Lunedì 30 ottobre

<u>Teorema</u> $(\lg x)' = \frac{1}{x} \quad \forall x > 0$

<u>Dim</u> $\lim_{y \to x} \frac{\lg(y) - \lg x}{y - x} = \lim_{h \to 0} \frac{\lg(h+x) - \lg x}{h}$

$y = h + x$

Ora vogliamo utilizzare $\lim_{h \to 0} \frac{\lg(1+h)}{h} = 1$

$= \lim_{h \to 0} \frac{\lg\left(\left(1+\frac{h}{x}\right)x\right) - \lg x}{h} = \lim_{h \to 0} \frac{\lg\left(1+\frac{h}{x}\right) + \cancel{\lg x} - \cancel{\lg x}}{h}$

$= \lim_{h \to 0} \frac{\lg\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \frac{1}{x} \qquad u = \frac{h}{x}$

$= \lim_{u \to 0} \frac{\lg(1+u)}{u} \frac{1}{x} = \frac{1}{x}$

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Teor (continuità nei punti dove esiste derivata)

Sia $f: I \to \mathbb{R}$, $x_0 \in \mathring{I}$. Supponiamo esista $f'(x_0)$.

Allora f è continua in x_0.

Dim La continuità di f in x_0 significa

$$\lim_{x \to x_0} f(x) = f(x_0) \iff \boxed{\lim_{x \to x_0} \left(f(x) - f(x_0) \right) = 0}$$

$$\lim_{x \to x_0} \left(f(x) - f(x_0) \right) = \lim_{x \to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\to f'(x_0)} \underbrace{(x - x_0)}_{\xrightarrow{x \to x_0} 0}$$

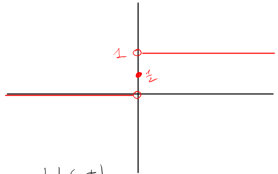
$$= f'(x_0) \cdot 0 = 0$$

$\square$

Qualche esempio di discontinuità

1) $$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \\ \frac{1}{2} & x = 0 \end{cases}$$

Heaviside

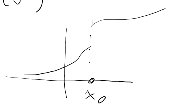


E' discontinua in 0

$$\lim_{x \to 0^+} H(x) = \lim_{\substack{x \to 0 \\ x > 0}} 1 = 1 = H(0^+)$$

$$\lim_{x \to 0^-} H(x) = \lim_{\substack{x \to 0 \\ x < 0}} 0 = 0 = H(0^-)$$

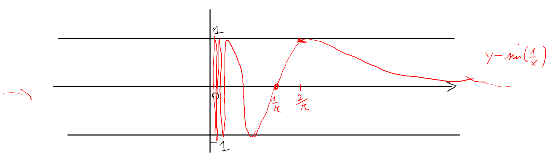
$$1 = H(0^+) \neq H(0^-) = 0$$



2) $$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\exists ?$ $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$? no !

$$\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right) = \lim_{y \to +\infty} \sin(y) \quad \text{non esiste} \qquad y = \frac{1}{x}$$



Ricordiamo $\lim_{y \to 0} \frac{\sin y}{y} = 1 \Rightarrow \frac{\sin y}{y} = 1 + o(1)$

$$\sin y = y\,(1 + o(1)) \Rightarrow \sin\left(\frac{1}{x}\right) = \frac{1}{x}\,(1 + o(1))$$

per $|x| >> 1$

Teorema (Algebra delle derivate)

Siano $f, g : I \to \mathbb{R}$, $x_0 \in \mathring{I}$ ed esistono $f'(x_0)$, $g'(x_0)$.

1) $$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

2) $$(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$$

3) $$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{g^2(x_0)}$$

$g(x) \neq 0 \quad \forall x \in I$

Dim di 1 e 2

1) $$\lim_{x \to x_0} \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \left[\underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\xrightarrow{x \to x_0} f'(x_0)} + \underbrace{\frac{g(x) - g(x_0)}{x - x_0}}_{\xrightarrow{x \to x_0} g'(x_0)} \right]$$

$$= f'(x_0) + g'(x_0)$$

2) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

$$\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} =$$

$$= \frac{\left(f(x)g(x) - f(x)g(x_0)\right) + \left(f(x)g(x_0) - f(x_0)g(x_0)\right)}{x - x_0}$$

$$= f(x)\,\frac{g(x) - g(x_0)}{x - x_0} + \frac{f(x) - f(x_0)}{x - x_0}\,g(x_0)$$

$$\xrightarrow{x \to x_0} f(x_0)\,g'(x_0) + f'(x_0)\,g(x_0)$$

$$f(x_0)g'(x_0) + f'(x_0)g(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

Esempi già sapevamo $(x^2)' = 2x$ $\qquad (x)' = 1\, x^0 = 1$

$$(x\, x)' = x \underbrace{(x)'}_{=1} + \underbrace{(x)'}_{=1} x = x + x = 2x$$

$$(x^n)' = n\, x^{n-1} \qquad \forall\, n \geq 2 \quad n \in \mathbb{N}$$

Per $n = 2$ è vero

Sia vero per $n \Longrightarrow$ è vero per $n+1$

$$(x^{n+1})' = (x^n\, x)' = (x^n)' x + x^n (x)' = n\, x^{n-1} x + x^n = n\, x^n + x^n = (n+1) x^n$$

$$(\tan x)' = \left(\frac{\text{sen}\, x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x\,(\cos x)'}{\cos^2(x)} =$$

$$= \frac{\cos x \, \cos x + \sin x\,(+\,\text{sen}\, x)}{\cos^2(x)} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2(x)} = \frac{1}{\cos^2(x)} =$$

$$= 1 + \tan^2(x)$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\,\text{sen}\, x$$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \left[\cos x \underbrace{\frac{\sin(h)}{h}}_{\to 1} + \text{sen}\, x \underbrace{\frac{\cos(h) - 1}{h}}_{\to 0} \right] = \cos x$$

$$\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0$$

$$= \lim_{h \to 0} \frac{1 - \cos(h)}{h} \cdot \frac{1 + \cos(h)}{1 + \cos(h)} = \lim_{h \to 0} \frac{\overbrace{1 - \cos^2(h)}^{\sin^2(h)}}{h} \cdot \frac{1}{1 + \cos(h)}$$

$$= \lim_{h \to 0} \underbrace{\frac{\text{sen}^2(h)}{h^2}}_{\to 1} \cdot \underbrace{h}_{\to 0} \cdot \underbrace{\frac{1}{1 + \cos(h)}}_{\to \frac{1}{2}} = 0$$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$$(f^2)' = 2 f f'$$

Osservazione: Non deve essere $\cancel{(\cos x)' = \sin x}$

$$1 = \sin^2(x) + \cos^2(x)$$

$$0 = (1)' = (\sin^2(x))' + (\cos^2(x))' = 2 \sin(x) \sin'(x) + 2\cos(x)\cos'(x)$$

$$= 2 \sin x \cos x + 2 \cos x (-\sin x)$$

$$= 0$$

$$(f^2)' = (f f)' = f' f + f f' = 2 f f'$$

$$\lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \left[-\sin(x) \frac{\sin(h)}{h} + \cos(x) \frac{\cos(h) - 1}{h} \right] = -\sin x$$

$\frac{\sin(h)}{h} \to 1$, $\frac{\cos(h)-1}{h} \xrightarrow{h \to 0} 0$

$$(x)' = (x^1) = 1 \, x^{1-1} = 1 \, x^0 = 1$$

$$\lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1 = 1$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$(\sinh(x))' = \cosh(x)$$

$$(\mathrm{ch}\, x)' = \mathrm{sh}(x)$$

$$(e^x)' = e^x$$

$$(e^{-x})' = \left(\frac{1}{e^x}\right)' = \frac{\overset{0}{\overset{\shortparallel}{(1)'}}\, e^x - 1\,(e^x)'}{e^{2x}} =$$

$$= -\frac{(e^x)'}{e^{2x}} = -\frac{e^x}{e^{2x}} = -\frac{1}{e^x}$$

$$= -e^{-x}$$

$$(cf)' = c f'$$

$$(cf)' = \underset{\underset{0}{\shortparallel}}{c'}\, f + c f' = c f'$$

$$\frac{d}{dx} cf = c \frac{d}{dx} f$$

$$\frac{d}{dx}(f+g) = \frac{d}{dx} f + \frac{d}{dx} g$$