} g \in L^2(\mathbb{R}^d)$

$$\int_{|\xi| \ge \rho} |\xi|^{d-2} |g|^2 d\xi \xrightarrow{\rho \to +\infty} 0$$

$$\leq \int_{|\xi|\geq \rho} |\xi|^{d-2} \sup_{0\leq t<T} |\hat{u}(t,\xi)|^2 \, d\xi \xrightarrow{\rho\to+\infty} 0$$

$$\sup_{0\leq t<T} \int_{|\xi|\geq\rho} |\xi|^{d-2} |\hat{u}(t,\xi)|^2 \, d\xi$$

so $\forall \ \varepsilon_0 > 0 \quad \exists \ \rho \quad s.t \quad \forall \ 0\leq t<T$

$$\int_{|\xi|\geq\rho} |\xi|^{d-2} |\hat{u}(t,\xi)|^2 \, d\xi < \varepsilon_0^2$$

$$= | \chi_{\sqrt{-\Delta}\geq\rho} \ u(t) |^2_{\dot{H}^{\frac{d}{2}-1}} \leq \varepsilon_0^2$$

We know also that $|u(t)|_{\dot{H}^{\frac{d}{2}-1}} \leq M$

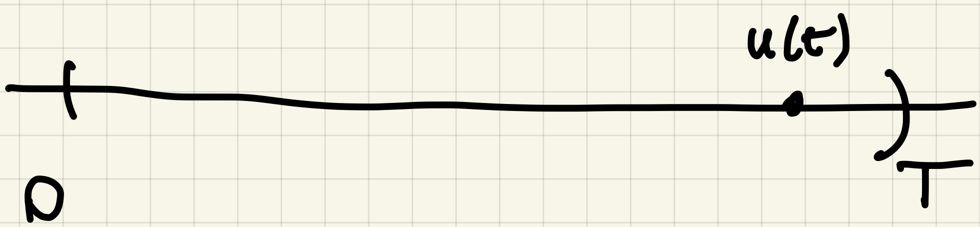
$$|u(t)|_{\dot{H}^{\frac{d}{2}-1}} \leq |u_0|_{\dot{H}^{\frac{d}{2}-1}} + |Q(u,u)|_{L^2((0,t),\dot{H}^{\frac{d}{2}-2})}$$

$$\partial_t u - \Delta u = \underbrace{Q(u,u)}_{f} \qquad t \nearrow T$$

$$\leq |u_0|_{\dot{H}^{\frac{d}{2}-1}} + |Q(u,u)|_{L^2((0,T),\dot{H}^{\frac{d}{2}-2})}$$

$$< M$$

$$\|u(t)\|_{\dot H^{\frac{d}{2}-1}} \le M \qquad \forall\, 0\le t<T$$



If $u(t)$ is the initial value, the solution will live at least a time τ_t $[0,\tau_t]$ $[0,\tau]$

$$\tau_t \le \left(\frac{1}{8\, \rho_t^{\frac12}\, C_d\, \|u(t)\|_{\dot H^{\frac{d}{2}-1}}} \right)^4$$

$$\|\chi_{\sqrt{-\Delta}\ge \rho_t}\, u(t)\|_{\dot H^{\frac{d}{2}-1}} < \frac{1}{8C_d}$$

$$\|u(t)\|_{\dot H^{\frac{d}{2}-1}} \le M$$

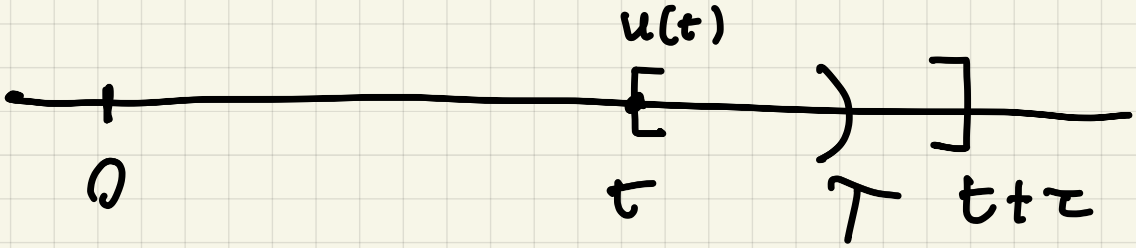
$$\rho_t \le \rho$$

$$\|\chi_{\sqrt{-\Delta}\ge \rho}\, u(t)\|_{\dot H^{\frac{d}{2}-1}} < \frac{1}{8C_d}$$

$$\forall \quad 0 \le t < T$$

So for any $0 \le t < T$ the initial value problem with initial value $u(t)$ has corresponding solution well defined $[0, \tau]$

$$0 < \tau \le \left(\frac{1}{8 \, \rho^{\frac{1}{2}} C_d M} \right)^4$$



if $\quad t > T - \tau \quad \Longrightarrow \quad t + \tau > T$

~~let~~ $\partial_t v(s) - \Delta v = Q(v, v)$

$$v(0) = u(t)$$

$$w(t') = \begin{cases} u(t') & 0 \le t' \le t \\ v(t' - t) & t \le t' \le t + \tau \end{cases}$$

$\Longrightarrow$ w solves

$$w(t) = e^{t\Delta} u_0 + B(w, w)$$

in $[0, t+\tau] \supset [0, T)$

$u(t)$

$w(t)$ by uniqueness are equal.

If u and v are solutions then

$$|u(t) - v(t)|^2_{\dot{H}^{\frac{d}{2}-1}} + \int_0^t |\nabla(u-v)|^2_{\dot{H}^{\frac{d}{2}-1}} dt'$$

$$\leq |u_0 - v_0|_{\dot{H}^{\frac{d}{2}-1}} \, e^{C_d \|(u,v)\|^4_{L^4((0,T), \dot{H}^{\frac{d-1}{2}})}}$$

$$\begin{cases} \partial_t u - \Delta u = Q(u,u) & u(0) = u_0 \\ \partial_t v - \Delta v = Q(v,v) & v(0) = v_0 \end{cases}$$

$w = u - v$ $\qquad u^2 - v^2 = (u-v)(u+v)$

$$\partial_t w - \Delta w = Q(u,u) - Q(v,v)$$

$$\partial_t w - \Delta w = Q(w, u+v)$$

$$\Delta_w(t) = |w(t)|^2_{\dot H^{\frac d2 -1}} + 2\int_0^t | w|^2_{\dot H^{\frac d2}} \, dt'$$

$$= |w_0|^2_{\dot H^{\frac d2-1}} + 2\int_0^t \langle Q(w,u+v), w\rangle_{\dot H^{\frac d2 -1}} \, dt'$$

<u>Claim</u> $|\langle Q(a,b),c\rangle_{\dot H^{\frac d2-1}}| \le C_d |a|_{\dot H^{\frac{d-1}{2}}} |b|_{\dot H^{\frac{d-1}{2}}} |c|_{\dot H^{\frac d2}}$

Pf $|\langle Q(a,b),c\rangle_{\dot H^{\frac d2-1}}| =$

$$= |\langle |\nabla|^{\frac d2-1} Q(a,b), |\nabla|^{\frac d2-1} c\rangle_{L^2}|$$

$$= |\langle |\xi|^{\frac d2-1} \hat Q, |\xi|^{\frac d2-1} \hat c\rangle|$$

$$= |\langle |\xi|^{\frac d2-2} \hat Q, |\xi|^{\frac d2} \hat c\rangle|$$

$$= |\langle |\nabla|^{\frac d2-2} Q, |\nabla|^{\frac d2} c\rangle|$$

$$\le \| |\nabla|^{\frac d2-2} Q(a,b)\|_{L^2} \, \| |\nabla|^{\frac d2} c\|_{L^2}$$

$$= |Q(a,b)|_{\dot H^{\frac d2-2}} \, |c|_{\dot H^{\frac d2}}$$

$$\le C_d \, |a|_{\dot H^{\frac{d-1}{2}}} \, |b|_{\dot H^{\frac{d-1}{2}}} \, |c|_{\dot H^{\frac d2}}$$

$$\Delta_w(t) = |w(t)|^2_{\dot H^{\frac d2 -1}} + 2\int_0^t |\nabla w|^2_{\dot H^{\frac d2 -1}}\, dt'$$

$$= |w_0|^2_{\dot H^{\frac d2 -1}} + 2\int_0^t \langle Q(w, u+v), w\rangle_{\dot H^{\frac d2 -1}}\, dt'$$

$$\leq |w_0|^2_{\dot H^{\frac d2 -1}} + 2C\int_0^t |w|_{\dot H^{\frac{d-1}{2}}} \underbrace{|u+v|_{\dot H^{\frac{d-1}{2}}}}_{\leq N(t')} |w|_{\dot H^{\frac d2}}\, dt'$$

$$|u+v|_{\dot H^{\frac{d-1}{2}}} \leq N(t)$$

$$N(t) = |u(t)|_{\dot H^{\frac{d-1}{2}}} + |v(t)|_{\dot H^{\frac{d-1}{2}}}$$

$$|w|_{\dot H^{\frac{d-1}{2}}} \leq |w|^{\frac12}_{\dot H^{\frac d2 -1}} |w|^{\frac12}_{\dot H^{\frac d2}}$$

by interpolation

$$\frac{d-1}{2} = \frac12\left(\frac d2 - 1\right) + \frac12\, \frac d2 = \frac d2 - \frac12$$

$$\Delta_w(t) = |w(t)|^2_{\dot H^{\frac d2 -1}} + 2\int_0^t |\nabla w|^2_{\dot H^{\frac d2 -1}}\, dt'$$

$$\leq |w_0|^2_{\dot H^{\frac d2 -1}} + 2C\int_0^t |w|_{\dot H^{\frac{d-1}{2}}} \underbrace{|u+v|_{\dot H^{\frac{d-1}{2}}}}_{\leq N(t')} |w|_{\dot H^{\frac d2}}\, dt'$$

$$\leq |w_0|^2_{\dot H^{\frac d2 -1}} + 2C\int_0^t N(t')\, |w|^{\frac12}_{\dot H^{\frac d2 -1}} |w|^{\frac32}_{\dot H^{\frac d2}}\, dt'$$

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

$$ab \le \frac{a^4}{4} + \frac{3}{4}\, b^{\frac{4}{3}}$$

$$2CN\, |w|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}}\, |w|^{\frac{3}{2}}_{\dot H^{\frac{d}{2}}} =$$

$$= \left(2CN\, |w|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}} \left(\frac{3}{4}\right)^{\frac{3}{4}}\right) \left(\frac{4}{3}\, |\nabla w|^{2}_{\dot H^{\frac{d}{2}-1}}\right)^{\frac{3}{4}}$$

$$\le \frac{1}{4}\left(2CN\, |w|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}}\, \frac{3}{4}\right)^4 + \cancel{\frac{3}{4}}\, \cancel{\frac{4}{3}}\, |\nabla w|^{2}_{\dot H^{\frac{d}{2}-1}}$$

$$\Delta_w(t) = |w(t)|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t |\nabla w|^2_{\dot H^{\frac{d}{2}-1}}\, dt'$$

$$\le |w_0|^2_{\dot H^{\frac{d}{2}-1}} + 2C\int_0^t N(t')\, |w|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}}\, |w|^{\frac{3}{2}}_{\dot H^{\frac{d}{2}}}\, dt'$$

$$\le |w_0|^2_{\dot H^{\frac{d}{2}-1}} + C_1\int_0^t |w|^2_{\dot H^{\frac{d}{2}-1}}\, N^4\, dt'$$

$$+ \int_0^t |\nabla w|^2_{\dot H^{\frac{d}{2}-1}}\, dt$$

$$X(t) \doteq |w(t)|^2_{\dot H^{\frac{d}{2}-1}} + \int_0^t |\nabla w|^2_{\dot H^{\frac{d}{2}-1}} dt'$$

$$\leq |w_0|^2_{\dot H^{\frac{d}{2}-1}} + C \int_0^t N^4(t') \, |w(t')|^2_{\dot H^{\frac{d}{2}-1}} dt'$$

$$X(t) \leq |w_0|^2_{\dot H^{\frac{d}{2}-1}} + C \int_0^t N^4(t') \, X(t') \, dt$$

$$\Rightarrow \quad X(t) \leq |w_0|^2_{\dot H^{\frac{d}{2}-1}} \, e^{C \int_0^t N^4(t') dt'}$$

by Gronwall