27 Ottobre

① $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e \iff \boxed{\forall \varepsilon>0\ \exists K_\varepsilon>0 \text{ t.c. } |x|>K_\varepsilon \Rightarrow \left|\left(1+\frac{1}{x}\right)^x - e\right| < \varepsilon}$

Altro limite importante e'

② $\lim_{n \to +\infty} \left(1+\frac{x}{n}\right)^n = e^x$

② significa che

$\cancel{\forall \varepsilon>0\ \exists k_\varepsilon>0 \text{ t.c. } n>k_\varepsilon} \Rightarrow \left|\left(1+\frac{x}{n}\right)^n - e^x\right| < \varepsilon$

$$\left(1+\frac{x}{n}\right)^{\frac{n}{x}x} = \left(\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x$$

Ora dimostro $\lim_{n \to +\infty} \left(1+\frac{x}{n}\right)^{\frac{n}{x}} = e$

$\forall \varepsilon>0\ \exists k_\varepsilon>0 \text{ t.c. } n>k_\varepsilon \Rightarrow \left|\left(1+\frac{x}{n}\right)^{\frac{n}{x}} - e\right| < \varepsilon$

So che se $|y|>K_\varepsilon \Rightarrow \left|\left(1+\frac{1}{y}\right)^y - e\right| < \varepsilon$

Pertanto se $\left|\frac{n}{x}\right| > K_\varepsilon \Rightarrow \left|\left(1+\frac{x}{n}\right)^{\frac{n}{x}} - e\right| < \varepsilon$

$\Updownarrow$

$n > K_\varepsilon |x|$

Se definisco $k_\varepsilon := K_\varepsilon |x|$ so che $n>k_\varepsilon \Rightarrow \left|\left(1+\frac{x}{n}\right)^{\frac{n}{x}} - e\right| < \varepsilon$

Abbiamo dimostrato $\lim_{n \to +\infty} \left(1+\frac{x}{n}\right)^{\frac{n}{x}} = e$

$$\left(1+\frac{x}{n}\right)^n = \left(\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x \xrightarrow{n \to +\infty} e^x$$

ricordare che

$x^a \in C^0((0,+\infty)) \quad \forall a \in \mathbb{R}$

Pertanto se $\lim_{n \to +\infty} x_n = e \Rightarrow \lim_{n \to +\infty} x_n^a = e^a$

③ $$\lim_{y \to 0} \frac{\lg(1+y)}{y} = 1$$

$$y = \frac{1}{x} \qquad y \to 0 \Rightarrow x \to \infty$$

$$x = \frac{1}{y}$$

$$\frac{\lg(1+y)}{y} = \frac{\lg\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = x \lg\left(1+\frac{1}{x}\right)$$

$$= \lg\left(1+\frac{1}{x}\right)^x$$

$$\lim_{y \to 0} \frac{\lg(1+y)}{y} = \lim_{x \to \infty} \lg\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{z \to e} \lg z$$

$$z = \left(1+\frac{1}{x}\right)^x \qquad = \lg e = 1$$

$$\boxed{\lg_a a = 1} \quad a > 0$$

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$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

$y = e^x - 1 \qquad y + 1 = e^x$

$x = \lg(y+1)$

$$y = e^x - 1 \qquad \lim_{x\to 0} y = e^0 - 1 = 1 - 1 = 0$$

$$\frac{e^x - 1}{x} = \frac{y}{\lg(y+1)}$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{y\to 0} \frac{y}{\lg(1+y)} = 1$$

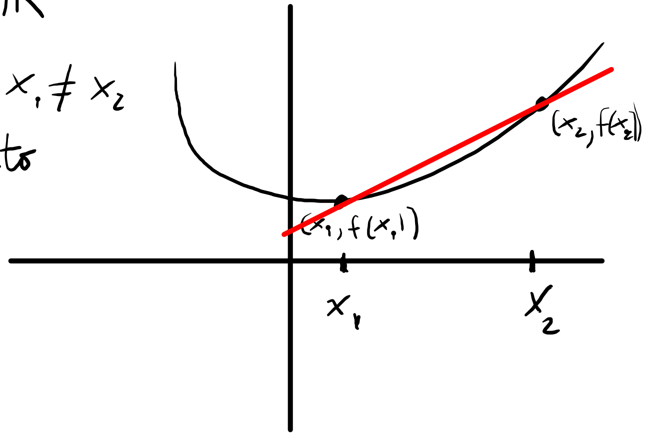
⑤ $$\lim_{x \to 0} \frac{(1+x)^{a} - 1}{x} = a \qquad \forall\, a \in \mathbb{R}$$

$$\frac{0}{x} \qquad\qquad \frac{0}{0}$$

Def Sia $f : I \to \mathbb{R}$

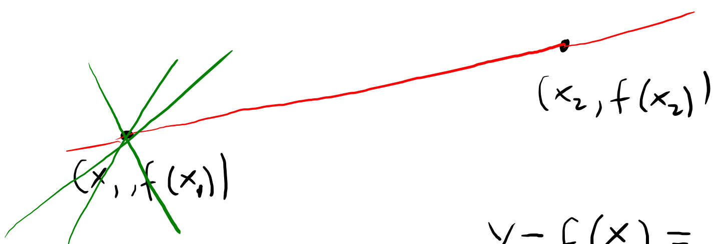
Dati $x_1, x_2 \in I$ $\quad x_1 \neq x_2$

resta definito il rapporto incrementale



$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Risulta che la retta del piano per $(x_1, f(x_1))$ e $(x_2, f(x_2))$ ha coefficente angolare dato da $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$



$$y - f(x_1) = m\,(x - x_1)$$

$$f(x_2) - f(x_1) = m\,(x_2 - x_1)$$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Def $f : I \to \mathbb{R}$ $\qquad x_0 \in \mathring{I}$

Supponiamo che

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \ell \in \mathbb{R}$$

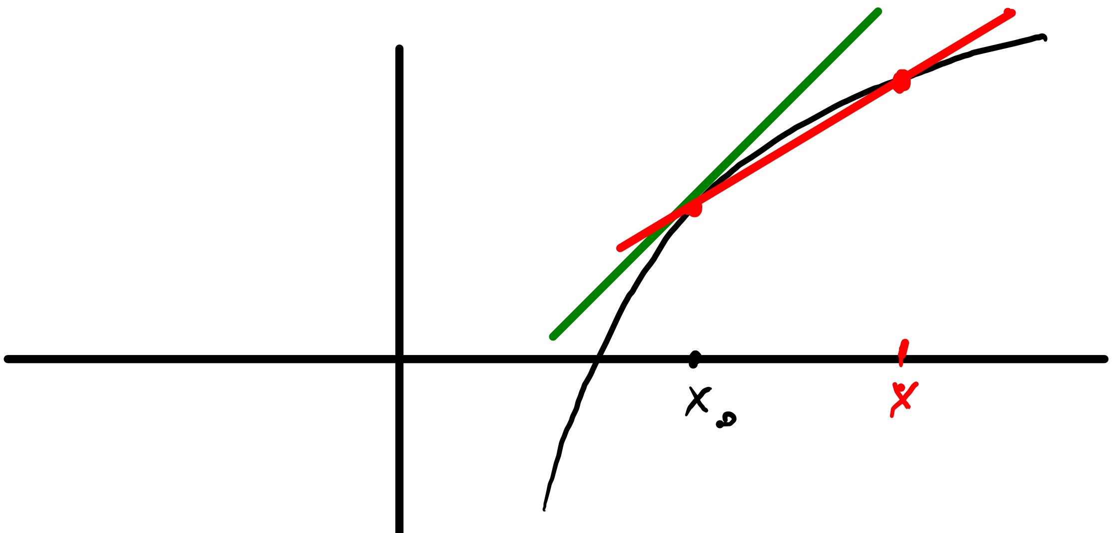
esista e sia finito. Allora si dice che f è derivabile o anche differenziabile nel punto x_0

Il limite ℓ viene chiamato la derivata di f nel punto x_0

$$\ell = f'(x_0) = \frac{df}{dx}(x) = \frac{d}{dx} f(x_0)$$

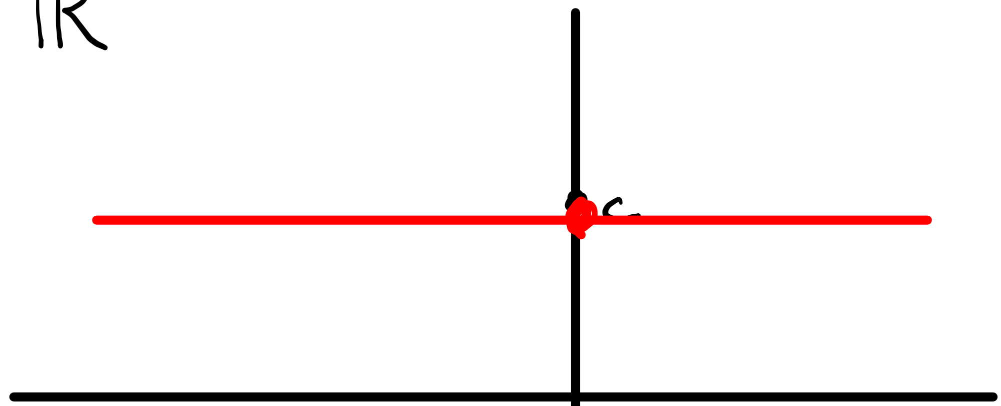
Def Sia $f : I \to \mathbb{R}$, $x_0 \in \mathring{I}$ e supponiamo esista $f'(x_0)$. Allora la retta tangente al grafico $y = f(x)$ nel punto $(x_0, f(x_0))$ è la retta

$$y - f(x_0) = f'(x_0)(x - x_0)$$



$$\frac{f(x) - f(x_0)}{x - x_0}$$

c definito in $\mathbb{R}$



$$\lim_{x \to x_0} \frac{\overbrace{c(x)}^{=c} - \overbrace{c(x_0)}^{=c}}{x - x_0} = \lim_{x \to x_0} \frac{0}{x - x_0} = \lim_{x \to x_0} 0 = 0$$

$$(c)' = 0$$

$$(e^x)'(x_0) = e^{x_0} \qquad \forall x_0 \in \mathbb{R}$$

$$\lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0}$$

$$h = x - x_0$$
$$x = h + x_0$$

$$= \lim_{h \to 0} \frac{e^{x_0+h} - e^{x_0}}{h} = $$

$$= \lim_{h \to 0} \frac{e^{x_0} e^h - e^{x_0}}{h} = \lim_{h \to 0} \underbrace{e^{x_0}}_{\downarrow\, e^{x_0}} \underbrace{\frac{e^h - 1}{h}}_{\downarrow\, 1 \ (h \to 0)} = e^{x_0}$$

dove ho usato $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

$$(e^x)' = e^x$$

$x^a : \mathbb{R}_+ \to \mathbb{R}$ $\qquad a \in \mathbb{R}$

$$(x^a)' = a\, x^{a-1}$$

$$\lim_{h \to 0} \frac{(x+h)^a - x^a}{h} = a\, x^{a-1}$$

$$\frac{(x+h)^a - x^a}{h} = \frac{x^a \left(1 + \frac{h}{x}\right)^a - x^a}{h}$$

$$= \frac{x^a}{x} \; \frac{\left(1 + \frac{h}{x}\right)^a - 1}{\frac{h}{x}}$$

ora vorrei usare $\displaystyle \lim_{y \to 0} \frac{(1+y)^a - 1}{y} = a$

$$\lim_{h \to 0} \frac{(x+h)^a - x^a}{h} = \lim_{h \to 0} x^{a-1} \frac{\left(1 + \frac{h}{x}\right)^a - 1}{\frac{h}{x}}$$

$$y = \frac{h}{x} \qquad = \lim_{y \to 0} x^{a-1} \frac{(1+y)^a - 1}{y}$$

$$= x^{a-1}\, a$$