8 November

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

$$\begin{cases} i\partial_t \hat{u}(t,\xi) - |\xi|^2 \hat{u}(t,\xi) = 0 \\ \hat{u}|_{t=0} = \hat{u}_0 \end{cases}$$

$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{u}_0$

$$\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{u}_0(\xi)$$

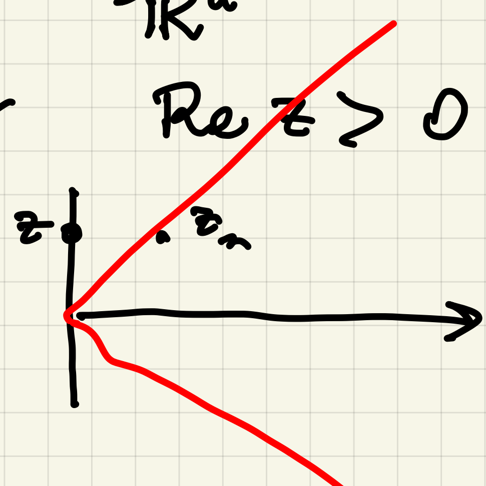
$$e^{-it|\xi|^2} = \hat{G}(t,\xi)$$

$$G(t,x) = (2ti)^{-\frac{d}{2}}\, e^{i\frac{|x|^2}{4t}}$$

$$e^{-\frac{|x|^2}{4t}}$$

$$e^{-z_n\frac{|\xi|^2}{2}} = (2\pi z_n)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x}\, e^{-\frac{|x|^2}{2z_n}}\,dx$$

holds for only $\operatorname{Re} z > 0$



z

$$e^{-z\frac{|\xi|^2}{2}} = (2\pi z)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} \, e^{-\frac{|x|^2}{2z}} \, dx$$

$z = 2it$

$$e^{-it|\xi|^2} = (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} \, e^{i\frac{|x|^2}{4t}} \, dx$$

$$= (2\pi)^{-\frac{d}{2}} \, (2it)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} \, \underbrace{e^{i\frac{|x|^2}{4t}}}_{G(t,x)} \, dx$$

$$\hat{u}(t,\xi) = e^{-it|\xi|^2} \, \hat{u}_0(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \, \widehat{G * u_0}(\xi)$$

$u(t,x)$

$$\widehat{f * g} = (2\pi)^{\frac{d}{2}} \, \hat{f} \, \hat{g}$$

$$u(t,x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int (2it)^{-\frac{d}{2}} \, e^{i\frac{|x-y|^2}{4t}} \, u_0(y) \, dy$$

$$u(t,x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int e^{i\frac{|x-y|^2}{4t}} \, u_0(y) \, dy$$

For heat

$$u(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int e^{-\frac{|x-y|^2}{4t}} u_0(y)\,dy$$

$$t \to it$$

Set $$e^{it\Delta} u_0 := \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int e^{i\frac{|x-y|^2}{4t}} u_0(y)\,dy$$

$$|e^{it\Delta} f|_{L^2} = |\cancel{e^{-it|\xi|^2}} \hat{f}|_{L^2} = |f|_{L^2}$$

$$|e^{it\Delta} f|_{L^\infty(\mathbb{R}^d)} =$$

$$= \left\| \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} f(y)\,dy \right\|_{L^\infty_x}$$

$$\le \frac{1}{(4\pi t)^{\frac{d}{2}}} \left\| \int_{\mathbb{R}^d} |f(y)|\,dy \right\|_{L^\infty_x}$$

$$|e^{it\Delta} f|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(4\pi t)^{\frac{d}{2}}} |f|_{L^1(\mathbb{R}^d)}$$

$$|e^{it\Delta} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)| \le \frac{1}{(4\pi t)^{\frac{d}{2}}}$$

Thm (Riesz – Thorin) Let T a linear map from $L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d) \to L^{q_0}(\mathbb{R}^d) \cap L^{q_1}(\mathbb{R}^d)$

with

$$|T : L^{p_j} \to L^{q_j}| \leq M_j \qquad j = 0, 1$$

Let $t \in (0,1)$ and

$$\frac{1}{p_t} = \frac{(1-t)}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{(1-t)}{q_0} + \frac{t}{q_1}$$

Then

$$|T : L^{p_t} \to L^{q_t}| \leq M_0^{1-t} M_1^t$$

$$|e^{it\Delta} : L^2 \to L^2| = 1$$

$$|e^{it\Delta} : L^1 \to L^\infty| \leq \frac{1}{(4\pi t)^{\frac{d}{2}}}$$

$1 < p < 2$

$$\frac{1}{p_s} = \frac{1}{p} = \frac{1-s}{2} + \frac{s}{1} = \frac{1}{2} s + \frac{1}{2}$$

$$s = 2\left(\frac{1}{p} - \frac{1}{2}\right)$$

$$\frac{1}{q_s} = \frac{1-s}{2} = \frac{1}{2} - \frac{s}{2} = \frac{1}{2} - \left(\frac{1}{p} - \frac{1}{2}\right) =$$

$$= 1 - \frac{1}{p} = \frac{1}{p'}$$

$$|e^{it\Delta} : L^p \to L^{p'}| \leq \left(\frac{1}{(4\pi t)^{\frac{d}{2}}}\right)^s =$$

$$= \frac{1}{(4\pi t)^{\frac{d}{\cancel{2}} \cancel{2}\left(\frac{1}{p} - \frac{1}{2}\right)}}$$

$$= \frac{1}{(4\pi t)^{d\left(\frac{1}{p} - \frac{1}{2}\right)}} \qquad \frac{1}{p} = 1 - \frac{1}{p'}$$

$$= \frac{1}{(4\pi t)^{d\left(\frac{1}{2} - \frac{1}{p'}\right)}}$$

Strichartz estimates

(q, r) is an admissible pair if

$$\frac{2}{q} + \frac{\cancel{2}\, \frac{d}{\cancel{2}}}{r} = \frac{d}{2}$$

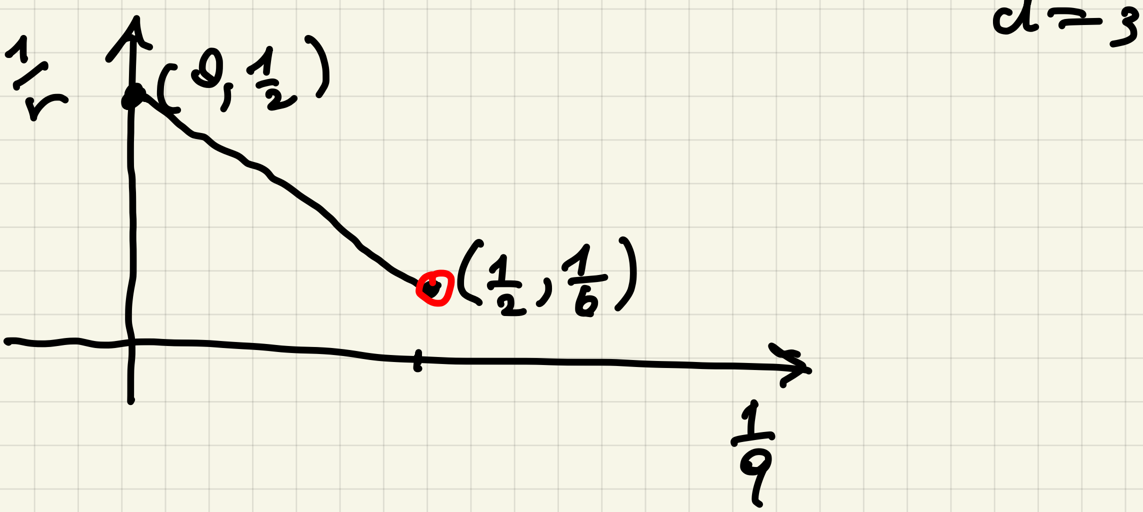
$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \text{is true}$$

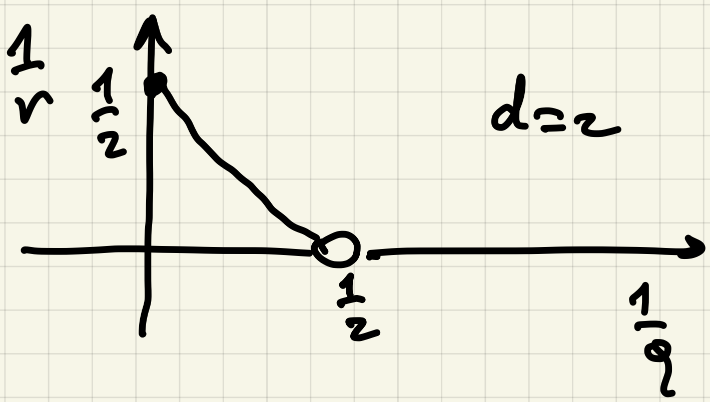
$r, q \geq 2$

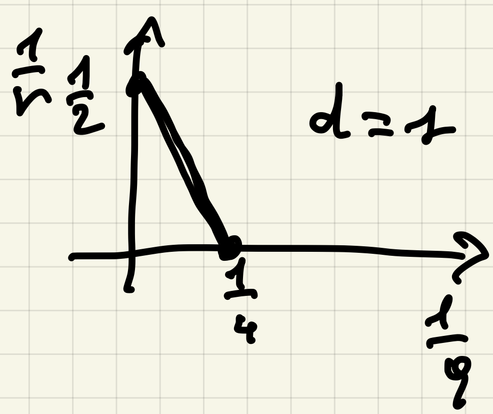
In $d=2$ we exclud case

$(q,r) = (2,\infty)$

For $d \geq 3$ is very important the pair $\left(2, \frac{d}{\frac{d}{2}-1}\right)$ $(2,6)$







Thm

1) If $u_0 \in L^2(\mathbb{R}^d)$ then

$$e^{it\Delta} u_0 \in L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cap C^0(\mathbb{R}, L^2(\mathbb{R}^d))$$

$\forall$ (q,r) admissible and $\exists$ C_{qr}

$$\| e^{i\Delta t} u_0 \|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C_{qr} \| u_0 \|_{L^2}$$

2) I $t_0 \in \bar{I}$ (γ, ρ)

$f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))$ then

$$Tf = \int_{t_0}^{t} e^{i\Delta(t-s)} f(s)\, ds$$

belongs to

$$L^q(I, L^r(\mathbb{R}^d)) \cap C^0(\bar{I}, L^2(\mathbb{R}^d))$$

$\forall$ (q,r)

and

$$\| Tf \|_{L^q(I, L^r(\mathbb{R}^d))} \leq C \| f \|_{L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))}$$

$$\underbrace{\left| t^{-\frac{d}{2}} \int e^{i\frac{|x-y|^2}{4t}} f_\lambda(y)\, dy \right|}_{u_\lambda}{}_{L^a_t(\mathbb{R}, L^q_x(\mathbb{R}^d))} \leq C \, | f_\lambda |_{L^2_x}$$

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_0 = f \end{cases} \qquad f_\lambda(x) = \lambda^{\frac{d}{2}} f(\lambda x)$$

$$u(t, x)$$

$$u_\lambda(t,x) = \lambda^{\frac{d}{2}} \, u(\lambda^2 t, \lambda x)$$

$$\lambda^{\frac{d}{2}} \quad | \; u(\lambda^2 t, \lambda x) \; |_{L^a(\mathbb{R}, L^b(\mathbb{R}^d))}$$

$$= \lambda^{\frac{d}{2}} \quad \lambda^{-\frac{d}{b}} \quad \lambda^{-\frac{2}{a}} \qquad |u|_{L^a L^b} \leq C \, |f|$$

$$\frac{d}{2} = \frac{d}{b} + \frac{2}{a} \qquad (a, b)$$

$$\langle f, g \rangle = \mathrm{Re} \int f \bar{g}$$

$$| \; e^{it\Delta} f \; |_{L^p(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C \; | f |_{L^2(\mathbb{R}^d)}$$

$$L^2(\mathbb{R}^d) \xrightarrow{e^{it\Delta}} L^p(\mathbb{R}, L^r(\mathbb{R}^d))$$

$$L^{p'}(\mathbb{R}, L^{r'}(\mathbb{R}^d)) \longrightarrow L^2(\mathbb{R}^d)$$

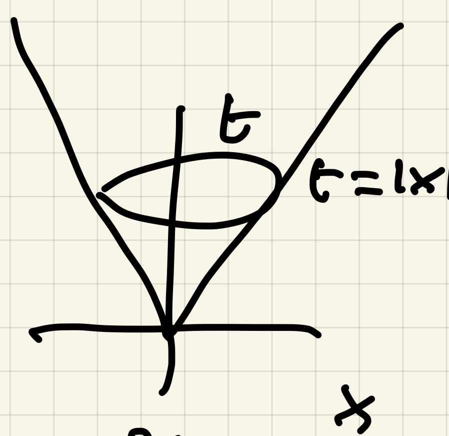
$$\int_{\mathbb{R}} \langle e^{it\Delta} f, G(t) \rangle \, dt =$$

$$= \int_{\mathbb{R}} \langle f, e^{-it\Delta} G(t) \rangle \, dt$$

$$= \left\langle f, \int_{\mathbb{R}} e^{-it\Delta} G(t) \, dt \right\rangle_{L^2(\mathbb{R}^d)}$$

$$\left| \int_{\mathbb{R}} e^{-it\Delta} G(t) \, dt \right|_{L^2_x} \leq$$

$$\leq C \, | G |_{L^{p'}_t L^{r'}_x}$$



$$\tau = |\xi|^2$$

$$\left| \left\langle \int_{\mathbb{R}} e^{-it\Delta} G(t) \, dt, \int_{\mathbb{R}} e^{-is\Delta} F(s) \, ds \right\rangle \right| \leq$$

$$\leq C^2 \; |G|_{L^{p'} L^{r'}} \; |F|_{L^{p'} L^{r'}}$$

$$\int_{t>s} dt\, ds \; \langle e^{-it\Delta} G(t), \; e^{-is\Delta} F(s) \rangle$$

$$= \int_{t>s} dt\, ds \; \langle e^{-i(t-s)\Delta} G(t), \; F(s) \rangle$$

$$|\langle e^{-i(t-s)\Delta} G, F \rangle| \leq |G(t)|_{L^2_x} \; |F(s)|_{L^2_x}$$

$$\leq |e^{-i(t-s)\Delta} G(t)|_{L^r} \; |F(s)|_{L^{r'}}$$

$$\leq |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} \; |G(t)|_{L^{r'}} \; |F(s)|_{L^{r'}}$$

(p', r')

$$\int_{\cancel{t>s}} dt\, ds \; |\langle e^{-it\Delta} G(t), \; e^{-is\Delta} F(s) \rangle|$$

$$\leq \int dt\, ds \; |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} \; |G(t)|_{L^{r'}_x} \; |F(s)|_{L^{r'}_x}$$

$$= \int dt \; |G(t)|_{L^{r'}_x} \; \int ds \; |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} \; |F(s)|_{L^{r'}_x}$$

$$\leq \| G \|_{L^{p'}_t L^{r'}_x} \left(\left\| \int ds\, |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} \| F(s) \|_{L^{r'}_x} \right\|_{L^{p}_t} \right)$$

?

$$\leq \| F \|_{L^{p'}_t L^{r'}_x}$$

$$|t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)} * : L^{p'}_t(\mathbb{R}) \to L^{p}_t(\mathbb{R})$$

$$\frac{1}{p'} = \frac{1}{p} + \left(1 - d\left(\frac{1}{2}-\frac{1}{r}\right)\right)$$

$$\cancel{1} - \frac{1}{p} = \frac{1}{p} + \cancel{1} - \frac{d}{2} + \frac{d}{r}$$

$$\boxed{\frac{d}{2} = \frac{2}{p} + \frac{d}{r}}$$