272SM: Introduction to Artificial Intelligence

Constraint Satisfaction Problems II

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Exercise: Formulating a CSP

Can you phrase the problem of Hamiltonian tour as a CSP (given a network of cities connected by roads, choose an order to visit all cities in a country without repeating any)?

Exercise: Reduction to binary constraints

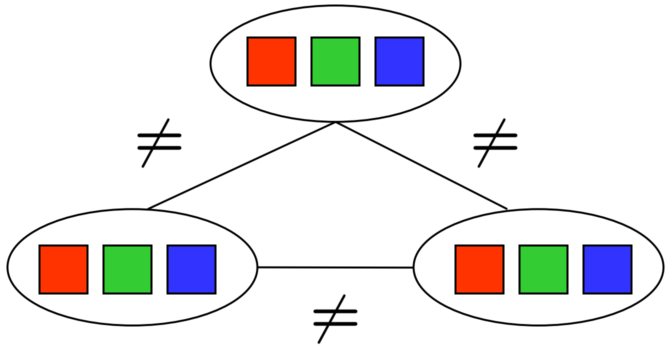
- *Show how a single ternary constraint such as "A+B = C" can be turned into three binary constraints by using an auxiliary variable. You may assume finite domains.*

Today

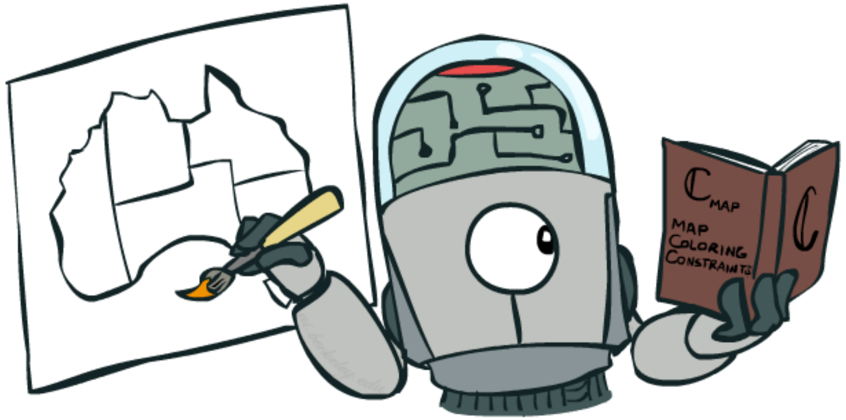
- Efficient Solution of CSPs
- Iterative Improvement

Review: CSPs

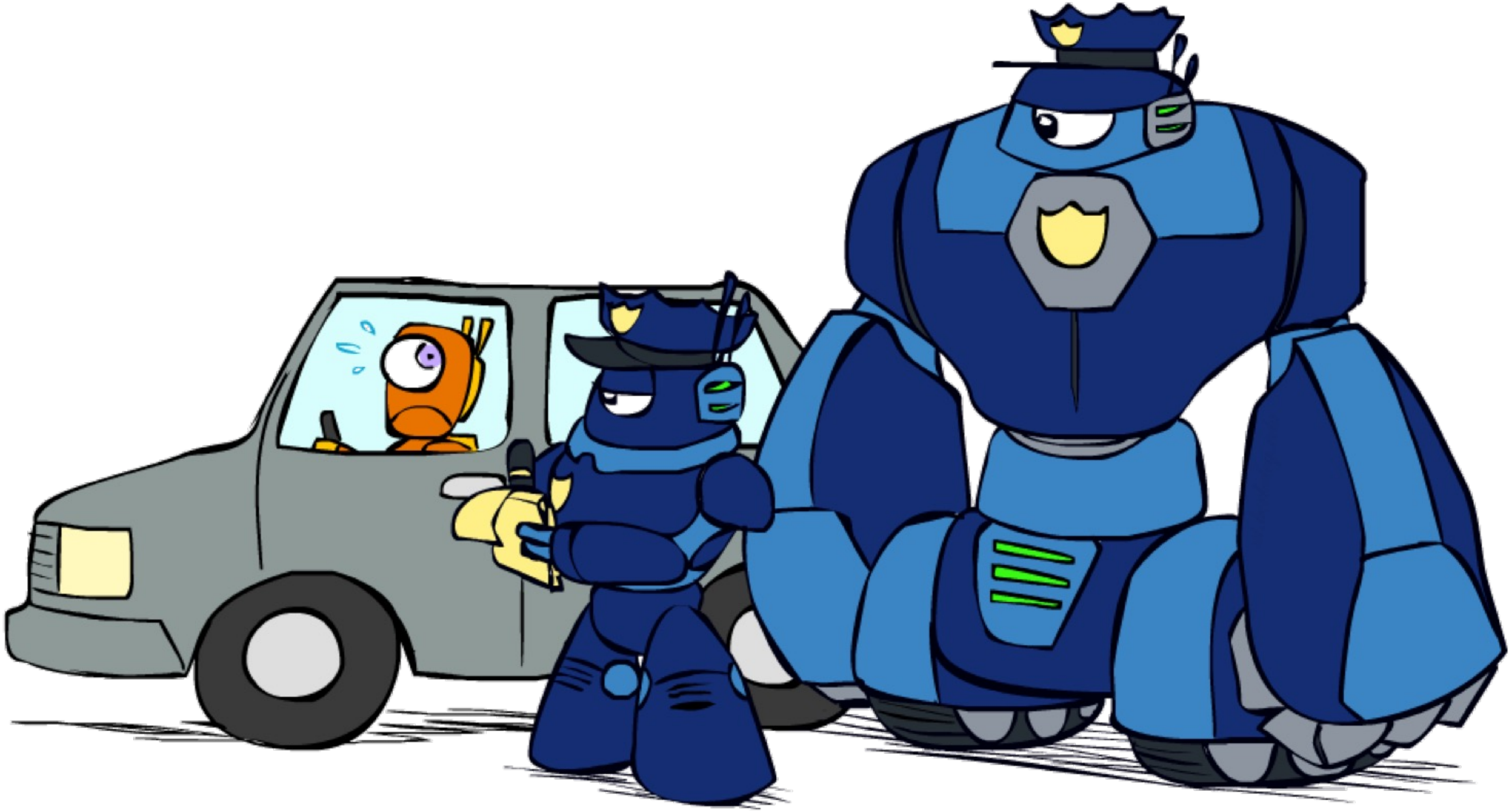
- CSPs:
  - Variables
  - Domains
  - Constraints
    - Implicit (provide code to compute)
    - Explicit (provide a list of the legal tuples)
    - Unary / Binary / N-ary



- Goals:
  - Here: find any solution
  - Also: find all, find best, etc.

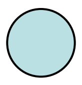


K-Consistency

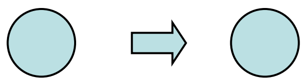


K-Consistency

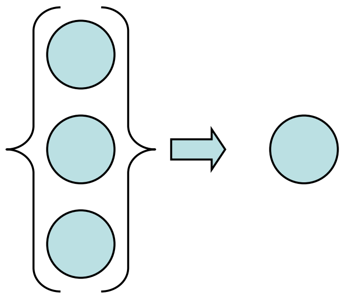
- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints



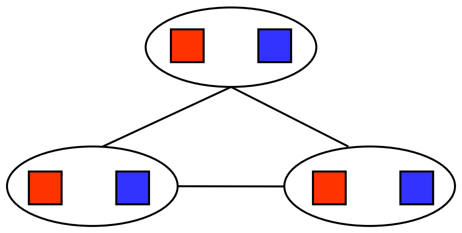
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other



  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.



- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



Strong K-Consistency

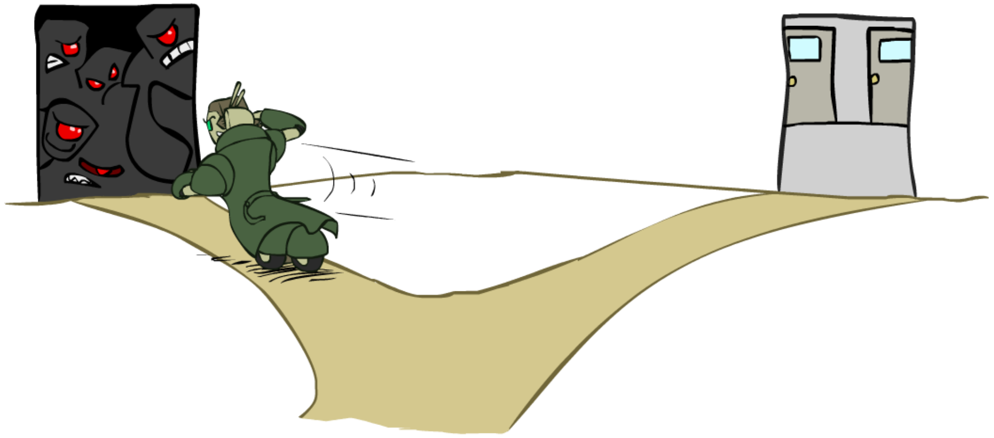
- Strong k-consistency: also k-1, k-2, … 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

Ordering

Ordering: Minimum Remaining Values

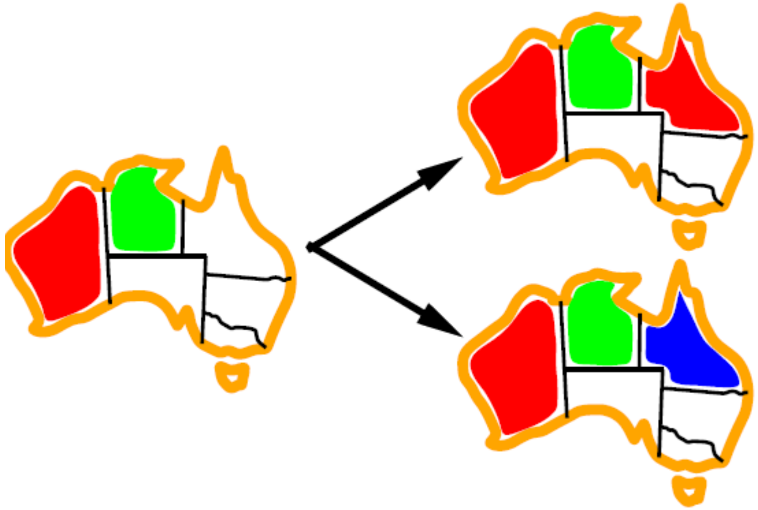
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain

- Why min rather than max?
- Also called "most constrained variable"
- "Fail-fast" ordering



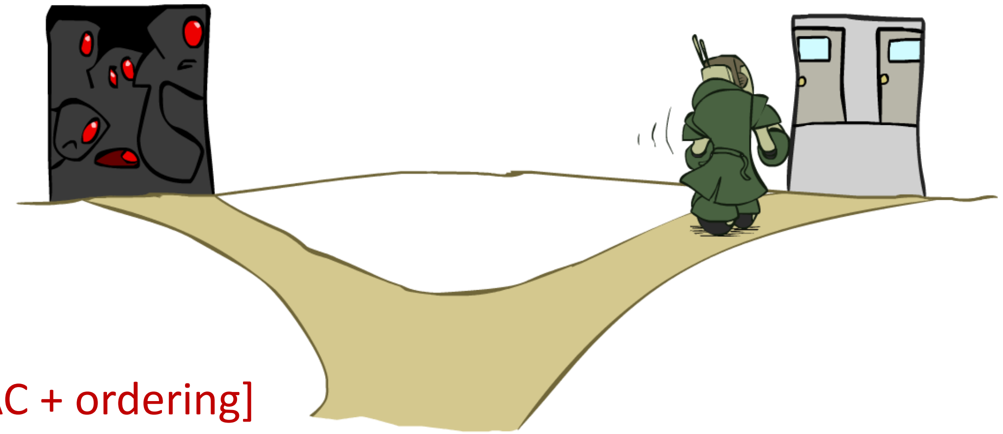
Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - I.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)



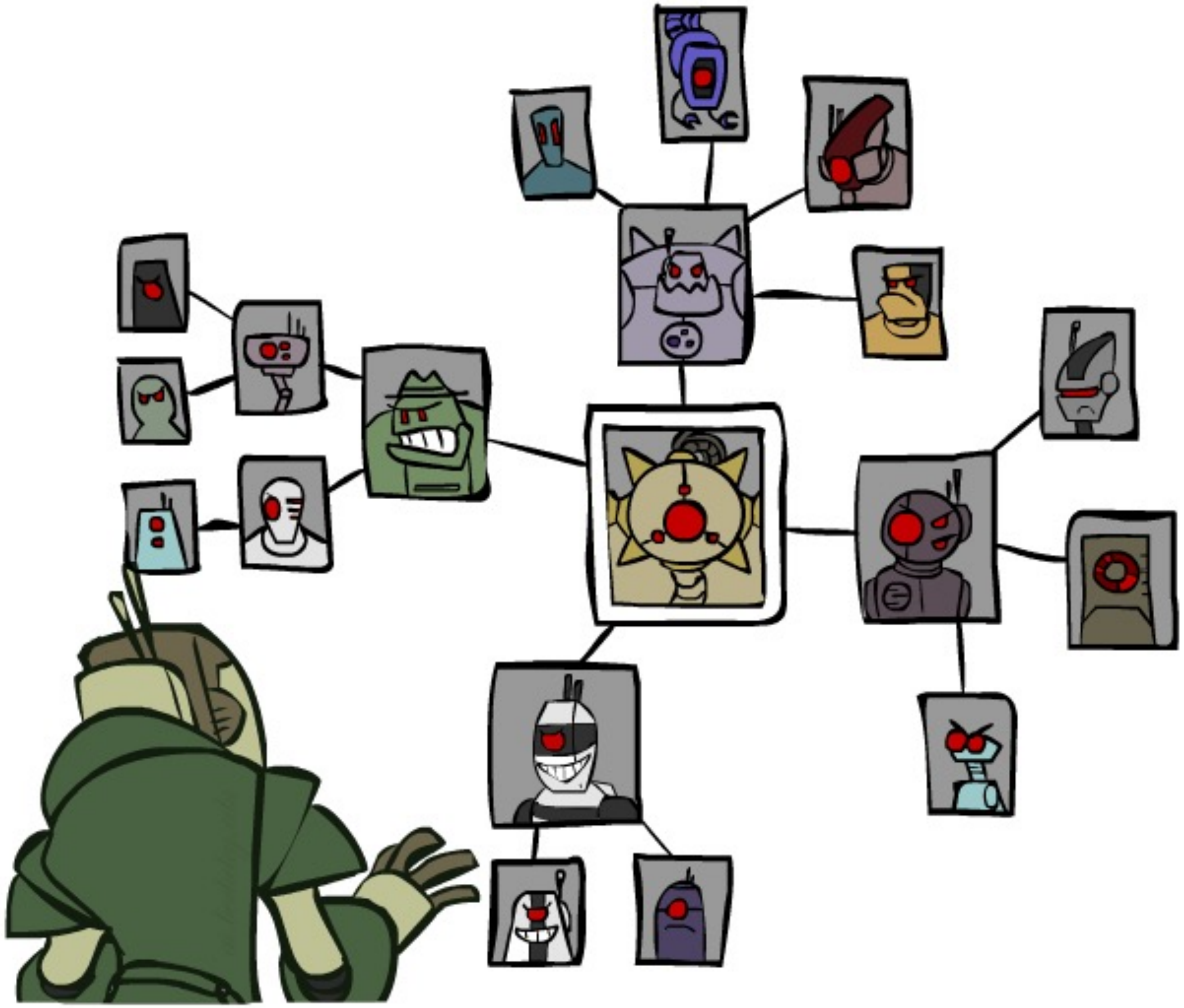
- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible



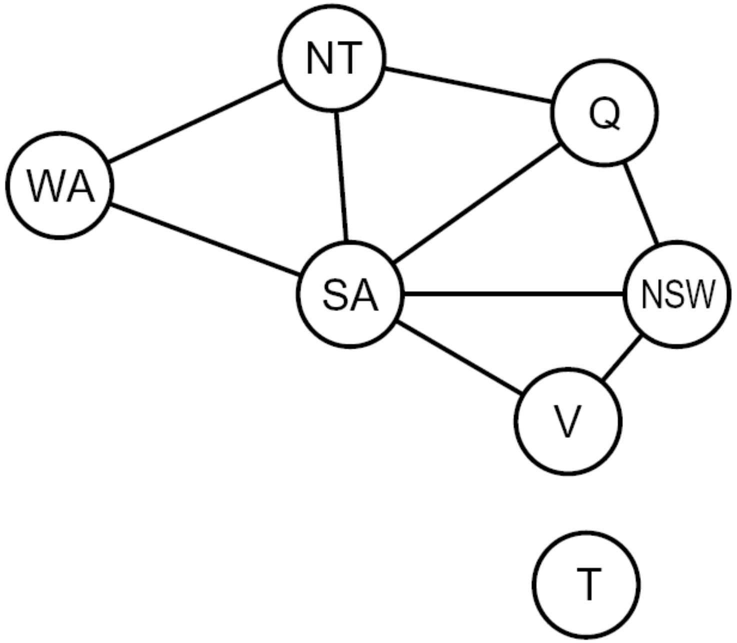
[Demo: coloring – backtracking + AC + ordering]

Structure

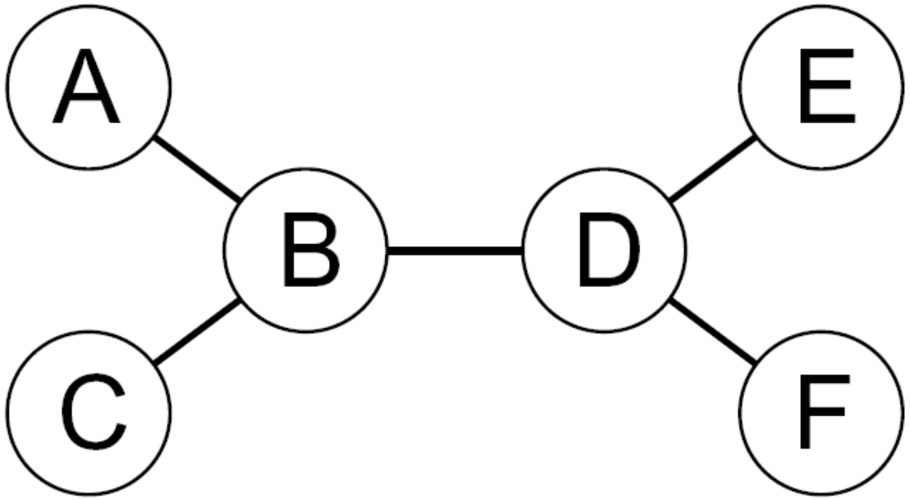


Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact
- Independent subproblems are identifiable as connected components of constraint graph
- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



Tree-Structured CSPs

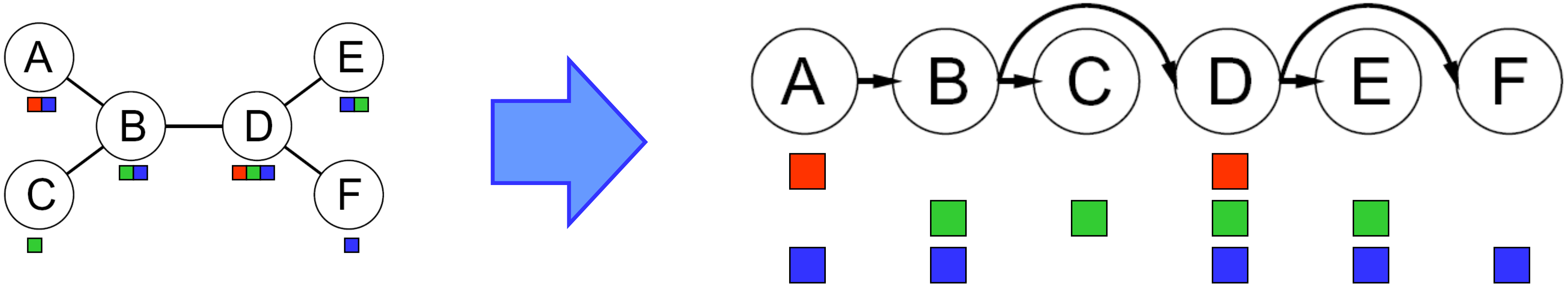


- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$

- This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning

Tree-Structured CSPs

- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children

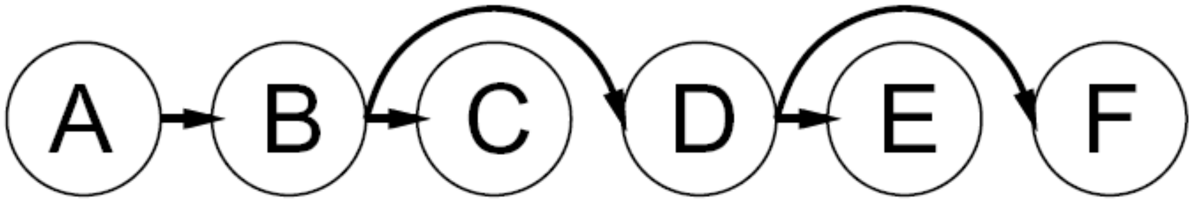


  - Remove backward: For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
  - Assign forward: For i = 1 : n, assign X_i consistently with Parent(X_i)

- Runtime: $O(n\ d^2)$

Tree-Structured CSPs

- Claim 1: After backward pass, all root-to-leaf arcs are consistent
- Proof: Each X→Y was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)

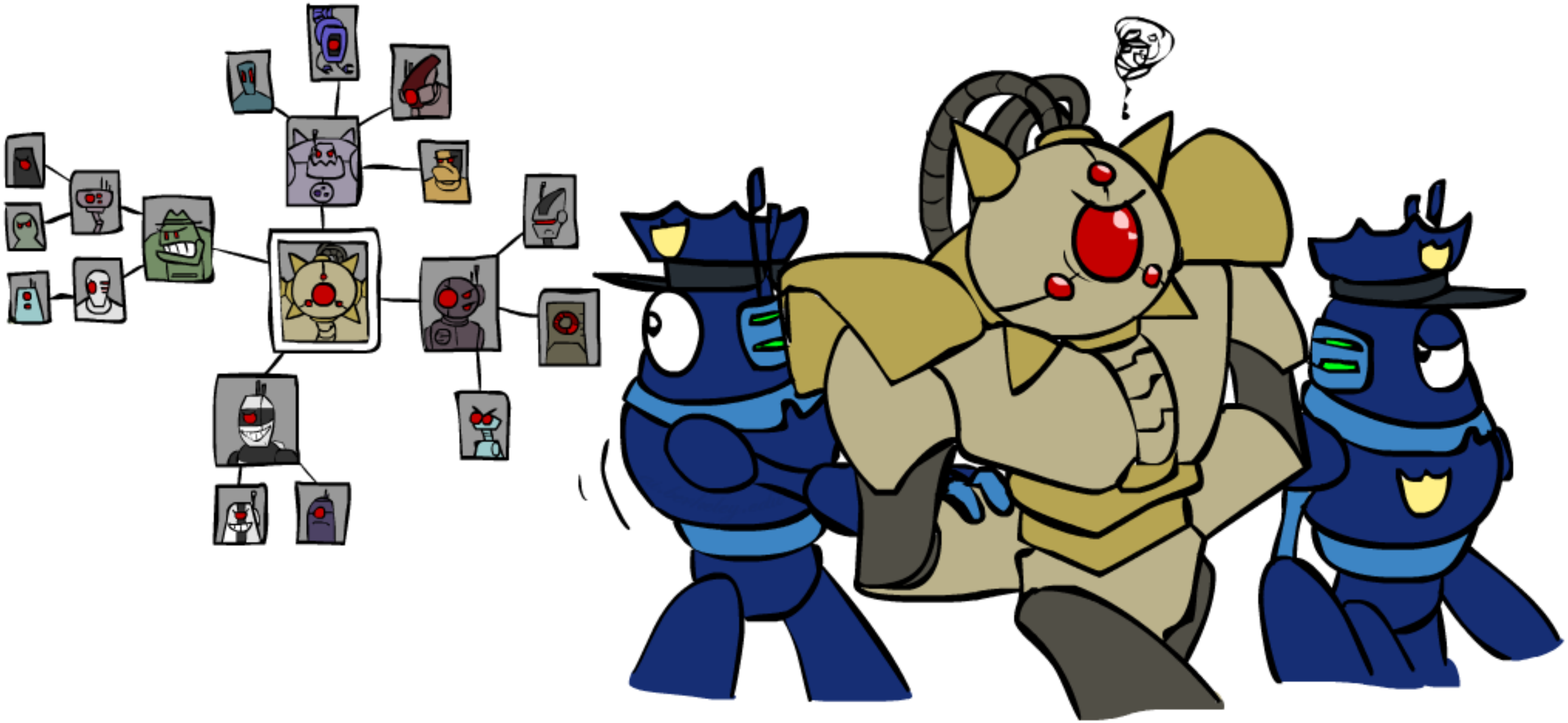


- Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack
- Proof: Induction on position

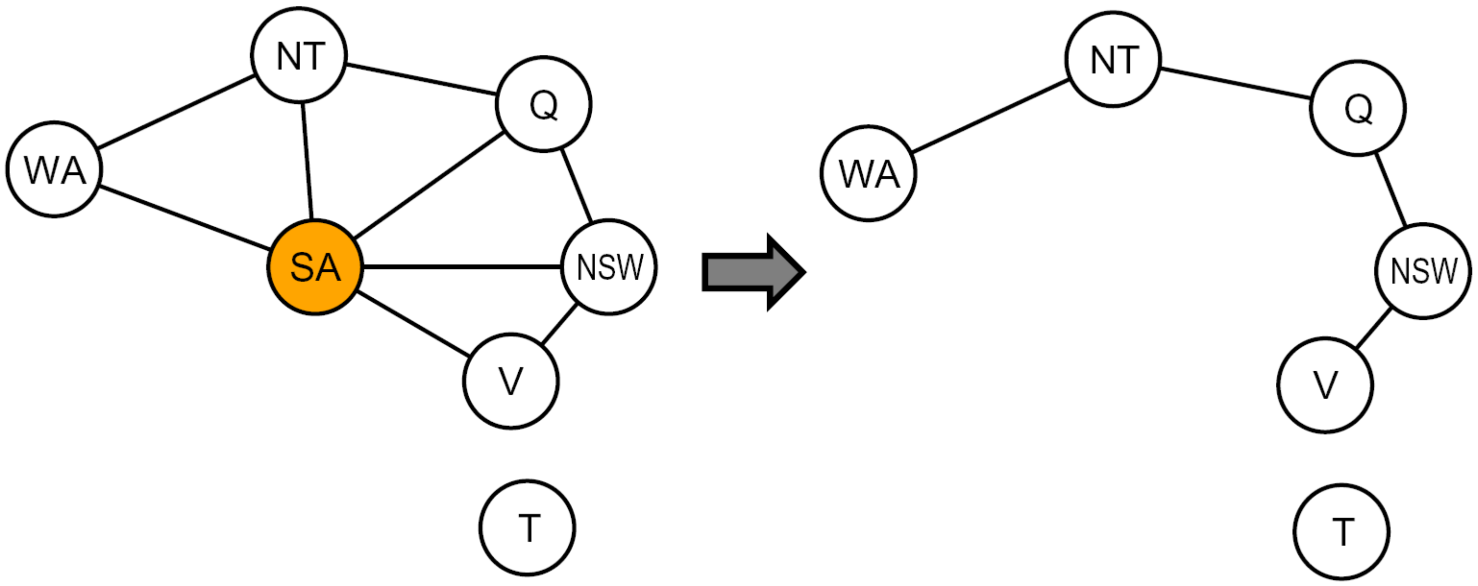
- Why doesn't this algorithm work with cycles in the constraint graph?

- Note: this basic idea is also used in Bayes' nets

Improving Structure



Nearly Tree-Structured CSPs



- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O((d^c) (n-c) d^2)$, very fast for small c

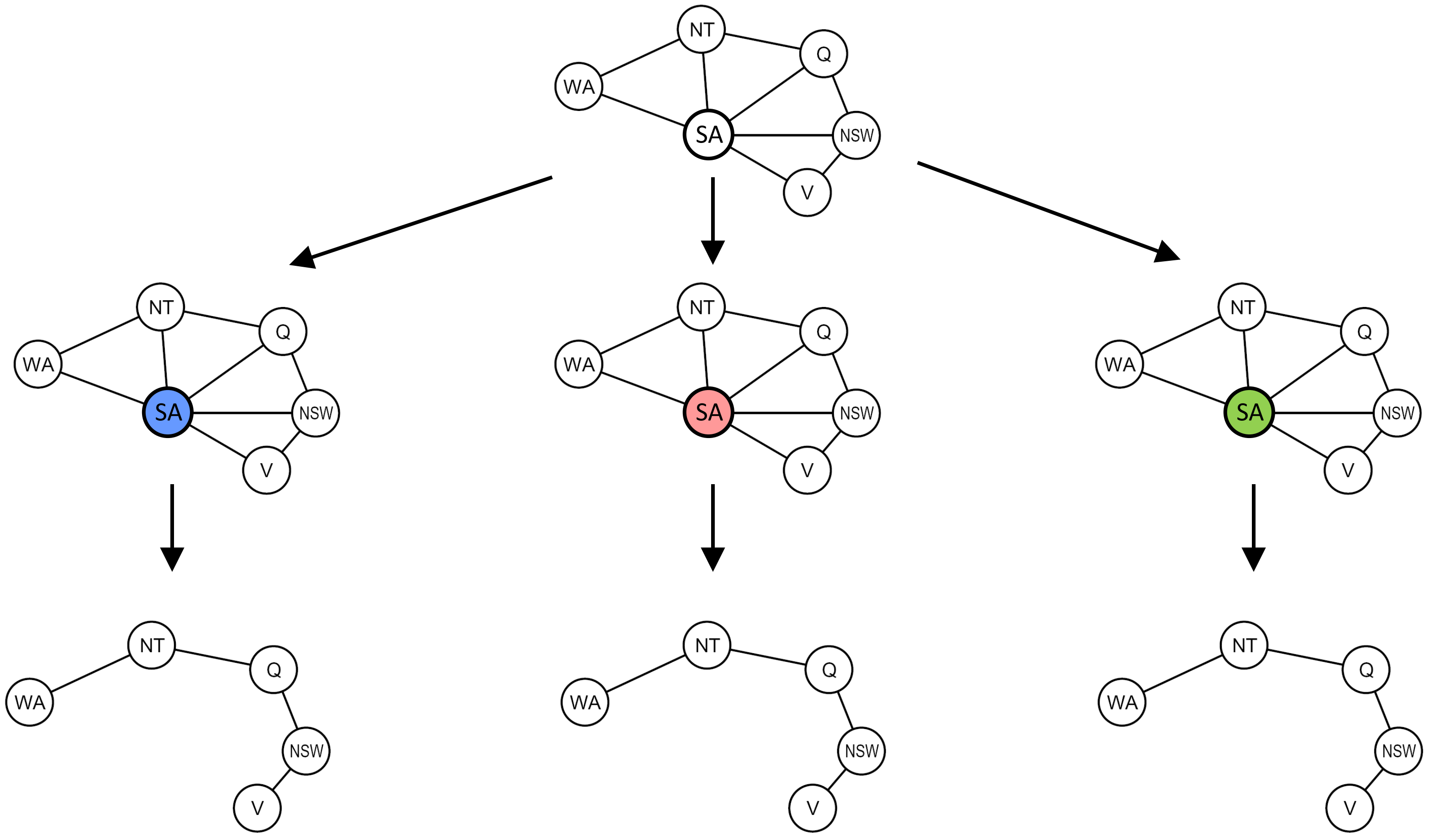
Cutset Conditioning

Choose a cutset

Instantiate the cutset (all possible ways)

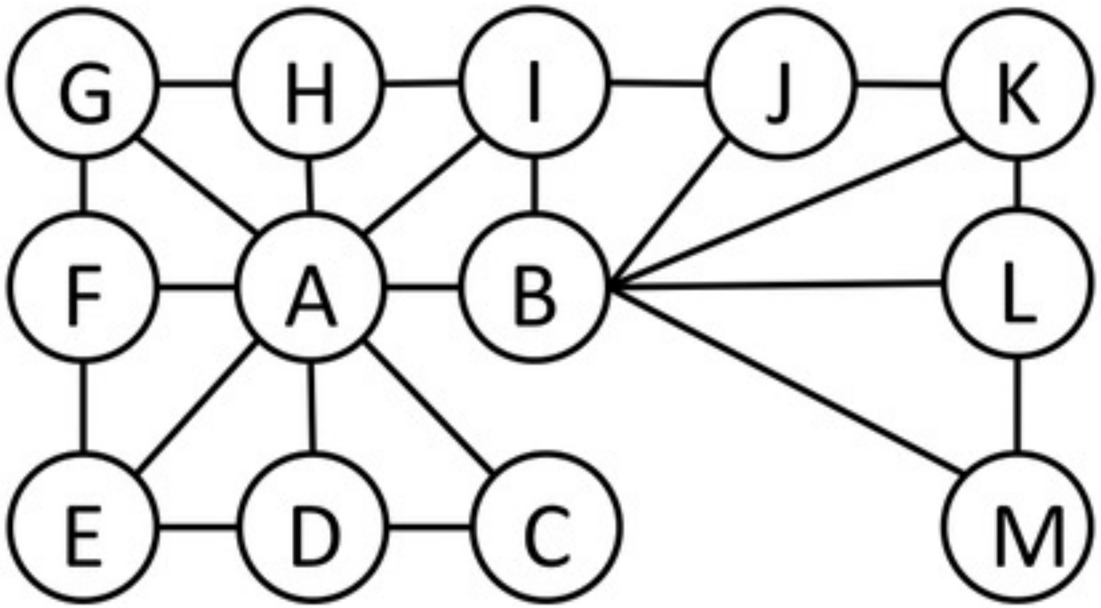
Compute residual CSP for each assignment

Solve the residual CSPs (tree structured)



Cutset Quiz

- Find the smallest cutset for the graph below.

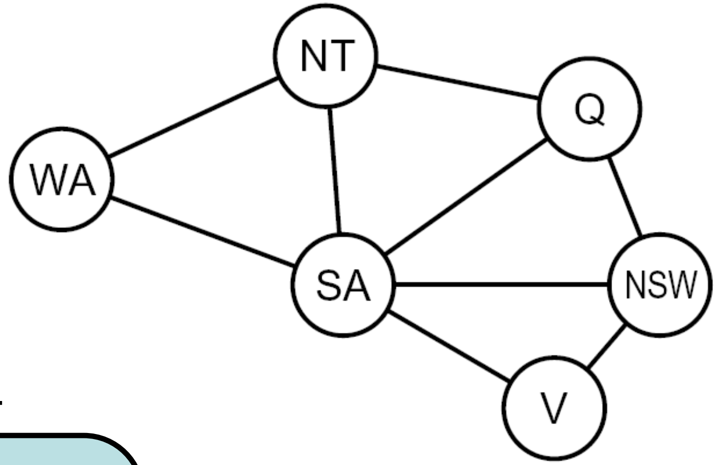


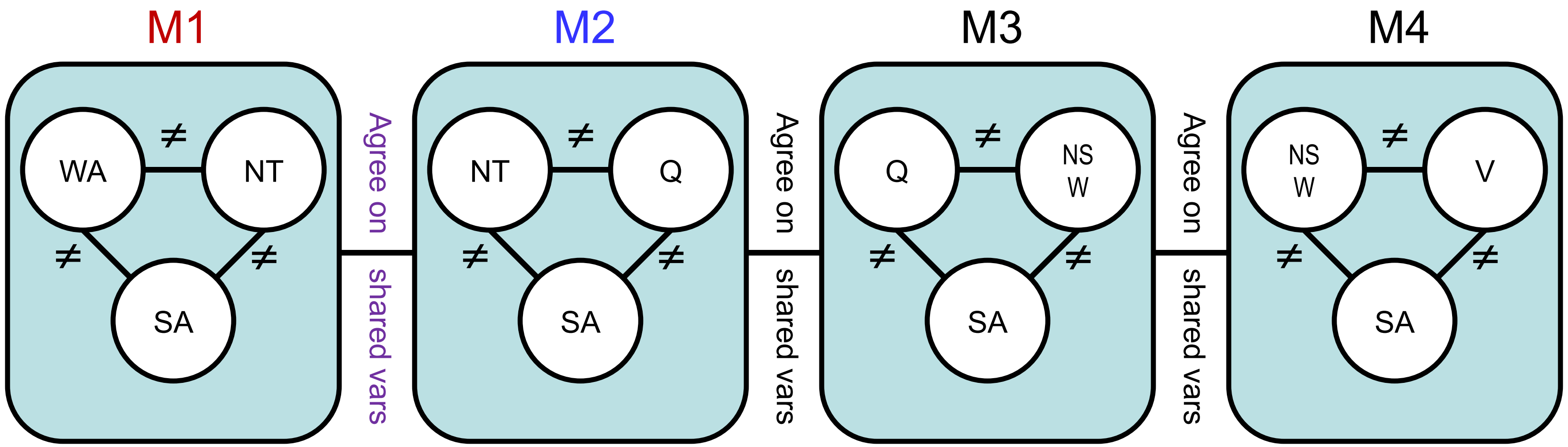
Exercise: Cutset

Consider a CSP with a constraint graph consisting of n variables arranged in a circle, where each variable has two constraints, one with each neighbor on either side. Explain how to solve this class of CSPs efficiently, in time O(n).

Tree Decomposition*

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions



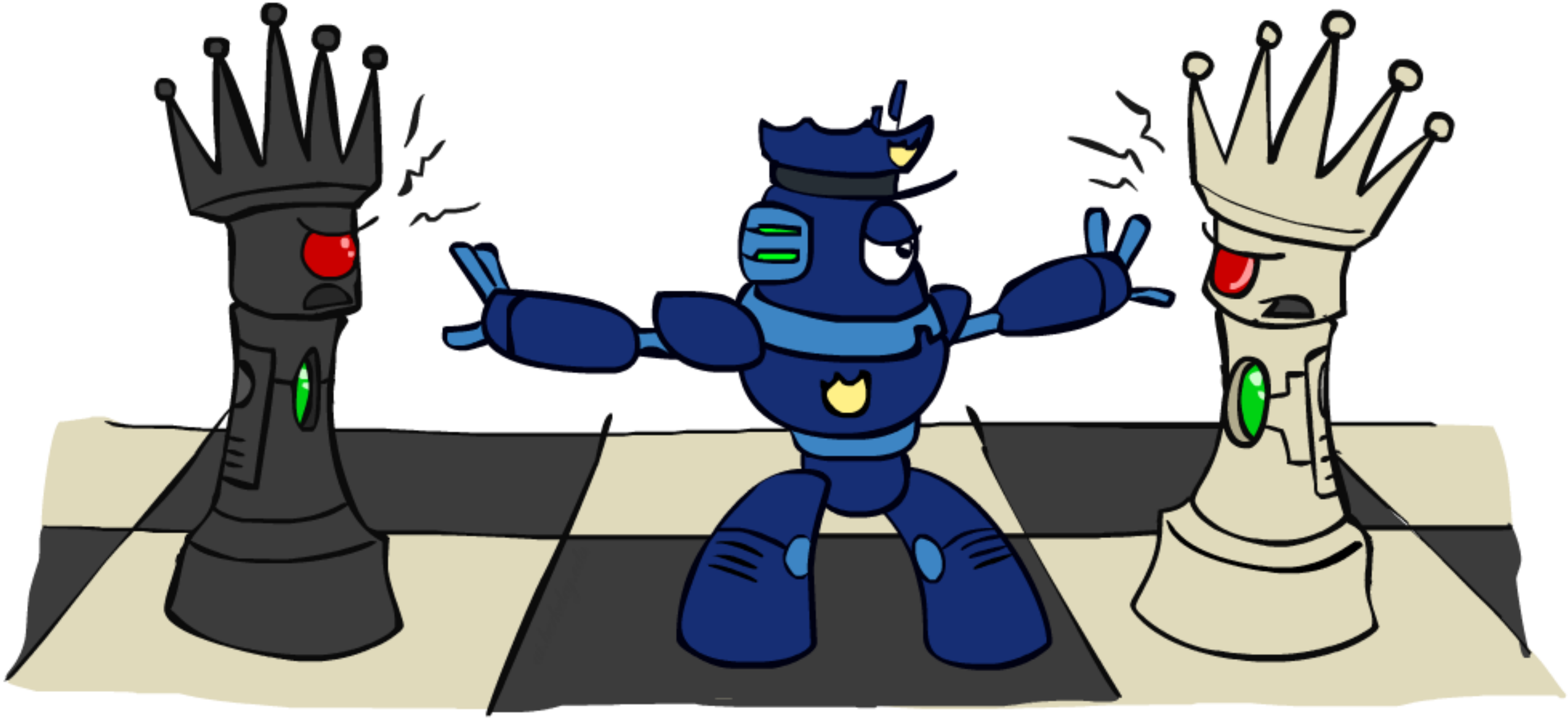


{(WA=r,SA=g,NT=b),
(WA=b,SA=r,NT=g),
…}

{(NT=r,SA=g,Q=b),
(NT=b,SA=g,Q=r),
…}

Agree: (M1,M2) ∈
{((WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)), …}

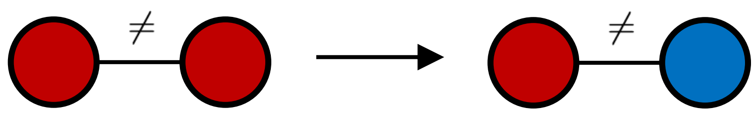
Iterative Improvement



Iterative Algorithms (Local search) for CSPs

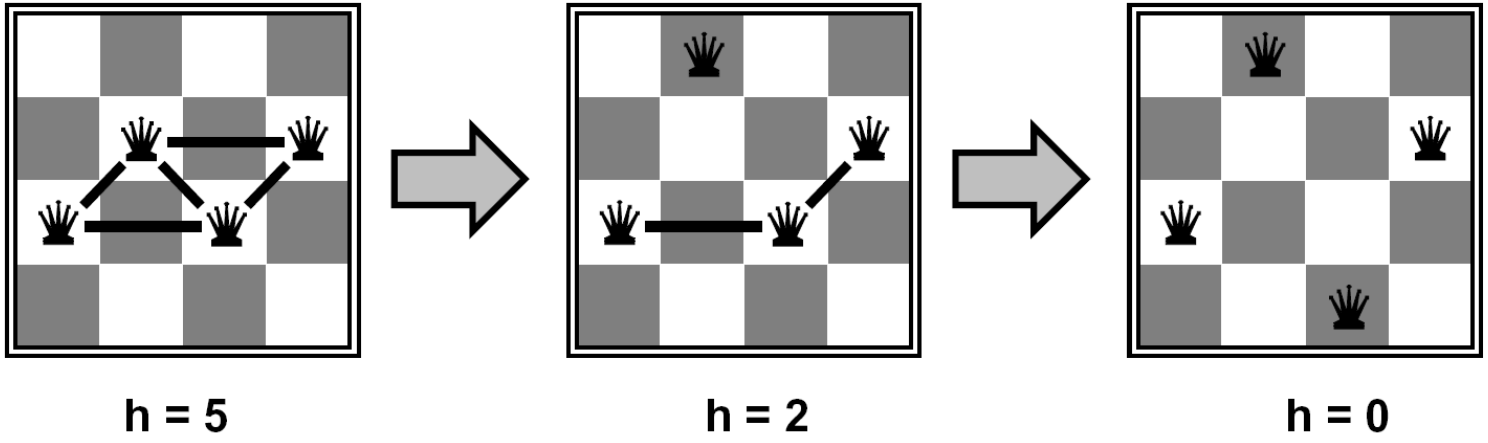
- Local search methods typically work with "complete" states, i.e., all variables assigned

- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.



- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with h(n) = total number of violated constraints

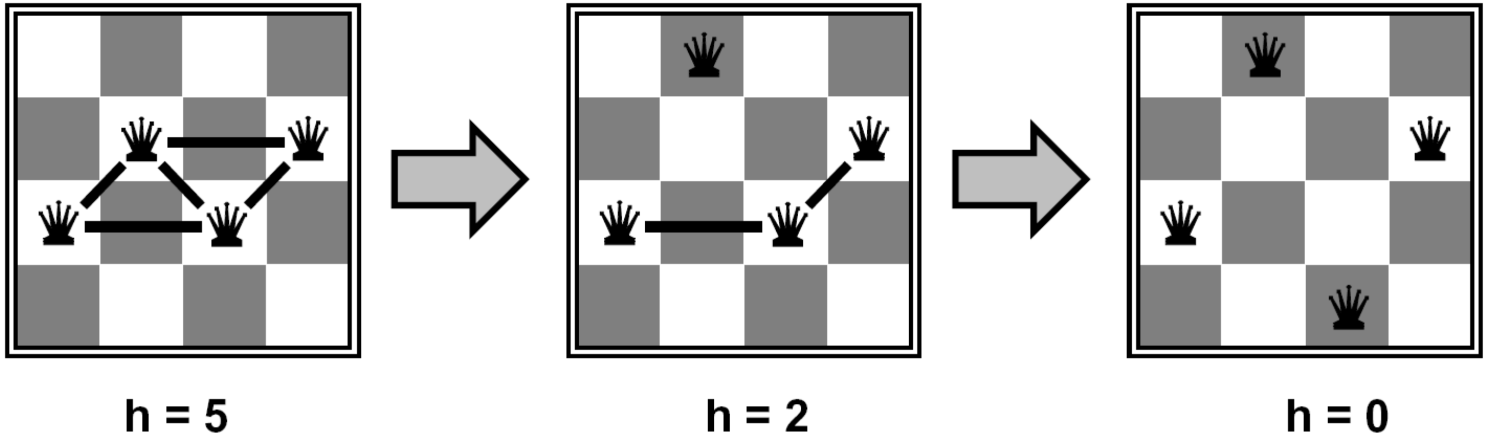
Example: 4-Queens



- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: c(n) = number of attacks

[Demo: n-queens – iterative improvement (L5D1)]
[Demo: coloring – iterative improvement]

Example: 4-Queens



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[Demo: n-queens – iterative improvement (L5D1)]
[Demo: coloring – iterative improvement]

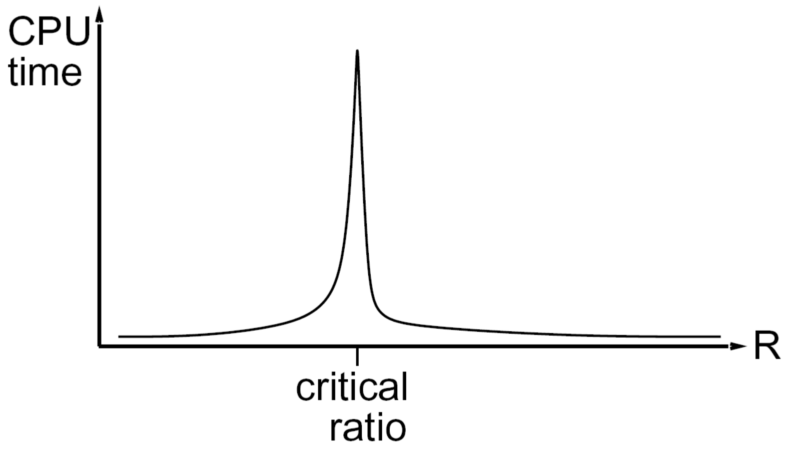
How does Min-conflicts work in practice?

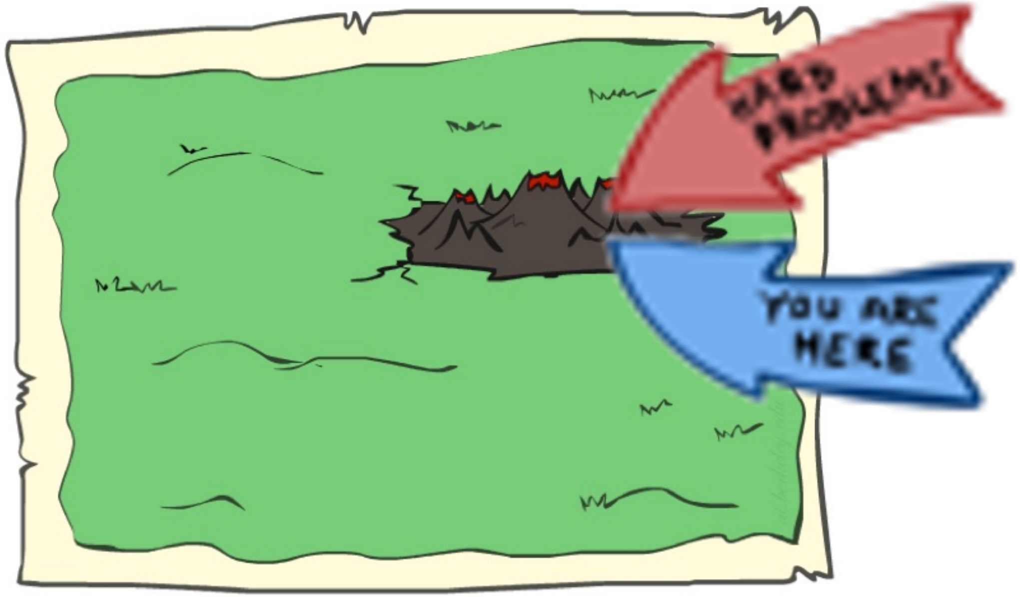
Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!

- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$





(your own illustration within Project 2)

Completeness of local search

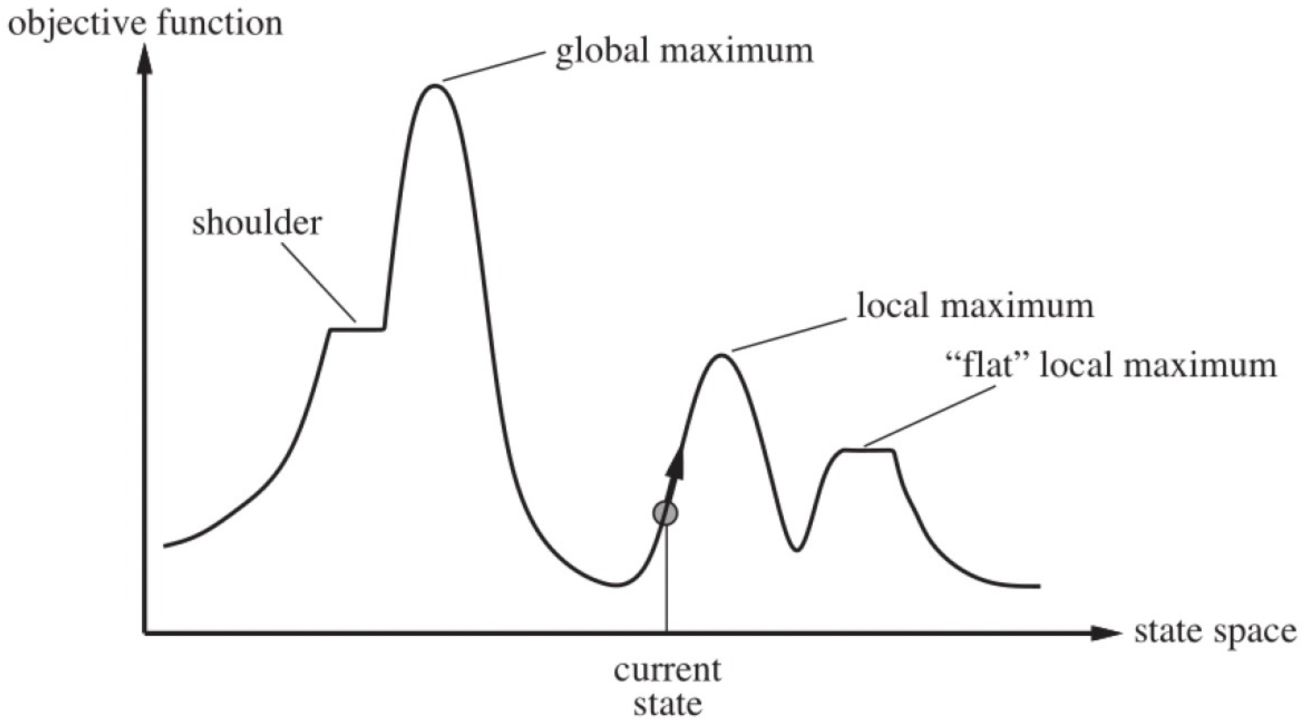
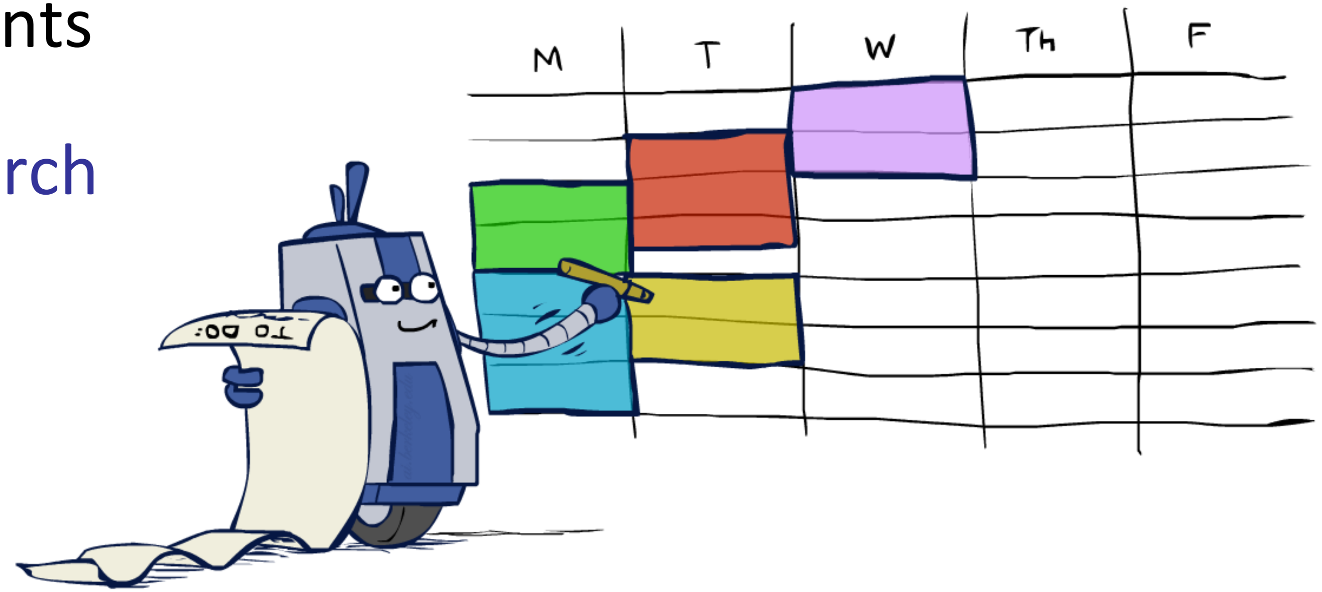


Figure 4.1 A one-dimensional state-space landscape in which elevation corresponds to the objective function. The aim is to find the global maximum. Hill-climbing search modifies the current state to try to improve it, as shown by the arrow. The various topographic features are defined in the text.

- Hill climbing with random restart, simulated annealing, genetic algorithms

Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - Structure
- Iterative min-conflicts is often effective in practice



HW: map coloring / performance

Generate random instances of map-coloring problems as follows: scatter n points on the unit square; select a point X at random, connect X by a straight line to the nearest point Y such that X is not already connected to Y and the line crosses no other line; repeat the previous step until no more connections are possible. The points represent regions on the map and the lines connect neighbors. Now try to find k-colorings of each map, for both k=3 and k=4, using min-conflicts, backtracking, backtracking with forward checking, and backtracking with MAC. Construct a table of average run times for each algorithm for values of n up to the largest you can manage. Comment on your results.

HW: Critical ratio

Using a CSP solver program and another program to generate random problem instances of CSPs, report on the time to solve the problem as a function of the ratio of the number of constraints to the number of variables.

Next Time: Adversarial Search!