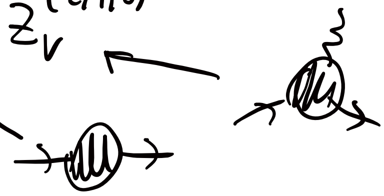
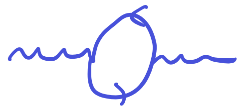
β- function

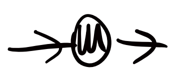
$$g_B = g_r \, \mu^{2-\omega} \, Z_A^{-1/2} \, Z_\psi^{-1} \, Z_V^{(2,1,0)}$$

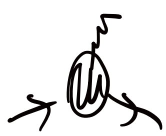


cost. necessaria per cancellare l'∞ in 

Per trovare la relaz. tra g_B e g_r dobbiamo calcolare 3 ampiezze 1PI (In QED, grazie a id. Ward, $Z_V^{(2,1)} = Z_\psi$ $\Rightarrow$ bastava calcolare , che a 1 loop è )

$Z_A \longleftrightarrow$  (4 diag. a 1-loop)

$Z_\psi \longleftrightarrow$  (1 diag. a 1-loop)

$Z_V^{(2,1,0)} \longleftrightarrow$  (2 diag. a 1-loop)

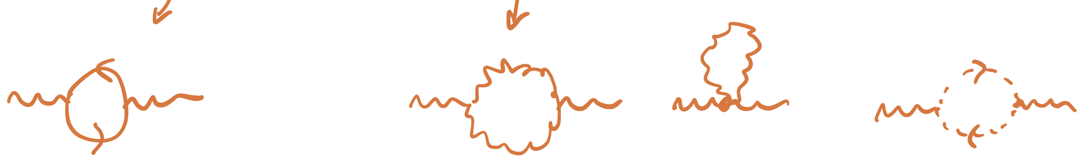
n° di FLAVOURS

Prendiamo una teoria con N_f fermioni di Dirac nella rapp. R di G $\left(i \bar\psi^I \not{D} \psi^I = i \bar\psi^I (\not\partial + i A_\mu^a \gamma^\mu t_R^a) \psi^I \right)$

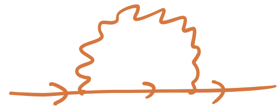
simmetria globale ("di flavor") è $SU(N_f)$

$Z = 1 + \delta Z$ (Vedi sotto per calcolo δZ)

$$\delta Z_A = -\frac{g_r^2}{16\pi^2} \left(\frac{4}{3} N_f \, c(R) - \frac{5}{3} c_2(G) \right) \frac{1}{2-\omega}$$

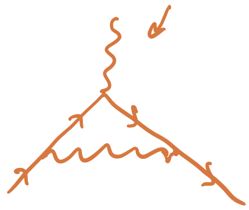


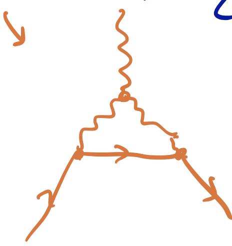
$$\delta Z_\psi = -\frac{g_r^2}{16\pi^2}\, C_2(R)\, \frac{1}{2-\omega} \rightarrow$$



$$\delta Z_V^{(2|1,0)} = -\frac{g_r^2}{16\pi^2}\left(C_2(R) + C_2(G) \right) \frac{1}{2-\omega}$$

$\delta Z_V^{(2|1,0)} \neq \delta Z_\psi$ contrariam. alla QED





$$g_B = g_r \mu^{2-\omega} \underbrace{\left(1 - \frac{1}{2}\delta Z_A\right)}_{\cong Z_A^{-1/2}} \underbrace{\left(1 - \delta Z_\psi\right)}_{\cong Z_\psi^{-1}} \underbrace{\left(1 + \delta Z_V^{(2|1,0)}\right)}_{Z_V^{(2|1,0)}}$$

conto perturbativo

$$= g_r \mu^{2-\omega}\left(1 - \frac{g_r^2}{16\pi^2}\left\{ -\frac{2}{3} N_f\, c(R) + \frac{5}{6} C_2(G) - \cancel{C_2(R)} + \cancel{C_2(R)} + C_2(G) \right\} \frac{1}{2-\omega}\right)$$

$$= g_r \mu^{2-\omega}\left(1 - \frac{g_r}{16\pi^2} \underbrace{\left\{ \frac{11}{6} C_2(G) - \frac{2}{3} N_f\, c(R) \right\}}_{\equiv \Delta} \frac{1}{2-\omega}\right)$$

$$\epsilon \equiv 2-\omega$$

non dip. da μ

$$g_B = g_r(\mu)\, \mu^\epsilon \left(1 - \frac{g_r(\mu)^2}{16\pi^2} \frac{\Delta}{\epsilon}\right) \qquad (0)$$

Vogliamo usare questa relazione per calcolare la dipendenza di g_r da μ. Si sceglie μ della scala a cui si vuole fare conto (o esperimento) affinchè i $\log(P^2/\mu^2)$ siano $O(1)$; $g_r(\mu)$ corrispondente mi dice se siamo in regime perturbativo o no (cioè $g_r(\mu) \ll 1$ o $\gg 1$?)

Prendiamo $\mu \frac{d}{d\mu}(0)$

$$0 = \mu \frac{d}{d\mu} g_r \cancel{\mu^\epsilon} \left(1 - \frac{g_r^2}{16\pi^2}\frac{\Delta}{\epsilon}\right) + \epsilon g_r \cancel{\mu^\epsilon}\left(1 - \frac{g_r^2}{16\pi^2}\frac{\Delta}{\epsilon}\right)$$
$$- g_r \cancel{\mu^\epsilon} \frac{g_r}{8\pi^2}\frac{\Delta}{\epsilon}\mu\frac{dg_r}{d\mu} + \text{higher order in } g_r$$

g_r e $\mu\frac{dg_r}{d\mu}$ sono finiti per $\epsilon \to 0$

$$= \epsilon g_r + \underline{\mu \frac{dg_r}{d\mu}} - \frac{g_r^3}{16\pi^2}\Delta - \frac{3 g_r^2}{16\pi^2}\frac{\Delta}{\epsilon}\underline{\mu\frac{dg_r}{d\mu}} + \dots$$

A ordine zero otteniamo $\mu\frac{dg_r}{d\mu} = -\epsilon g_r + \underbrace{\beta(g_r)}_{\text{subleading}}$

($\epsilon \to \mu\frac{dg_r}{d\mu}$ per $\epsilon \to 0$)

$$\Rightarrow \quad 0 = \cancel{\epsilon g_r} - \cancel{\epsilon g_r} + \beta(g_r) - \frac{g_r^3}{16\pi^2}\Delta + \frac{3g_r^3}{16\pi^2}\Delta + \text{higher order in } g_r$$

$$\Rightarrow \quad \beta(g_r) = -\frac{2g_r^3\Delta}{16\pi^2} = -\frac{g_r^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}N_f\, c(R)\right)$$

(#)

Prendiamo $G = SU(N)$ e $R = N \Rightarrow \begin{cases} C_2(G) = N \\ c(R) = 1/2 \end{cases}$

$$\beta = -\frac{g_r^3}{16\pi^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right)$$

Per QCD $\quad N = 3 \quad N_f = 6 \quad (u, d, s, c, t, b)$

$$\beta_{QCD} = -\frac{7}{16\pi^2}g_r^3 < 0 \quad \Rightarrow \quad \text{ASINTOTICAM. LIBERA in UV}$$

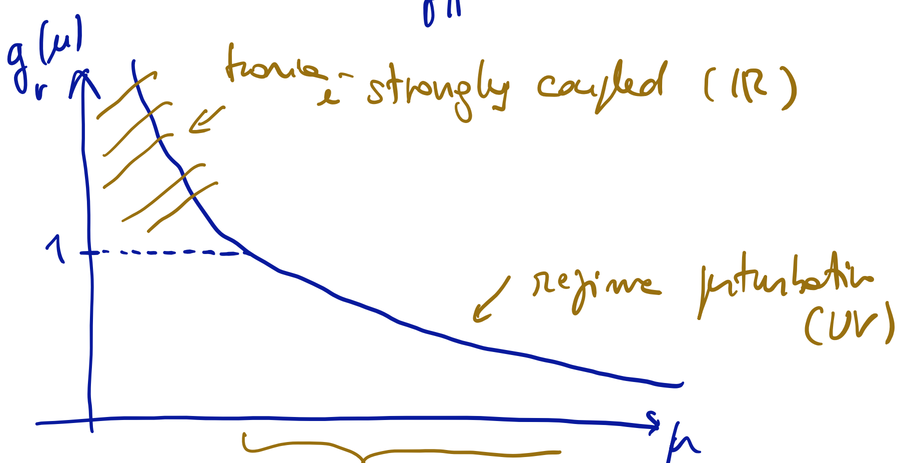
Per YM $\quad N_f = 0$

$$\beta_{YM} = -\frac{11}{48} C_2(G)\, g_r^3 < 0 \Rightarrow \text{ASINTOTICAM. LIBERA in UV}$$

(QED ha $\beta_{QED} > 0 \Rightarrow$ AS. LIB. in IR)

Ricordiamo che la relazione (#) può essere integrata (*)

$$g_r^2(\mu) = \frac{g_r^2(\mu_0)}{1 + \frac{g_r^2(\mu_0)}{8\pi^2}\left(\frac{11}{3} C_2(G) - \frac{4}{3} N_f\, C(R)\right) \ln \mu/\mu_0}$$



qui possiamo usare la teoria delle perturbazioni

(*) Si può integrare $\beta(g) = \mu \frac{dg}{d\mu}$:

$$\int_{g(\mu_0)}^{g(\mu)} \frac{dg}{\beta(g)} = \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} = \ln(\mu/\mu_0)$$

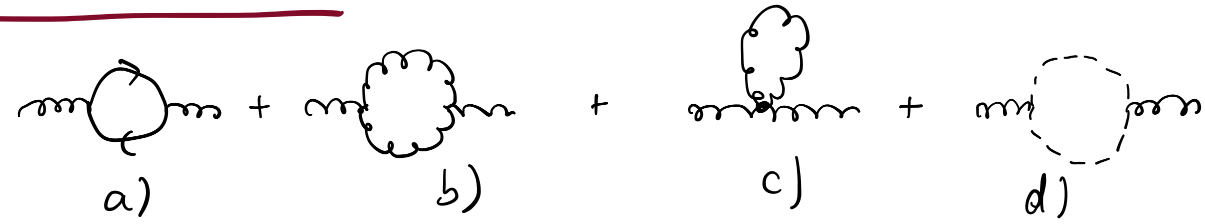
In regime perturbativo $\beta(g) = \beta_0 g^3$ $\qquad \beta_0 = -\frac{1}{16\pi^2}\left(\frac{11}{3} C_2(G) - \frac{4}{3} N_f\, C(R)\right)$ (#)

$$\frac{1}{\beta_0}\int_{g(\mu_0)}^{g(\mu)} \frac{dg}{g^3} = \frac{1}{\beta_0}\left[-\frac{1}{2g^2}\right]_{g(\mu_0)}^{g(\mu)} = \frac{1}{2\beta_0 g^2(\mu_0)} - \frac{1}{2\beta_0 g^2(\mu)}$$

$$\Rightarrow \frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \beta_0 \ln\left(\frac{\mu_0^2}{\mu^2}\right)$$

Calcolo dei δZ

$i\Pi_{\mu\nu}^{cb} =$



$$\equiv i\,(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2)\,\delta^{ab}$$

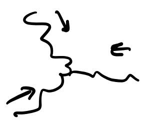
a) come in QED con in più il fattore

$$(t^a_R)_{ij}\,(t^b_R)_{ji} = tr_R\,(t^a_R t^b_R) = c(R)$$

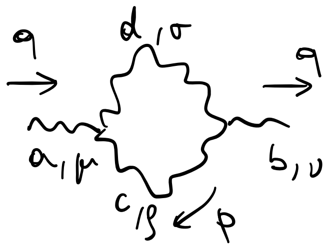
quando ci sono N_f fermioni, va moltiplicato per N_F.

$$\Rightarrow \Pi_{1/(2-\omega)} \supset -\frac{g^2 N_F\, c(R)}{12\pi^2(2-\omega)}$$

b) Ricordiamo



$$= -g f^{a_1a_2a_3}\{(k_1-k_2)_{\mu_3}\eta_{\mu_1\mu_2} + \text{cyclic}\}$$



$$= \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\,\frac{-i}{p^2}\,\frac{-i}{(p+q)^2}\,g^2\,\underbrace{f^{acd}f^{bcd}}_{=Tr(t^a_{Ad}t^b_{Ad}) = c(G)\delta^{ab} = c_2(G)\delta^{ab}}$$

$$\cdot\left\{(q-p)_\sigma\eta_{\rho\mu} + (p-(-q-p))_\mu\eta_{\rho\sigma} + ((-q-p)-q)_\rho\,\eta_{\mu\sigma}\right\}$$

$$\cdot\left\{(-q+p)^\sigma\eta_\nu^{\ \rho} + (-p-(q+p))_\nu\,\eta^{\sigma\rho} + (q+p-(-q))^\rho\eta_\nu^{\ \sigma}\right\}$$

- Risolviamo denominatori

$$\frac{1}{p^2}\cdot\frac{1}{(p+q)^2} = \int_0^1 dx\,\frac{1}{((1-x)p^2 + x(p+q)^2)^2} = \int_0^1 dx\,\frac{1}{(k^2-\Delta^2)^2}$$

con $k = p + xq$ e $\Delta = -x(1-x)q^2$

- Cambio variab. integr. $\int d^dp \to \int d^dk$
- Riscriviamo numeratore in termini di k, q e x.
- Inoltre eliminiamo termini dispari e sostituiamo

$$k^\mu k^\nu \to \frac{k^2}{d}\eta^{\mu\nu}$$

[$\int d^dk\, f(k^2)\, k^\mu k^\nu = \gamma\, \eta^{\mu\nu}$ (by sym.) $\Rightarrow \int d^dk\, f(k^2)k^2 = d\cdot\gamma \Rightarrow$

$\Rightarrow \gamma = \int d^dk\, f(k^2)\,\frac{k^2}{d}$.]

(See Peskin-Schröder)

Facendo qto, troviamo

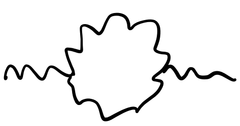
$$= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\, \frac{1}{(k^2-\Delta)^2}\cdot\Big\{ -\eta_{\mu\nu}k^2\cdot 6\left(1-\frac{1}{d}\right) - \eta_{\mu\nu}q^2\left[(2-x)^2+(1-x)^2\right]$$
$$+\, q_\mu q_\nu\left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right]\Big\}$$

- Now we Wick rotate, $k_0 = ik_4$, and use

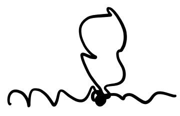
$$\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2-\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2}$$

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{k^2}{(k^2-\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\cdot\frac{d}{2}\cdot\frac{\Gamma(n-d/2-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2-1}$$

Mettendo tutto insieme:


$$= \frac{ig^2}{(4\pi)^{d/2}}\, C_2(G)\,\delta^{ab}\int_0^1 dx\,\frac{1}{\Delta^{2-d/2}}\,\cdot$$

$$\cdot\Big\{\Gamma\left(1-\frac{d}{2}\right)\eta^{\mu\nu}q^2\left[\frac{3}{2}(d-1)\,x(1-x)\right] + \Gamma\left(2-\frac{d}{2}\right)\eta^{\mu\nu}q^2\left[\frac{1}{2}(2-x)^2+\frac{1}{2}(1+x)^2\right]$$
$$-\,\Gamma\left(2-\frac{d}{2}\right)q^\mu q^\nu\left[\left(1-\frac{d}{2}\right)(1-2x)^2 + (1+x)(2-x)\right]\Big\}$$

c)  $= -g^2 c_2(G)\delta^{ab}\int \frac{d^dp}{(2\pi)^d}\ \frac{1}{p^2}\ \eta^{\mu\nu}(d-1) =$

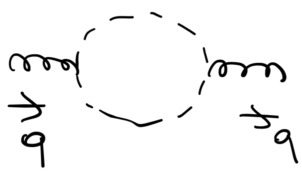
analogous steps as before

$$= \frac{ig^2}{(4\pi)^{d/2}}\ c_2(G)\,\delta^{ab}\int_0^1 dx\ \frac{\eta^{\mu\nu}q^2}{\Delta^{2-d/2}}\ \cdot$$

$$\cdot\left\{-\Gamma\left(1-\frac{d}{2}\right)\frac{1}{2}d(d-1)\,x(1-x) - \Gamma\left(2-\frac{d}{2}\right)(d-1)(1-x)^2\right\}$$

polo a $d=2$
($\rightsquigarrow$ quadratic div. in $d=4$)

polo a $d=4$

d)  $= (-1)\int \frac{d^dp}{(2\pi)^d}\ \frac{i}{p^2}\ \frac{i}{(p+q)^2}\, g^2 f^{dac} f^{cbd}(p+q)^\mu p^\nu =$

$$= \ldots = \frac{ig^2}{(4\pi)^{d/2}}\ c_2(G)\,\delta^{ab}\int_0^1 \frac{dx}{\Delta^{2-d/2}}\ \cdot$$

$$\cdot\left\{-\Gamma\left(1-\frac{d}{2}\right)\eta^{\mu\nu}q^2\,\frac{1}{2}x(1-x) + \Gamma\left(2-\frac{d}{2}\right)q^\mu q^\nu x(1-x)\right\}$$

- Combinando b, c, d, il coeff. di $\Gamma\left(1-\frac{d}{2}\right)$ è
  $\sim (3d-3-d^2+d-1) = -2\left(1-\frac{d}{2}\right)(2-d)$

Il polo $\frac{1}{1-\frac{d}{2}}$ della funzione Γ è CANCELLATO
($\Rightarrow$ no quadratic div. in $d=4$)

- Potendo rimpiazzare $x \leftrightarrow (1-x)$ in ogni termine e svolgendo i conti, otteniamo da b, c, d :

$$\frac{ig^2}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab}\int_0^1 dx\, \frac{\Gamma(2-\frac{d}{2})}{\underbrace{\Delta^{2-d/2}}_{\substack{=1\\ d=4}}}\,(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\left[(1-\tfrac{d}{2})(1-2x)^2+2\right].$$

• Estraiamo ora il contributo del polo $\frac{1}{2-\omega}$ $(d=2\omega)$, sapendo che $\Gamma(z)\sim\frac{1}{z}$.

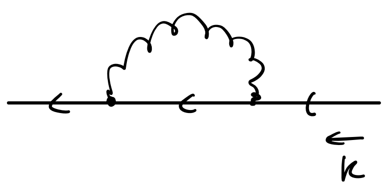
$$\frac{ig^2}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab}\,\frac{1}{2-\omega}\,(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\underbrace{\int_0^1\left[1+4x-4x^2\right]dx}_{5/3}$$

• Mettendo assieme anche il contributo da a), otteniamo

$$\Pi_{1/\epsilon} = -\frac{g^2 N_F}{6\pi^2(2-\omega)}c(R) - \frac{g^2}{(4\pi)^2}\left(-\frac{5}{3}\right)C_2(G)\frac{1}{2-\omega} =$$

$$= -\frac{g^2}{16\pi^2}\left[\frac{4}{3}N_F\, c(R) - \frac{5}{3}C_2(G)\right] = \delta Z_A$$

Passiamo a

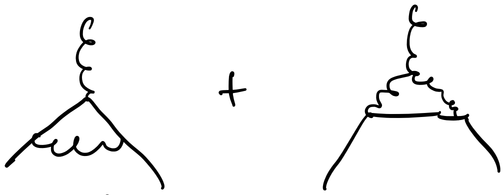


$$= (ig)^2\int\frac{d^dp}{(2\pi)^d}\,\gamma^\mu t^a_R\,\frac{i}{\not p-\not k}\,\gamma_\mu t^a_R\,\frac{-i}{p^2}$$

• Usiamo $(t^a_R t^a_R)_{ij} = C_2(R)\,\delta_{ij}$

$$\rightarrow \quad \delta Z_\psi = -\frac{g^2}{16\pi^2}C_2(R)\,\frac{1}{2-\omega}$$

E ora:



a) b)

a) 

$$= g^3 \int \frac{d^d p}{(2\pi)^d}\, t_R^b t_R^a t_R^b\, \gamma^\nu \frac{1}{\not p + \not k'} \gamma^\mu \frac{1}{\not p + \not k} \gamma_\nu \cdot \frac{1}{p^2}$$

- come in QED, con l'unica aggiunta del fattore

$$t^b t^a t^b = t^b t^b t^a + t^b [t^a, t^b] =$$

$$= C_2(R)\, t^a + \underbrace{i\, t^b f^{abc} t^c}_{i f^{abc} \frac{1}{2}[t^b, t^c] = \frac{1}{2} \underbrace{i f^{abc} i f^{bcd}}_{-C_2(G)\delta^{ab}} t^d} = \left[C_2(R) - \frac{1}{2} C_2(G)\right] t^a$$

b) 

- Il color factor è

$$f^{abc} t^b t^c = \frac{1}{2} f^{abc}\, i f^{bcd} t^d = \frac{i}{2} C_2(G)\, t^a$$

Mettendo assieme a) e b):

$$\delta Z_\psi = -\frac{g^2}{16\pi^2} \frac{1}{2-\omega} \left(C_2(R) + C_2(A) \right)$$