Image Processing for Physicists

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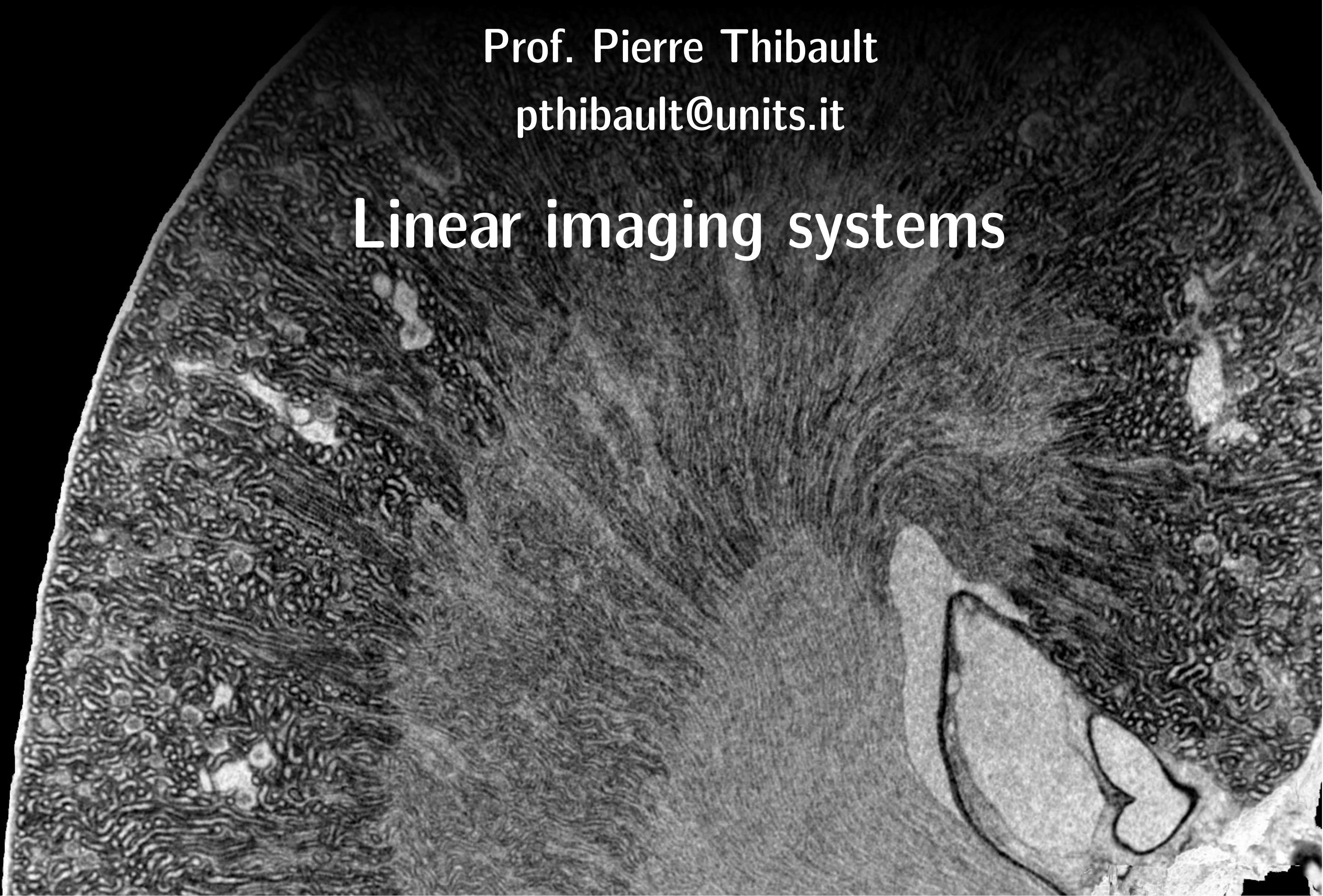
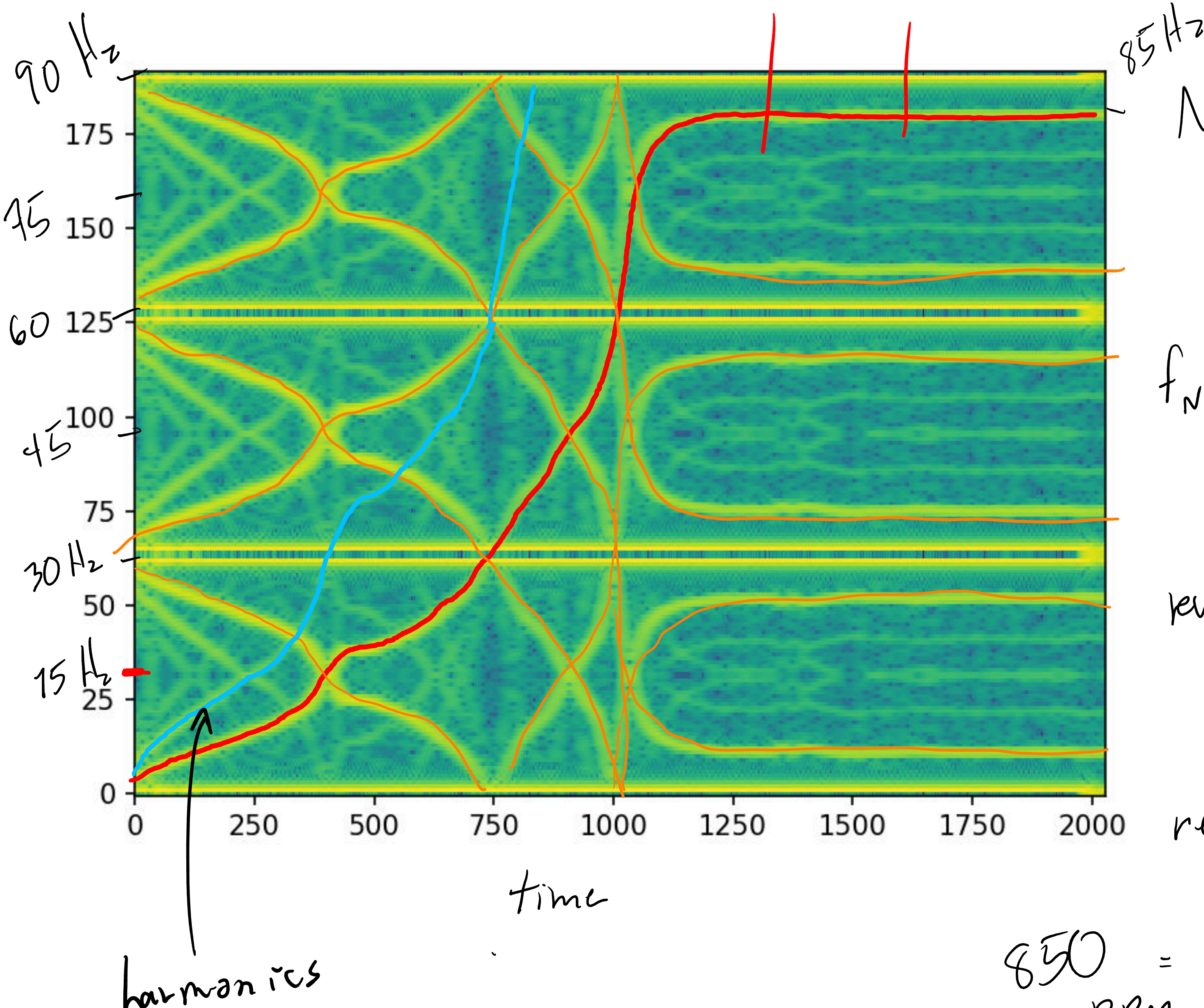
Linear imaging systems





Nyquist: $\frac{1}{25}$

$= \frac{1}{2}$ frame rate

$f_N = 15$ Hz

revolution/s =

85 Hz/6

revolution/min

850 RPM = 85 Hz/6 · 60

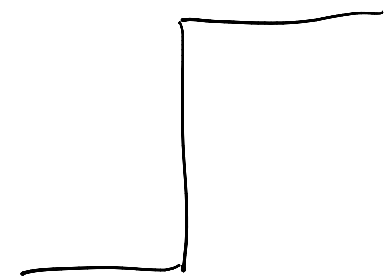
Overview

- Definition of resolution
- Imaging systems:
  - Linear transfer model
  - Noise

Resolution

"the smallest detail that can be distinguished" ← not a mathematical definition!

- No unique definition
  - Numerical aperture ← microscopy, photography, telescopes, ...
  - Pixel size ← detector-limited imaging
  - Other criteria (PSF, MTF)
- What is "detail"?
- What is "distinguish"?



Resolution

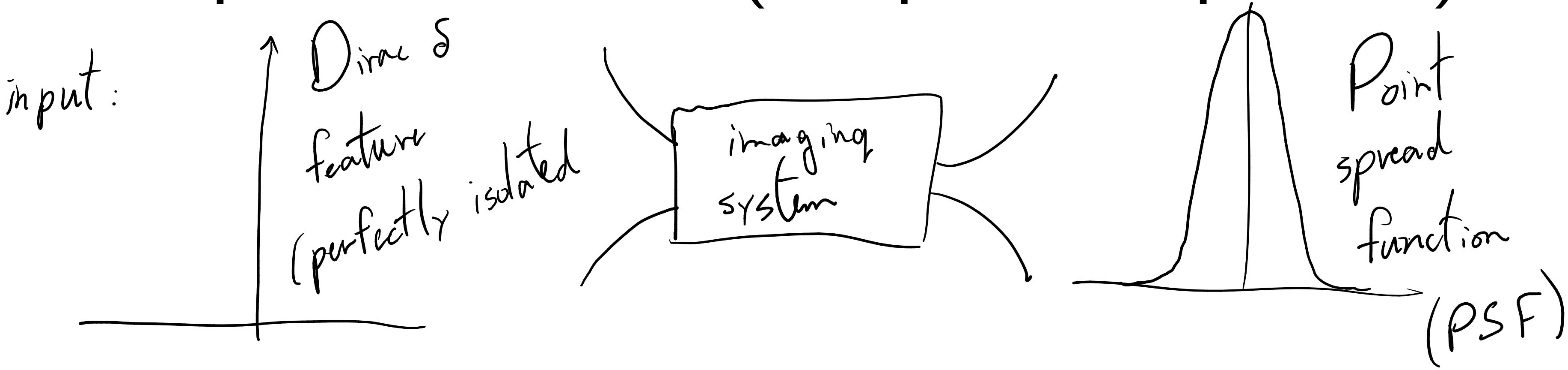
1280 x 1280

640 x 640

- **not** simply given by pixel size (i.e. sampling rate)
- light quality, optics quality, detector quality, algorithm quality, noise, ...

Linear translation-invariant systems

- Point spread function ("impulse response")



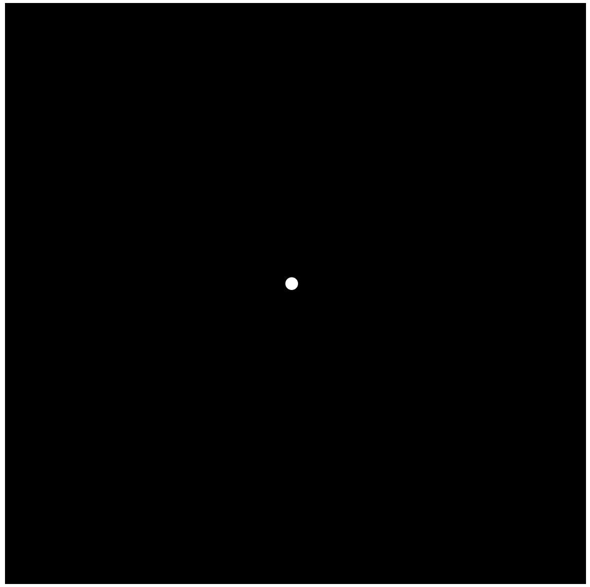
- LTI system: convolution with PSF

$$f(x,y) = \int dx'dy'\, f(x',y')\, \underbrace{\delta(x-x')\,\delta(y-y')}$$

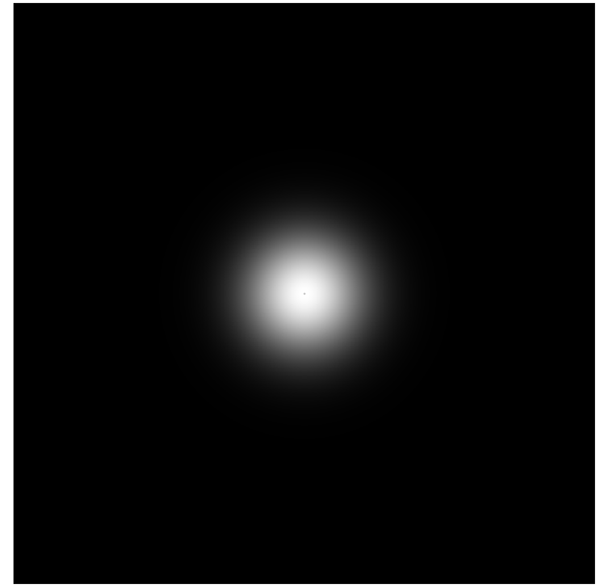
$\downarrow$ Imaging system

$$\text{output} \quad S\{f\} = \int dx'dy'\, f(x',y')\, \overbrace{h(x-x',y-y')} = f * \underset{\text{PSF}}{h}$$

Point spread function



"point source"



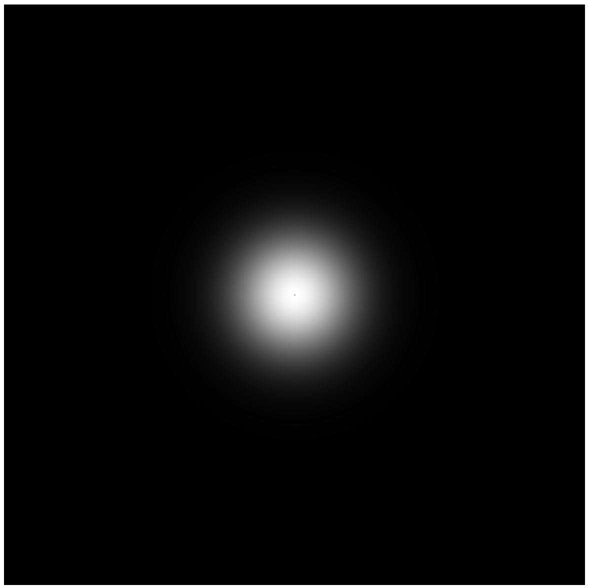
PSF

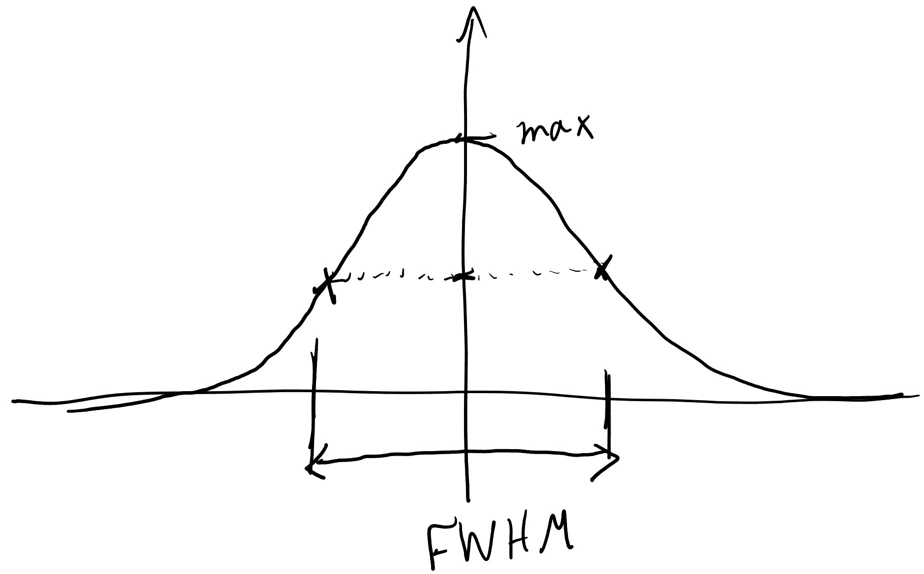
"true image"

"measured image"

PSF and resolution

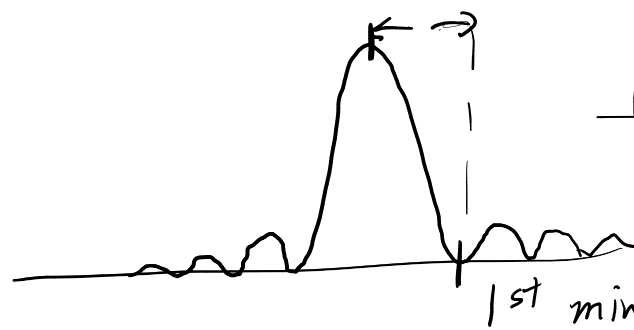
Common way to describe the PSF width is the "full-width at half-maximum" FWHM

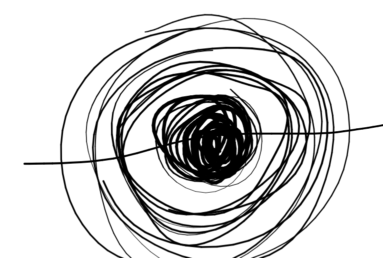




Rayleigh criterion: applies to imaging systems with a circular aperture:

PSF: Airy disc





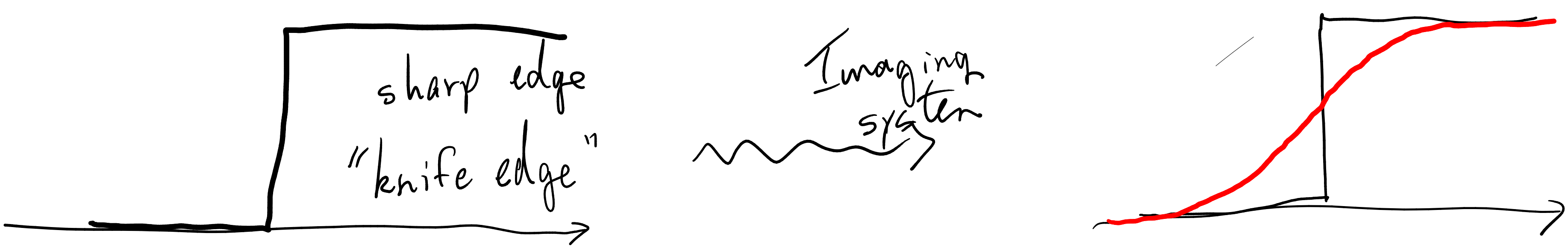
Measurement of the PSF

- Direct measurement from impulse

Generate a sharp point → output = PSF

Astronomy: just pick a star!

- ~~Line~~ edge-spread function

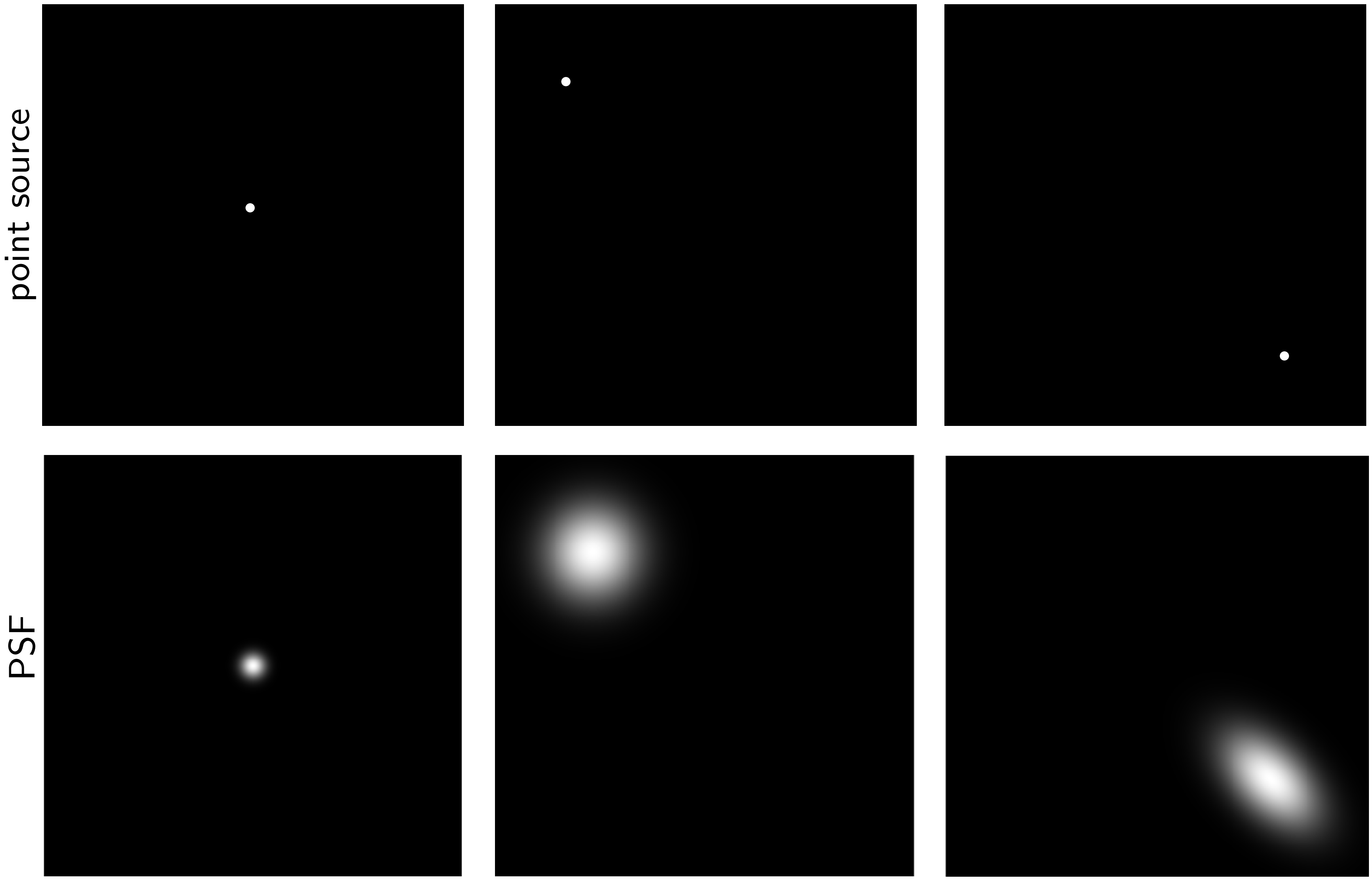


$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\frac{\partial H}{\partial x} = \delta(x)$$

PSF = derivative of edge-spread function

PSF and translation invariance



- Not translation invariant $\rightarrow$ PSF depends on position $\rightarrow$ not a convolution
- Useful to model system imperfections, lens aberrations, ...

The Fourier picture

F.T. of the PSF

$$\mathcal{F}\{f * h\} = F(u) \cdot H(u)$$

$\parallel$

"Optical transfer function"

OTF

Consider a single spatial frequency u_0

$$A e^{2\pi i u_0 x} \xrightarrow{\text{Imaging system}} \underbrace{H(u_0) A}_{} e^{2\pi i u_0 x}$$

original amplitude of this spatial frequency

modulated amplitude (reduced by some amount most likely, and possibly with a phase)

* observation: pure oscillations of the form $e^{2\pi i u_0 x}$ are eigenfunctions of a LTI imaging system

Optical transfer function

Response of a system to an oscillating signal with well-defined frequency

complex valued in general

$$OTF(\nu) = \mathcal{F}\{PSF(x)\}$$

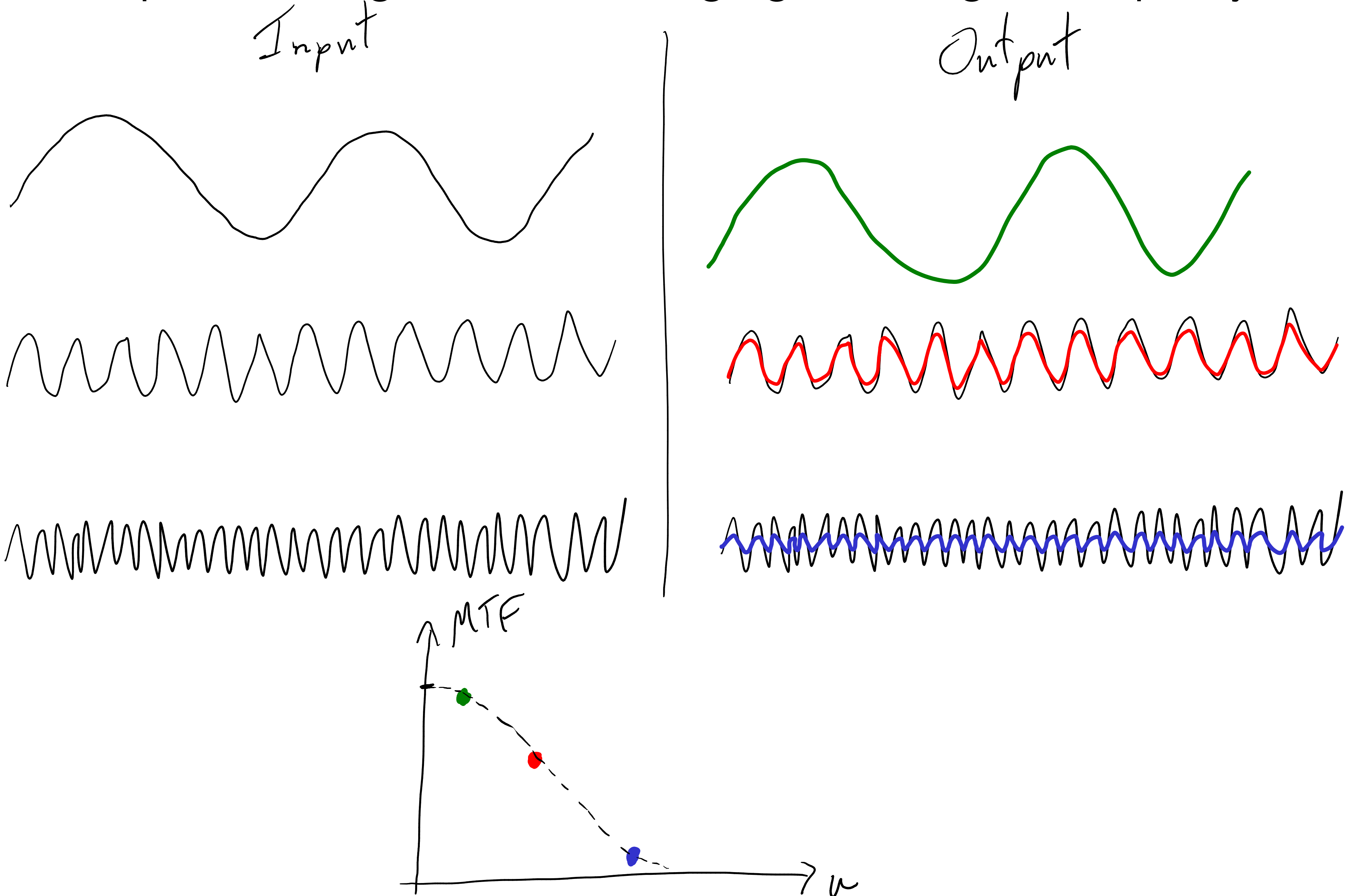
Amplitude : $|OTF| = MTF$ "modulation transfer function"

Phase : $\arg\{OTF\} = PTF$ "phase transfer function"

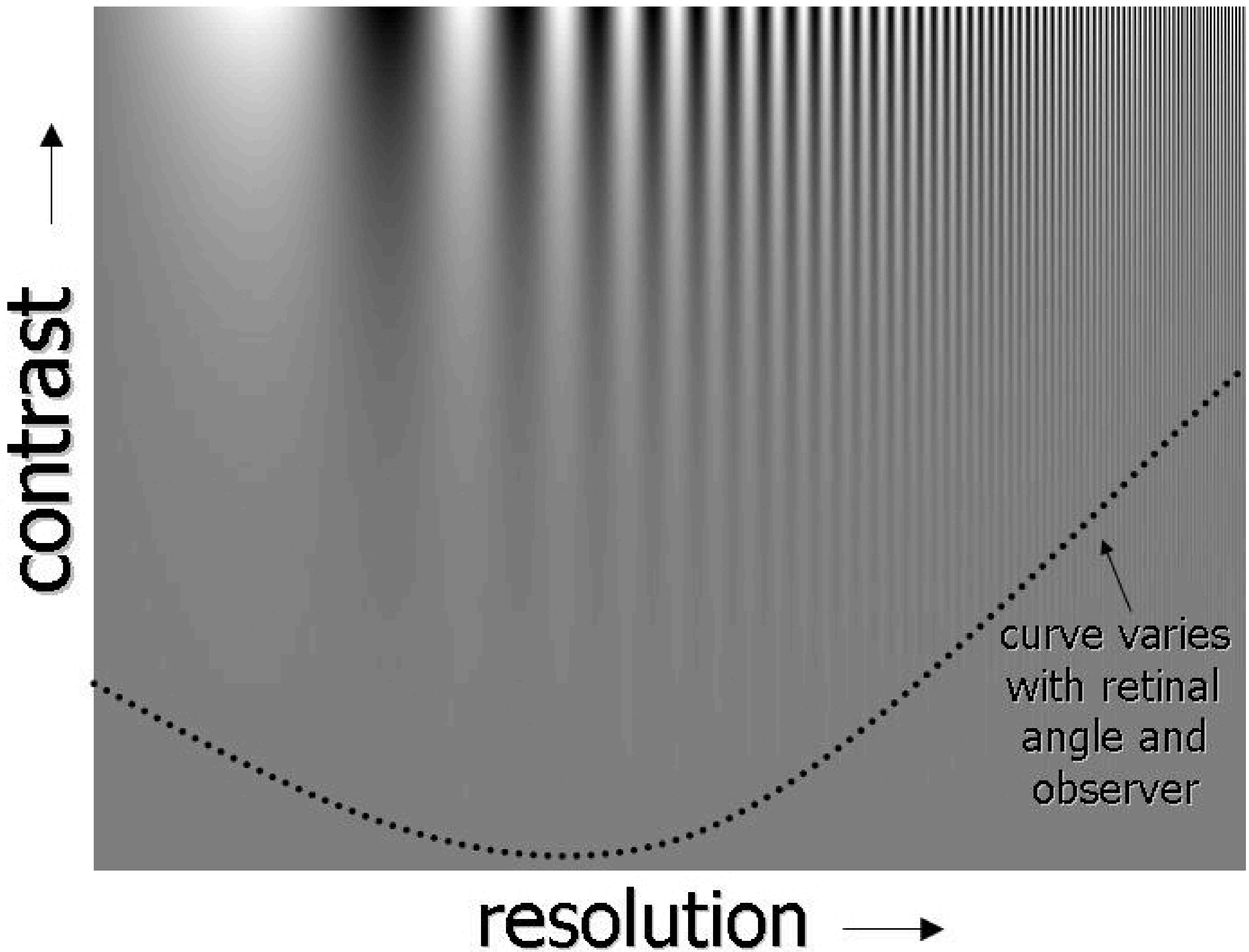
$$OTF = MTF\, e^{i\,PTF}$$

Modulation transfer function

Amplitude change of an oscillating signal for a given frequency



Eye MTF


contrast
curve varies
with retinal
angle and
observer
resolution

Campbell-Robson curve

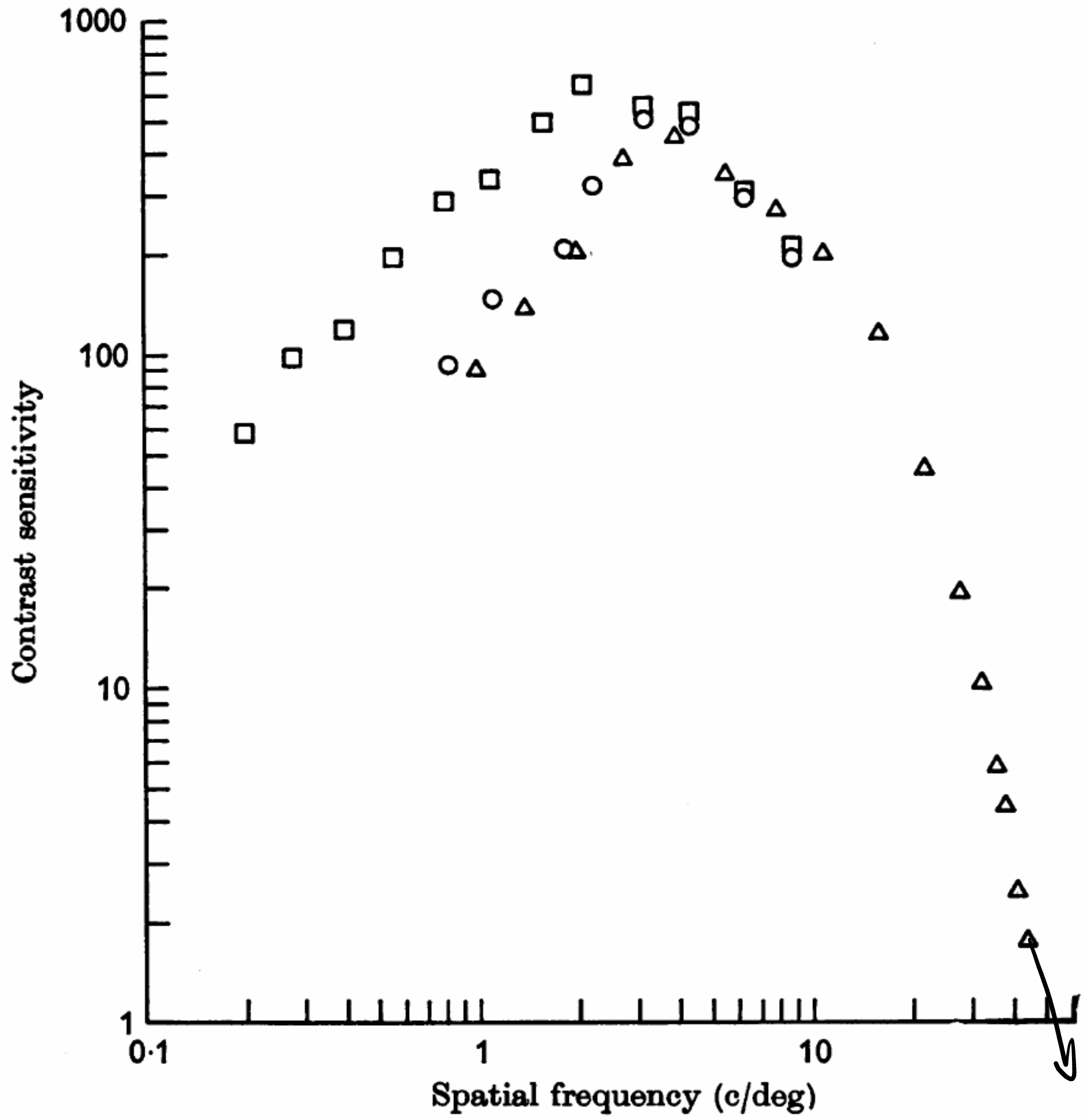


Fig. 2. Contrast sensitivity for sine-wave gratings. Subject F.W.C., luminance 500 cd/m². Viewing distance 285 cm and aperture 2° × 2°, △; viewing distance 57 cm, aperture 10° × 10°, □; viewing distance 57 cm, aperture 2° × 2°, ○.

20/20 vision

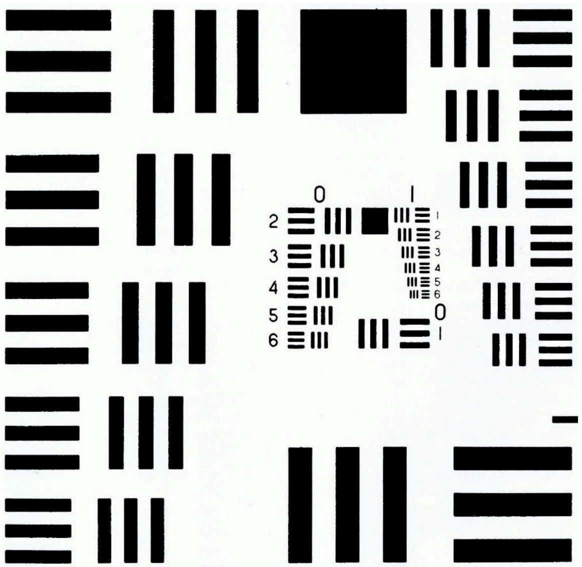
$\Downarrow$

60 cycles/degree

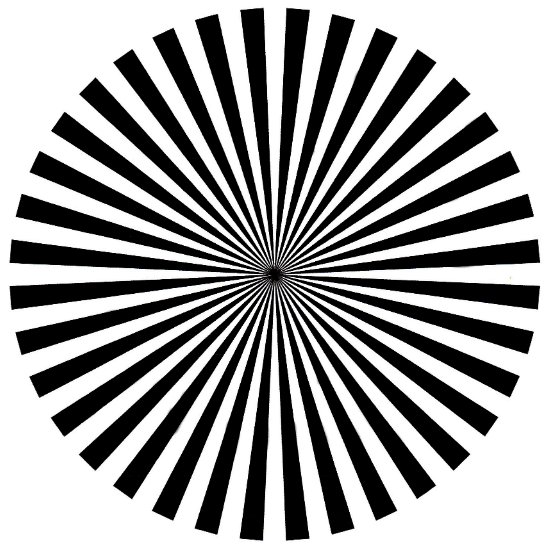
$\Updownarrow$

1 period ≃ 3×10^{-4} rad

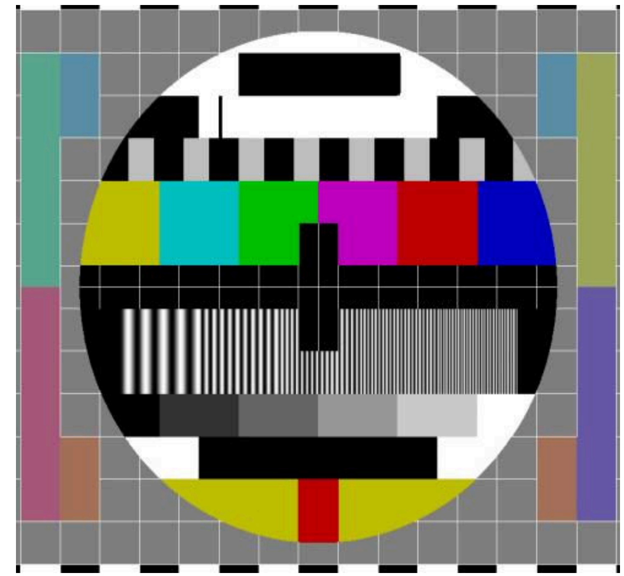
Measurement of MTF



USAF
resolution
reference



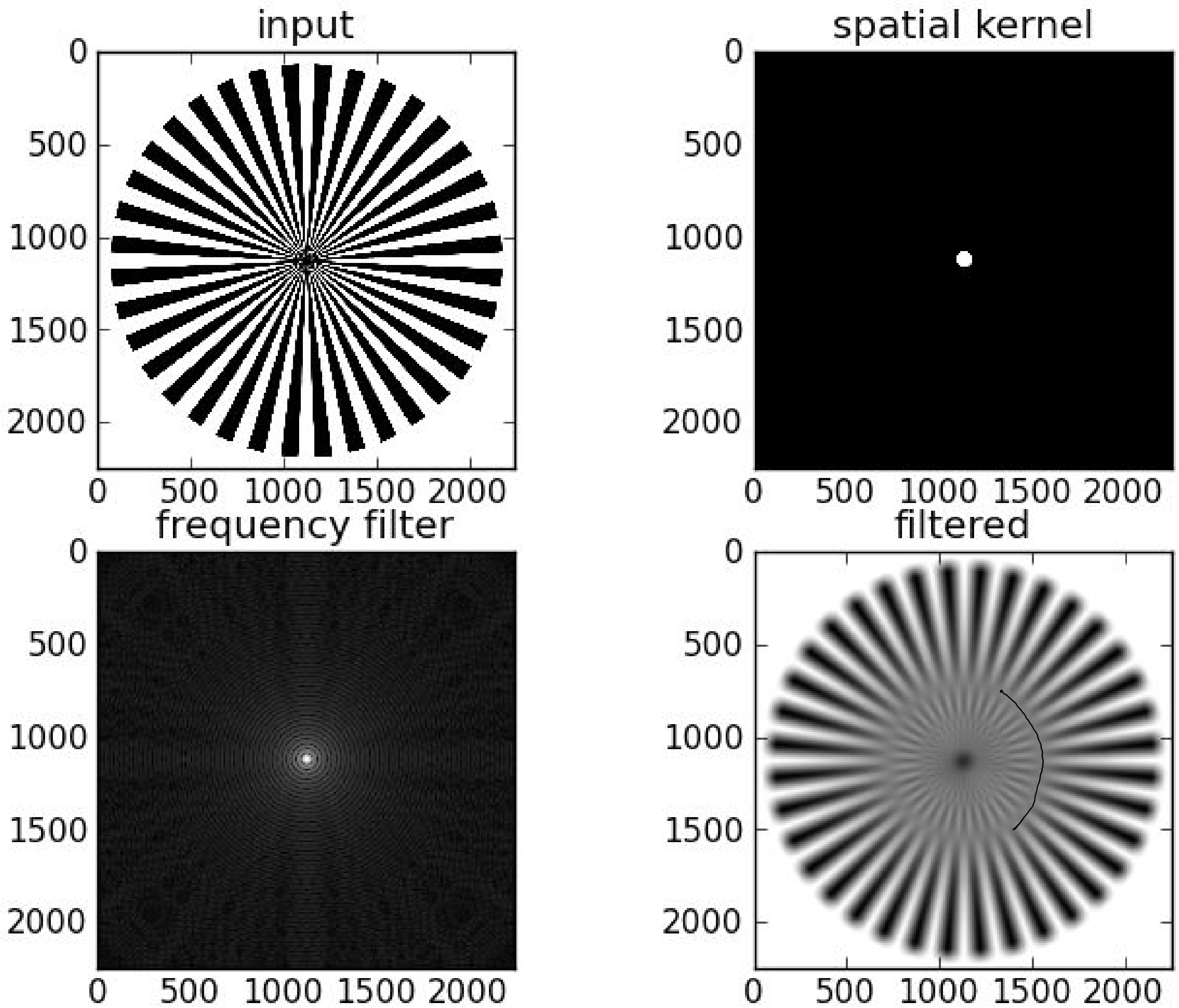
Siemens star



source: http://fotomagazin.de

Phase transfer function

describes how an oscillating signal changes in phase due to system



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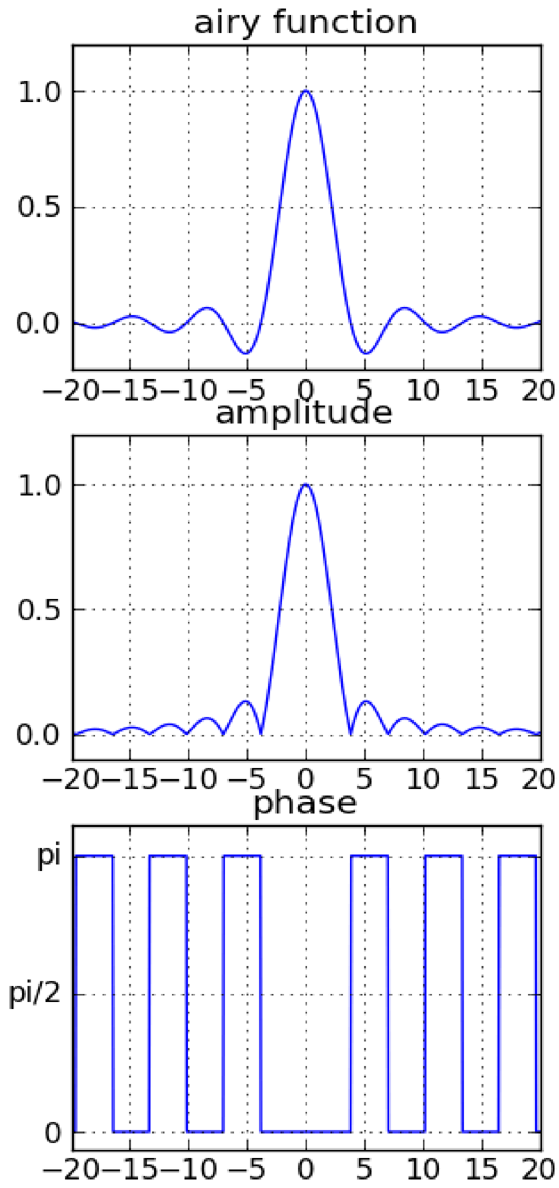
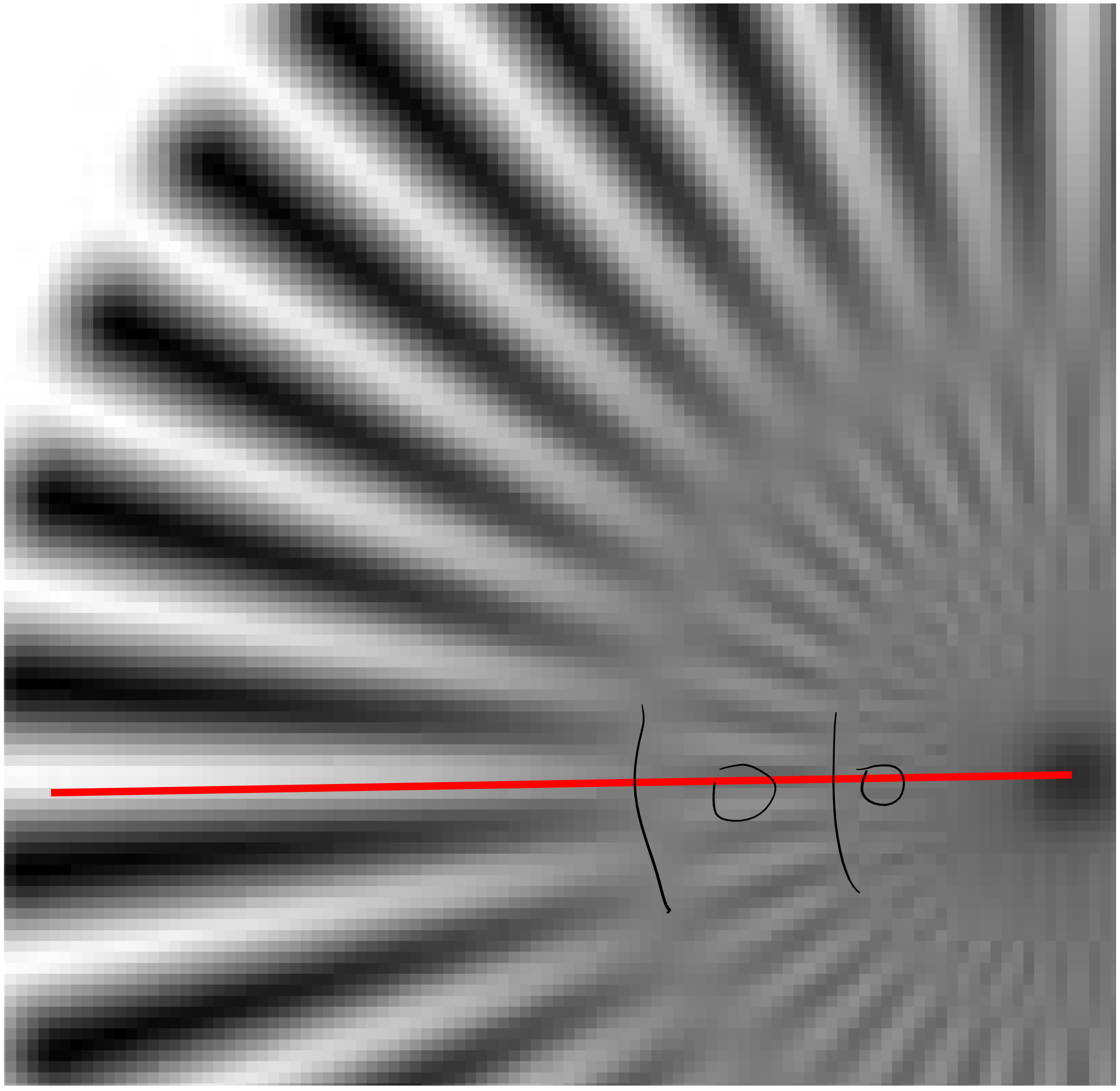
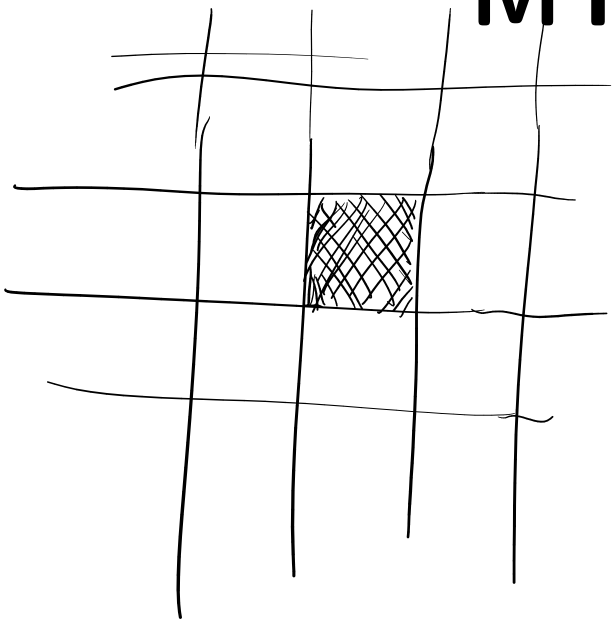
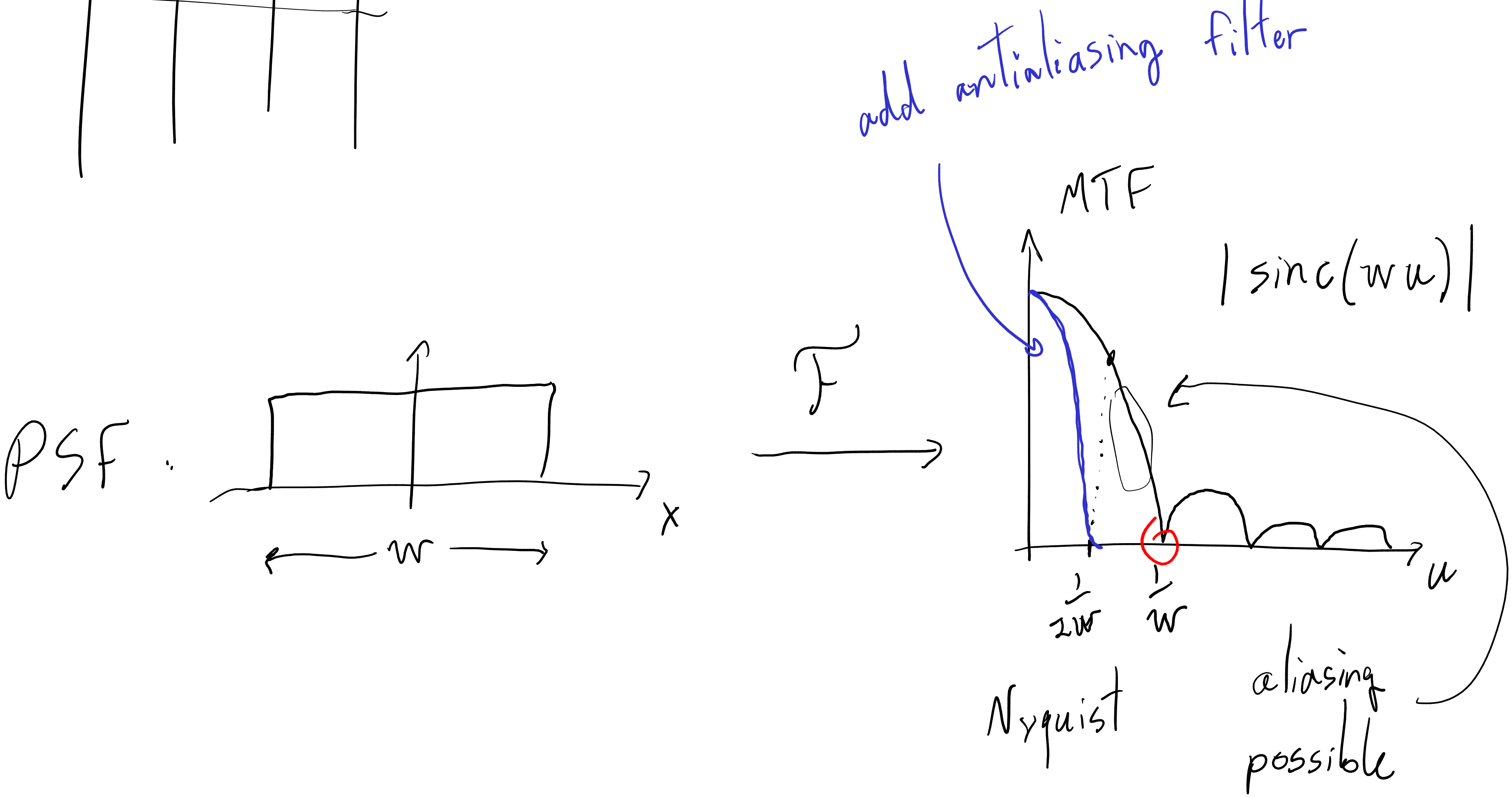




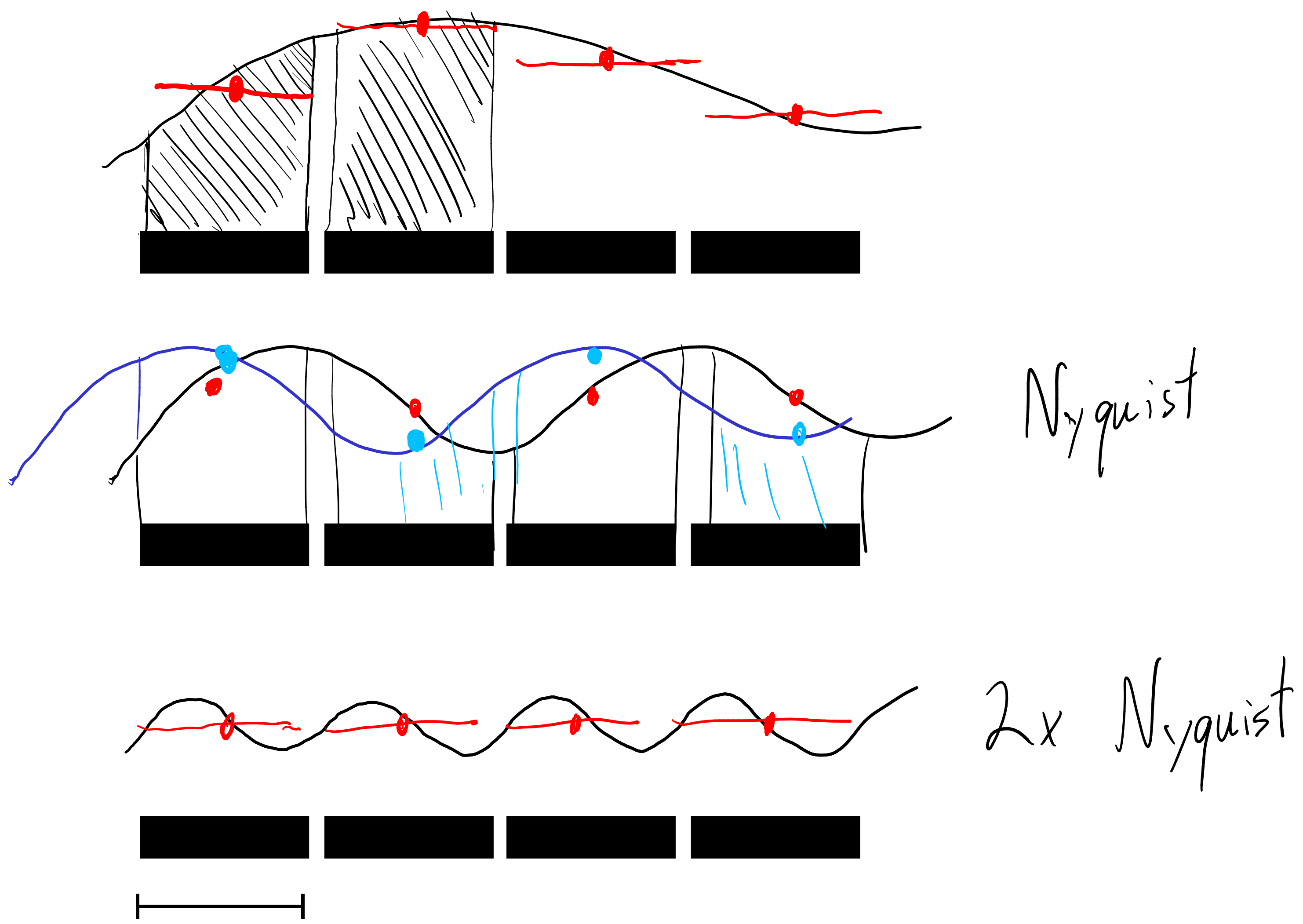
MTF of an ideal pixel





Pixel MTF

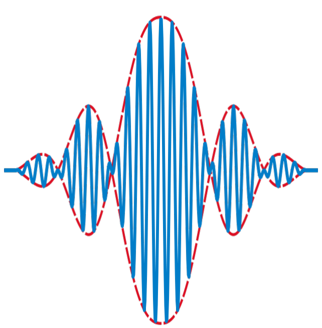
Modulation transfer function of a single detector pixel



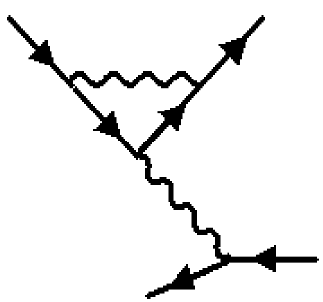
Imaging as a linear filter

$$\mathrm{Output}(u) = \mathrm{Input}(u) \cdot \mathrm{MTF}_{\mathrm{optics}}(u) \cdot \mathrm{MTF}_{\mathrm{detector}}(u) \cdot \mathrm{MTF}_{\mathrm{algorithm}}(u) \cdot \ldots$$

input



interaction



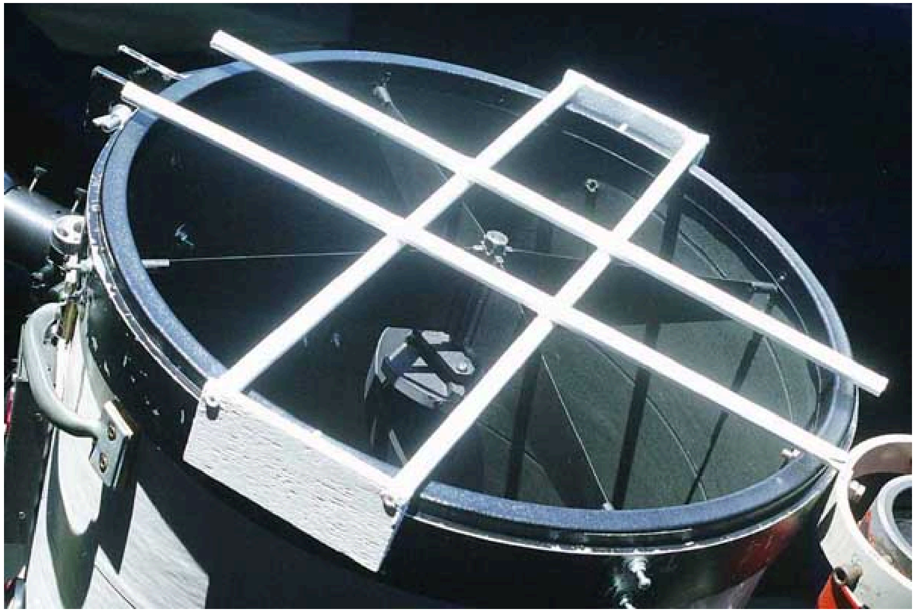
detection

analysis & processing

display

PSF examples

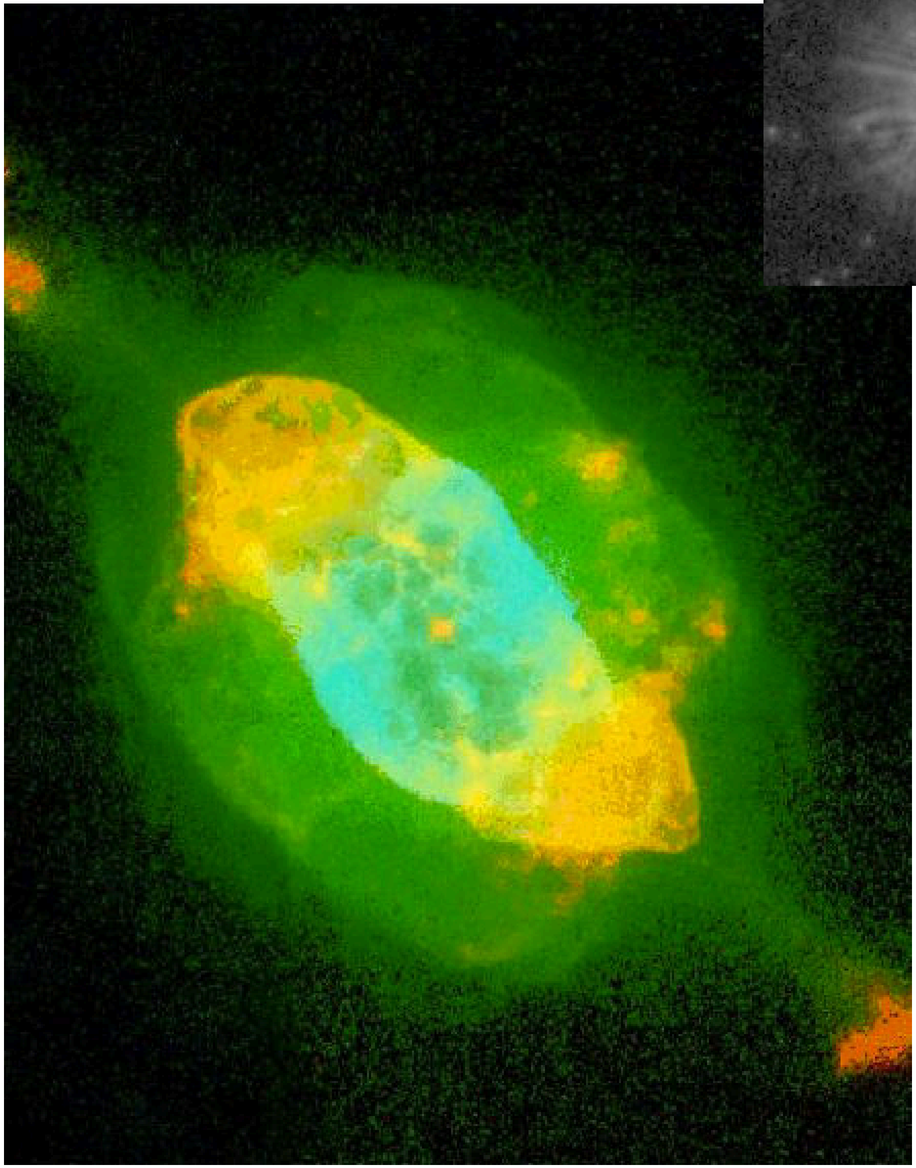
- images of isolated stars are essentially PSFs

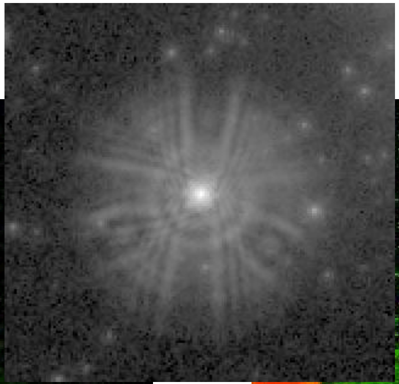


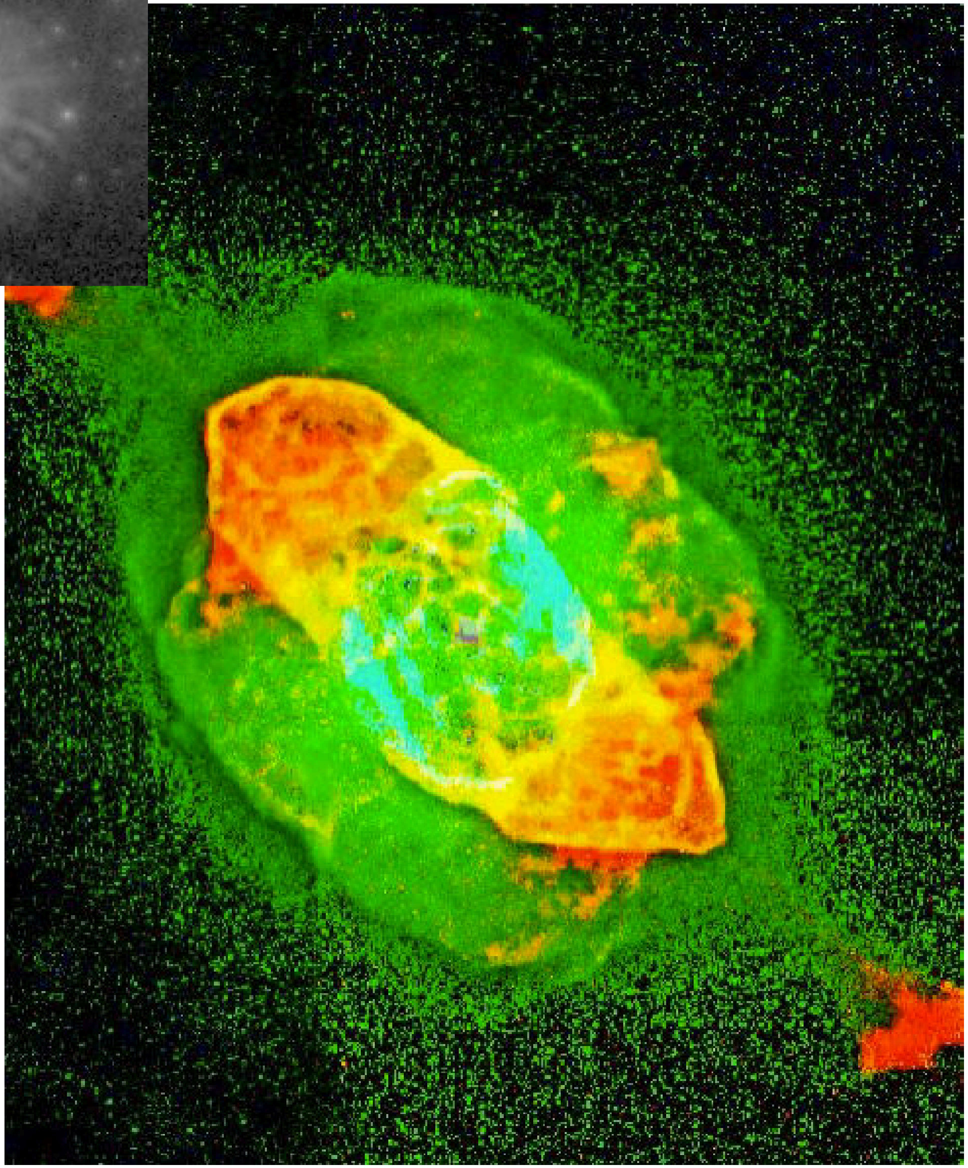
source: www.apod.nasa.gov

PSF examples

Hubble flawed mirror deconvolution (correction for spherical aberration)



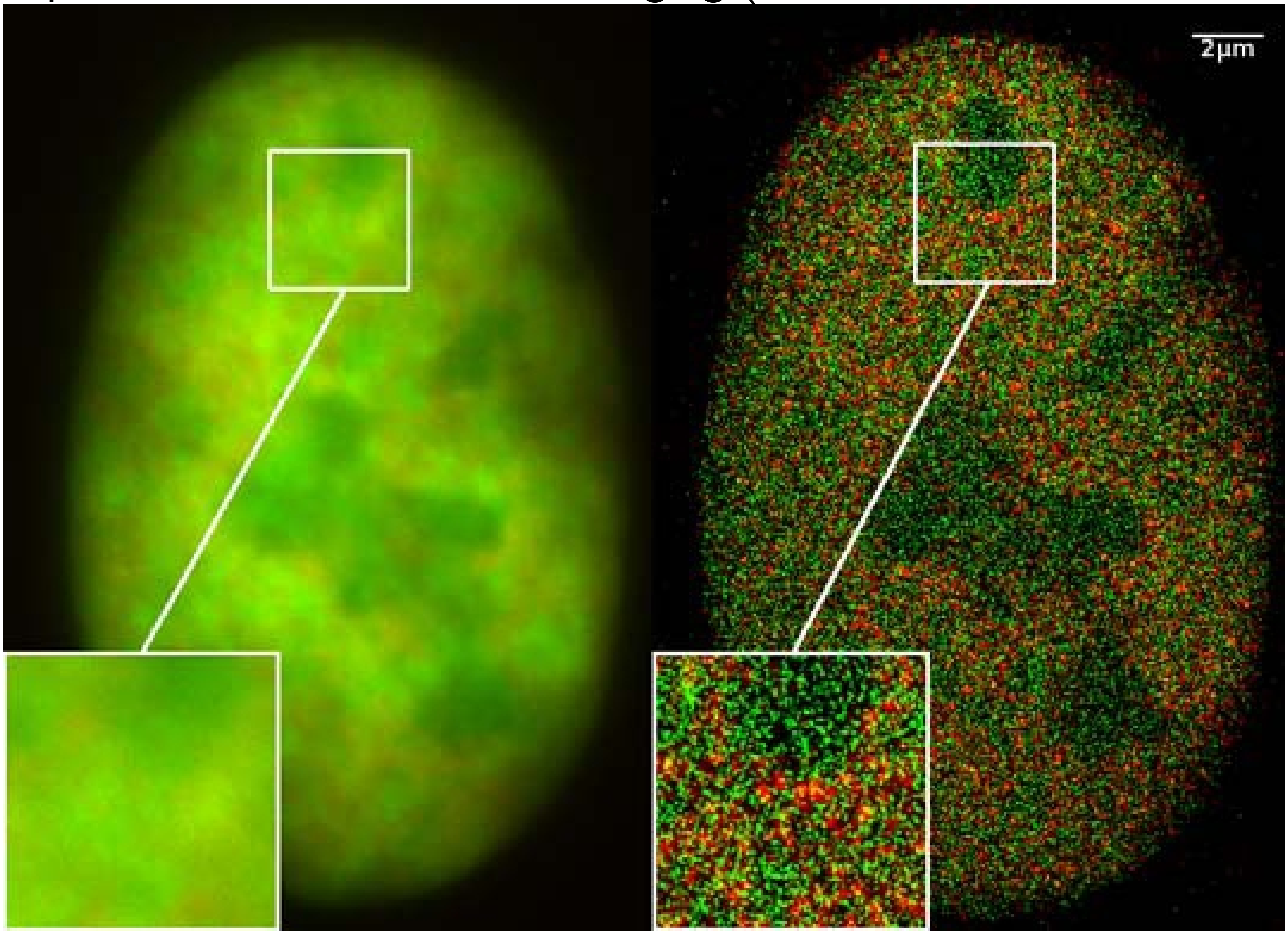




source: www.wikipedia.org

PSF examples

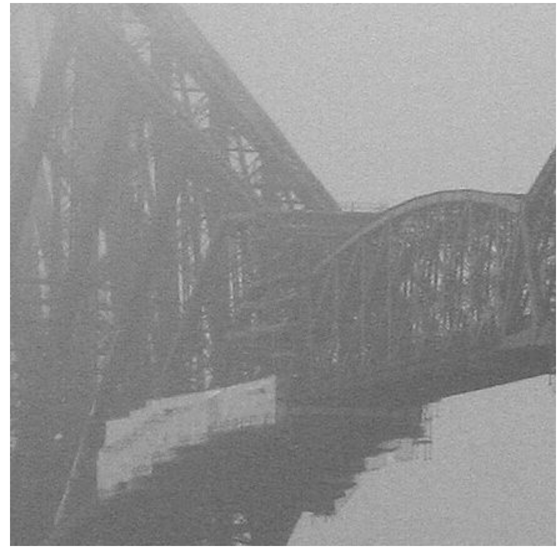
Super-resolution fluorescence imaging (STORM, STED, PALM, ...)

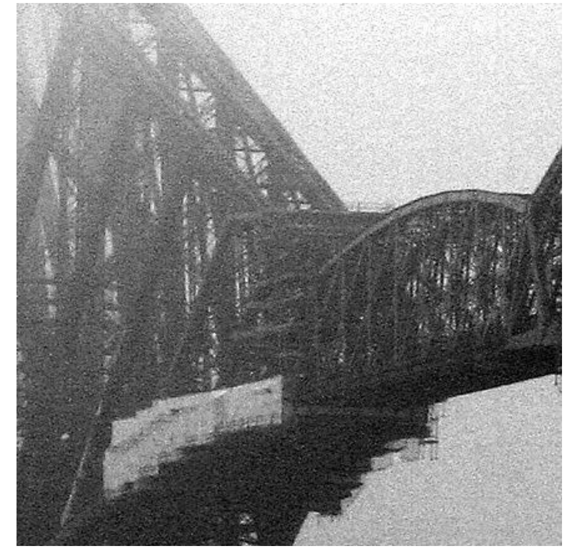


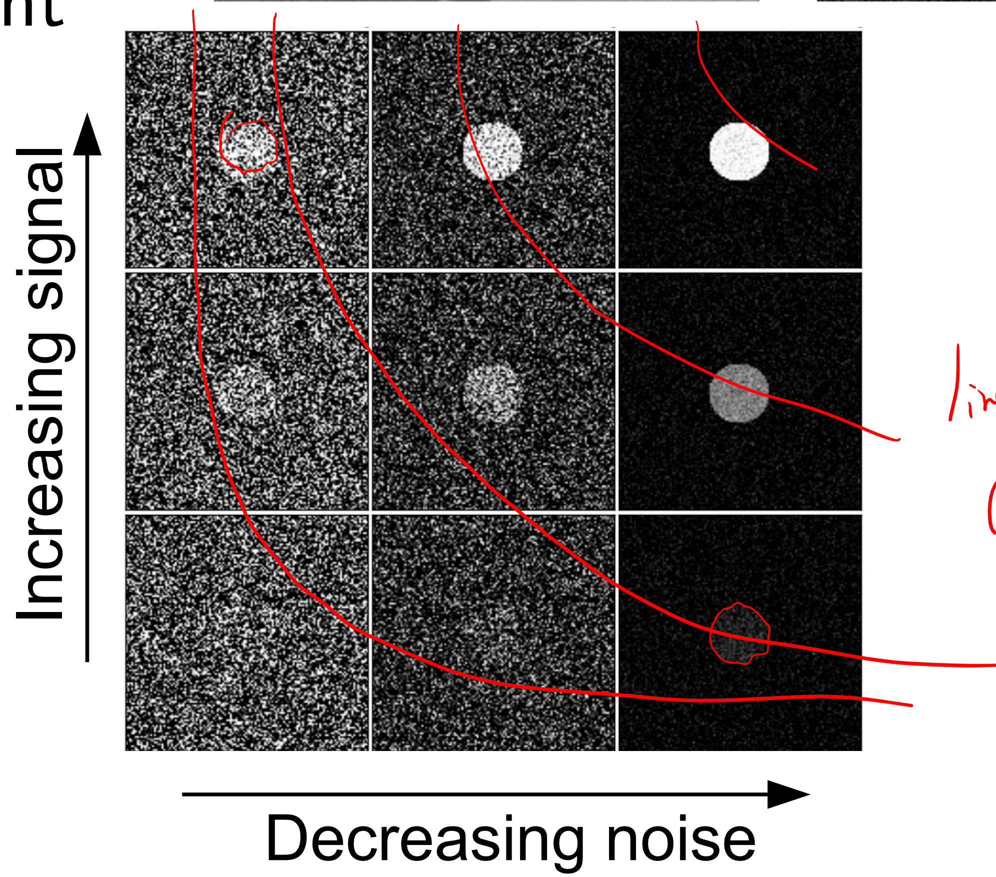
source: www.wikipedia.org

Contrast and noise

- Intensity operation: higher contrast, higher noise
- Contrast-to-noise remains constant







lines of equal CNR

Random variables

- random variable, sample space

X Ω probability of measuring x : $p(x)$

$$p(\Omega) = 1$$

- probability density function

probability density

$$p(a < x < b) = \int_a^b \rho(x)\,dx$$

$$\int_\Omega \rho(x)\,dx = 1$$

- expectation value

$$\langle f \rangle = \int_\Omega f(x)\,\rho(x)\,dx$$

special case: $\langle x \rangle = \int_\Omega x\,\rho(x)\,dx$ $= \mu$ "mean"

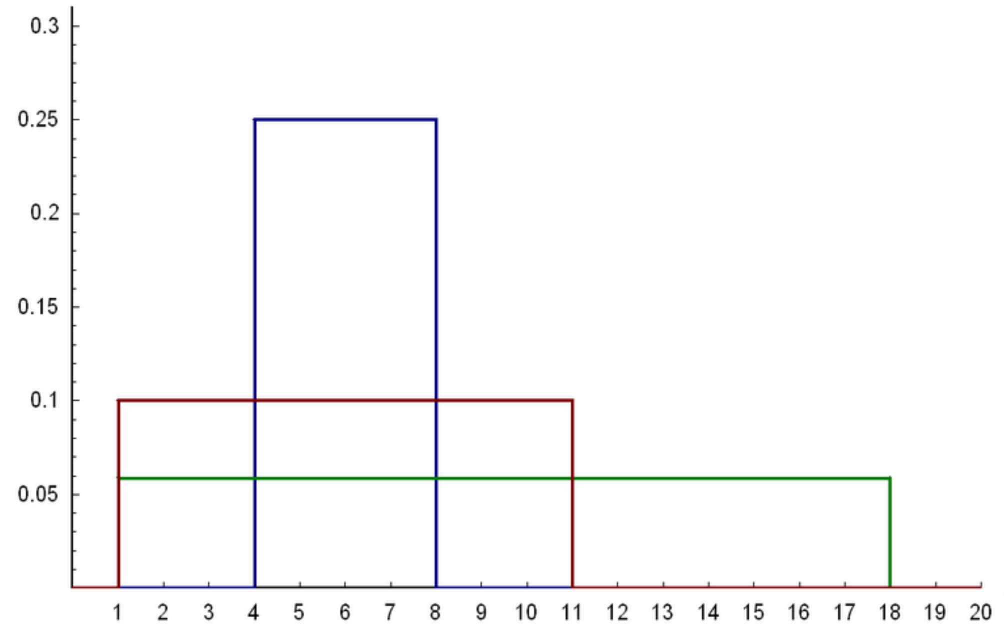
- variance

$$\mathrm{var}(x) = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

Uniform distribution

- probability density function

$$\rho(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \end{cases}$$



- expectation value (mean)

$$\langle x \rangle = \frac{1}{2}(a+b)$$

- variance

$$\text{var}\, x = \frac{(b-a)^2}{12}$$

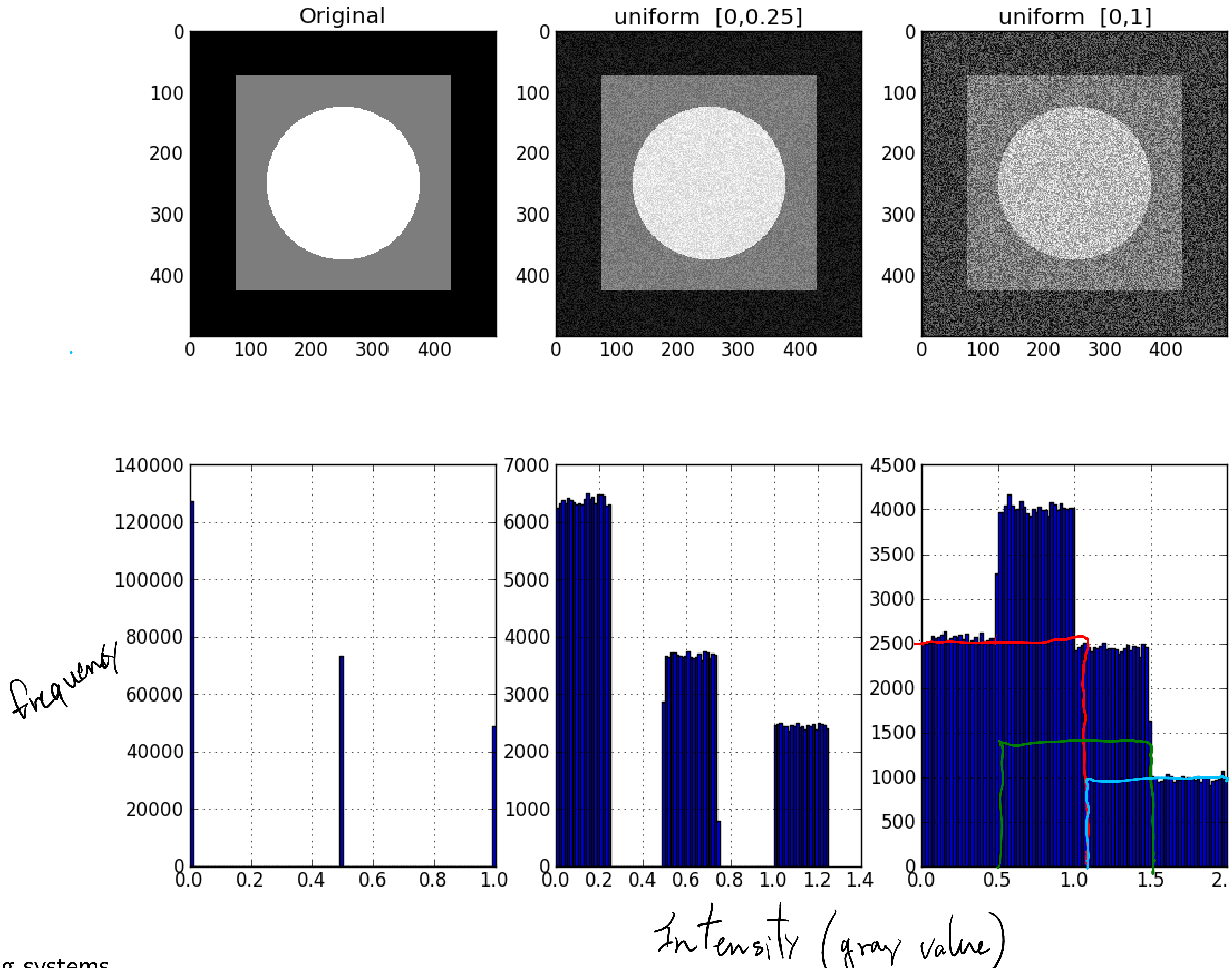
$$\text{var}\, x = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \int_a^b x^2 \rho(x)\, dx - \frac{1}{4}(a+b)^2$$

$$= \frac{x^3}{3}\Big|_a^b \frac{1}{b-a} \ldots\ldots$$

- occurrence not very common in imaging, but useful to build other probability distributions

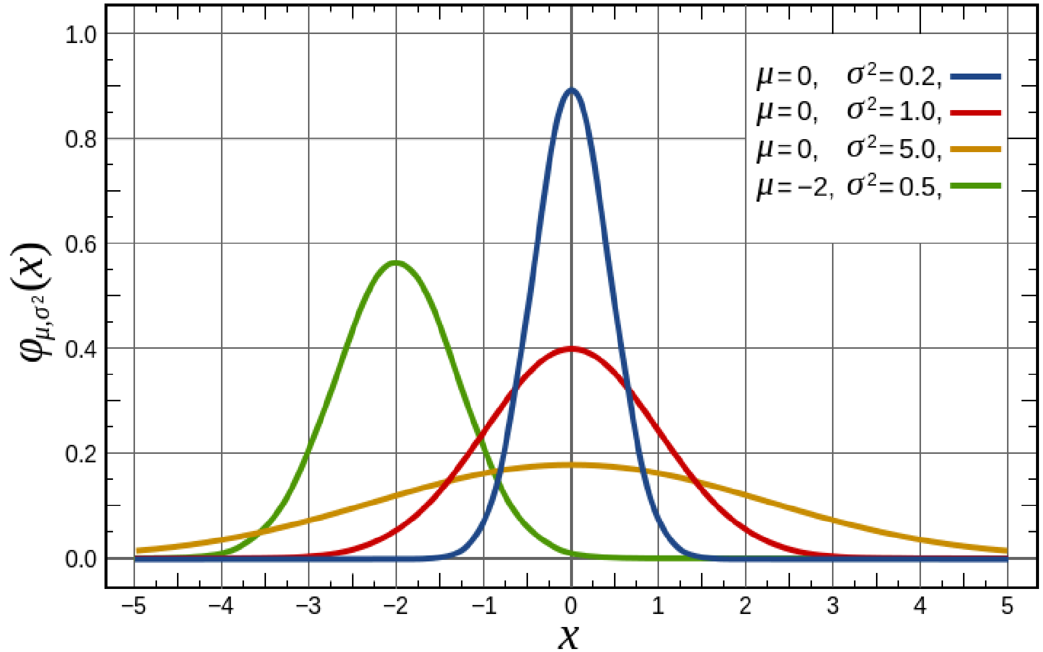
Uniform distribution



Gaussian distribution

- probability density function

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$



- expectation value

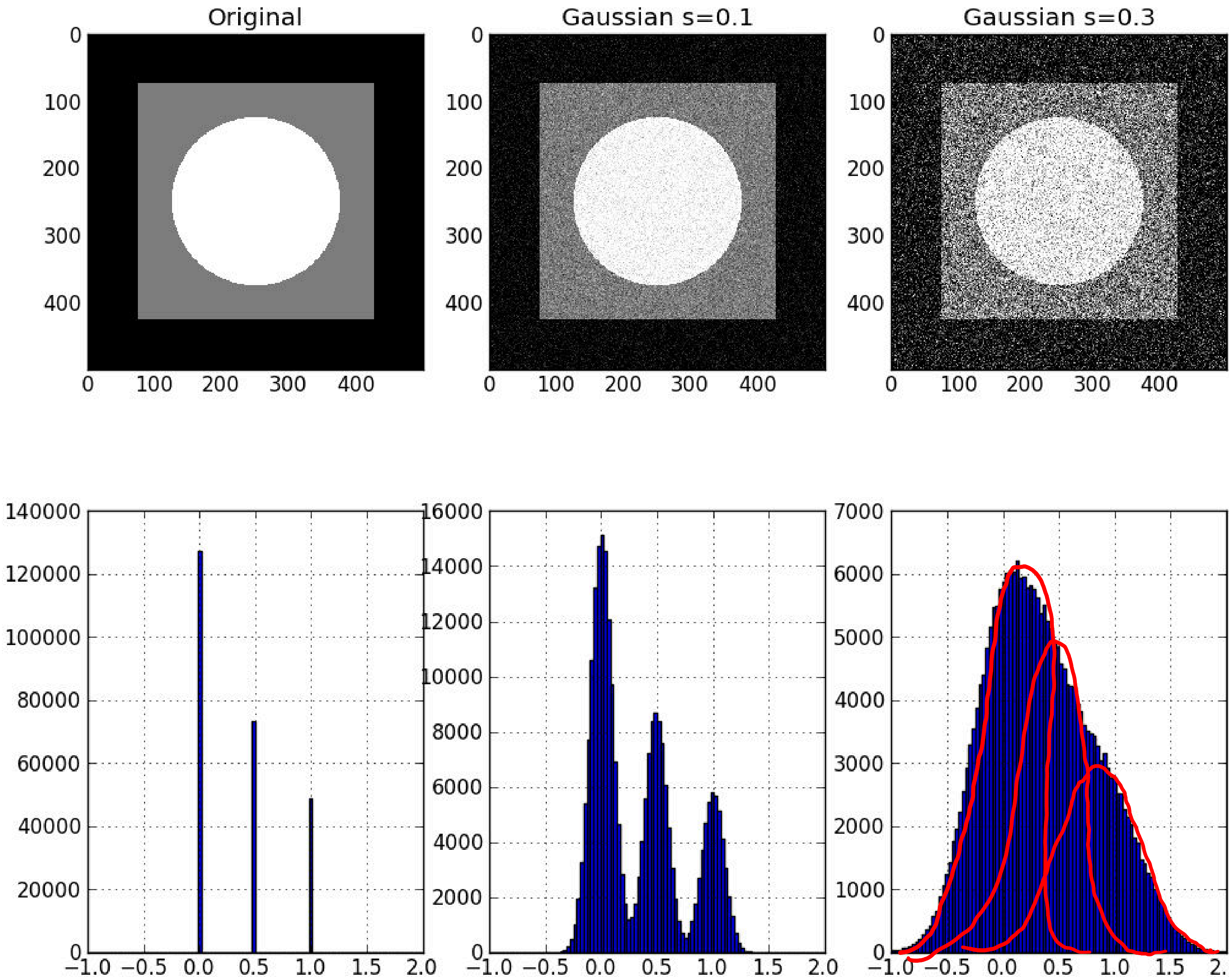
$$\langle x \rangle = \mu$$

- variance

$$\mathrm{var}\, x = \sigma^2$$

- occurrence very common (central limit theorem)

Gaussian distribution

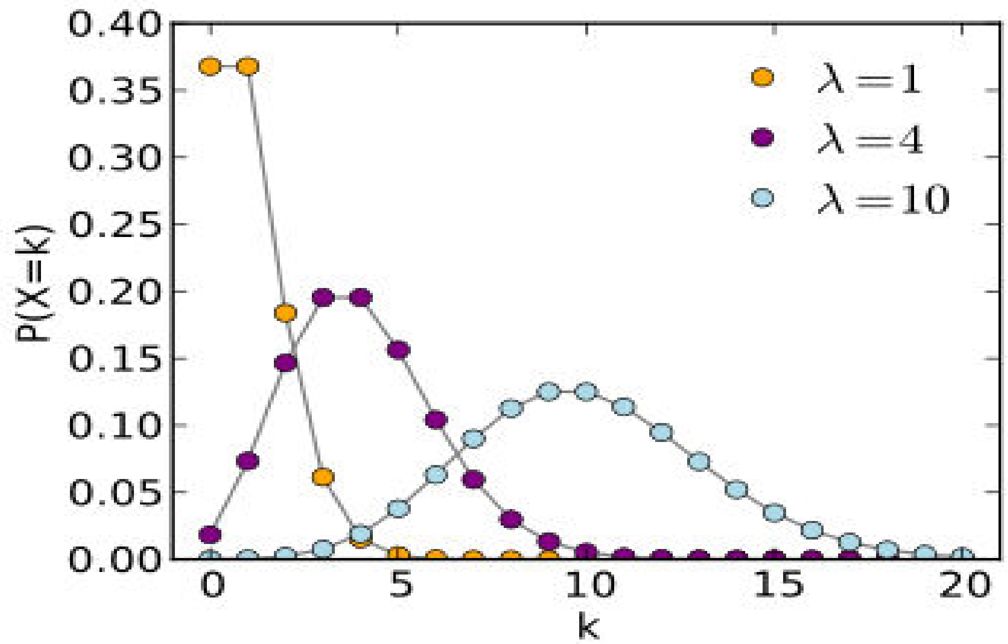


Poisson distribution

- probability mass function

$$p(n) = \frac{1}{n!} \lambda^n e^{-\lambda}$$

λ : only parameter



- expectation value

$$\langle n \rangle = \lambda$$

S/N ratio : $\frac{\langle n \rangle}{\sqrt{\mathrm{var}\, n}} = \frac{\lambda}{\sqrt{\lambda}} = \sqrt{\lambda}$

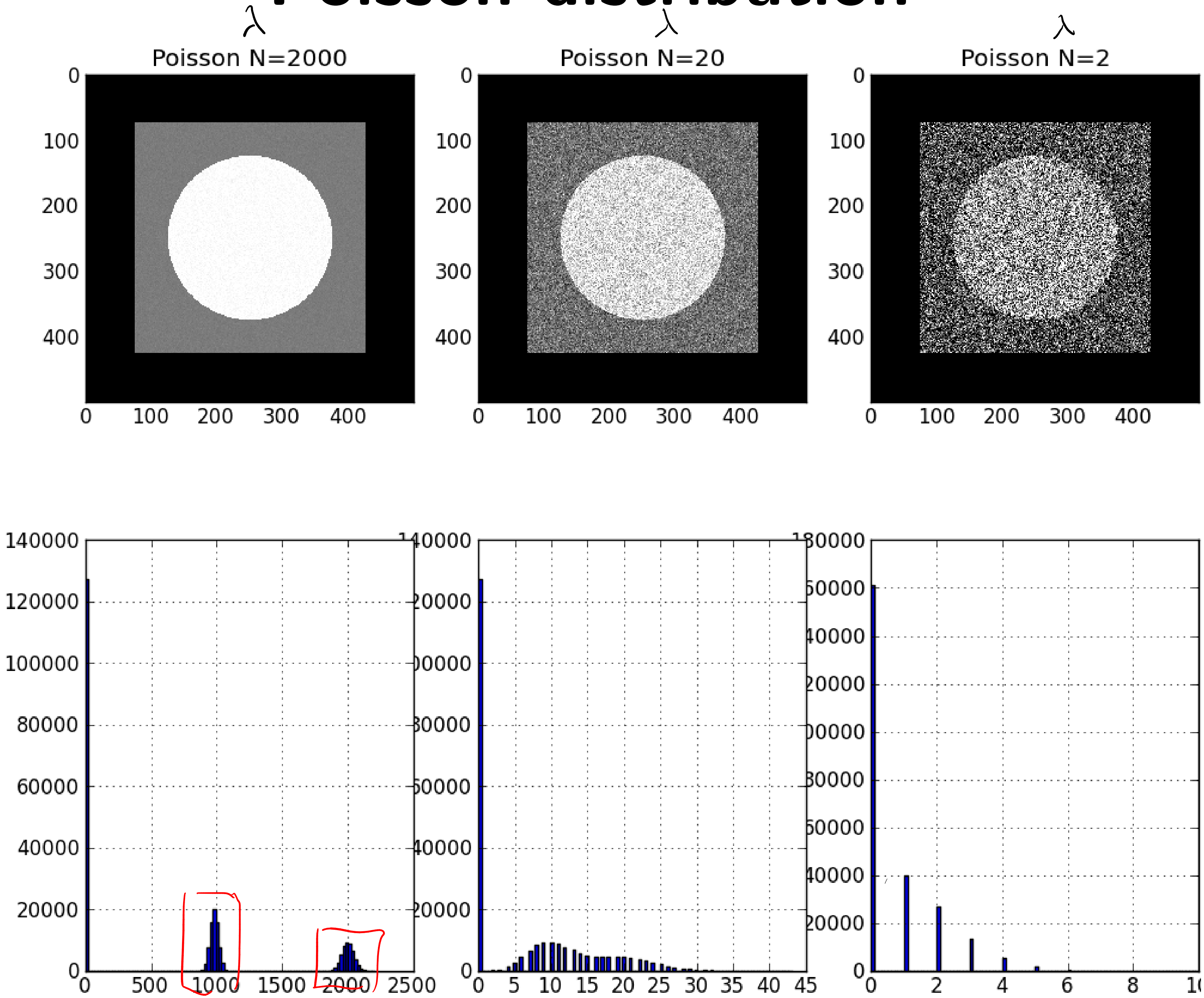
- variance

$$\mathrm{var}\, n = \lambda$$

- occurrence counting process (photons, electrons, ...)

"shot noise"

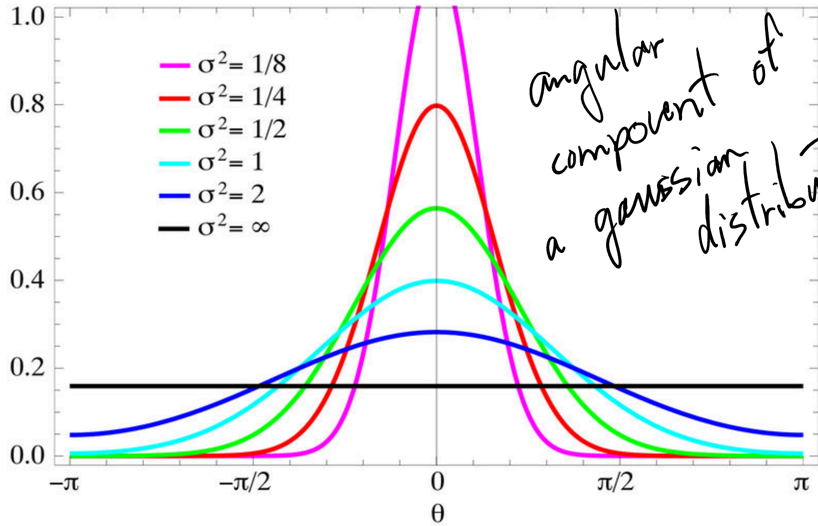
Poisson distribution



Poisson distribution

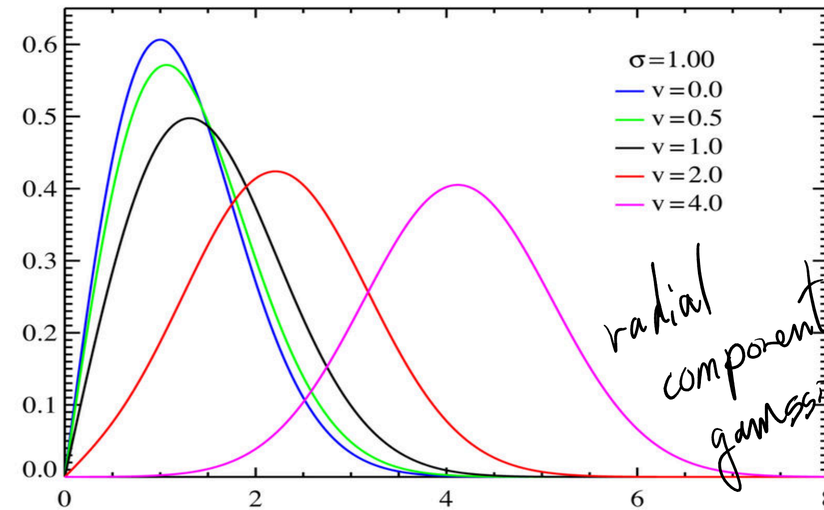
Many other distributions

Wrapped normal distribution



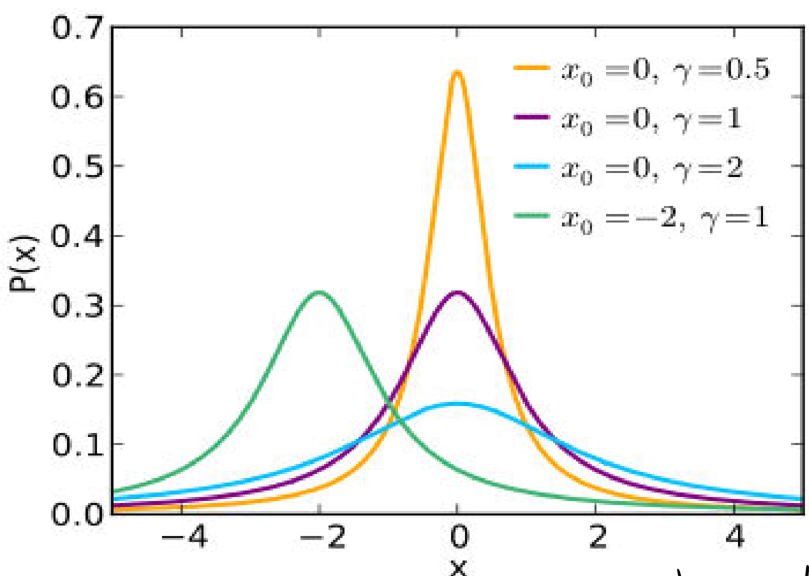
angular component of a gaussian distribution

Rice distribution

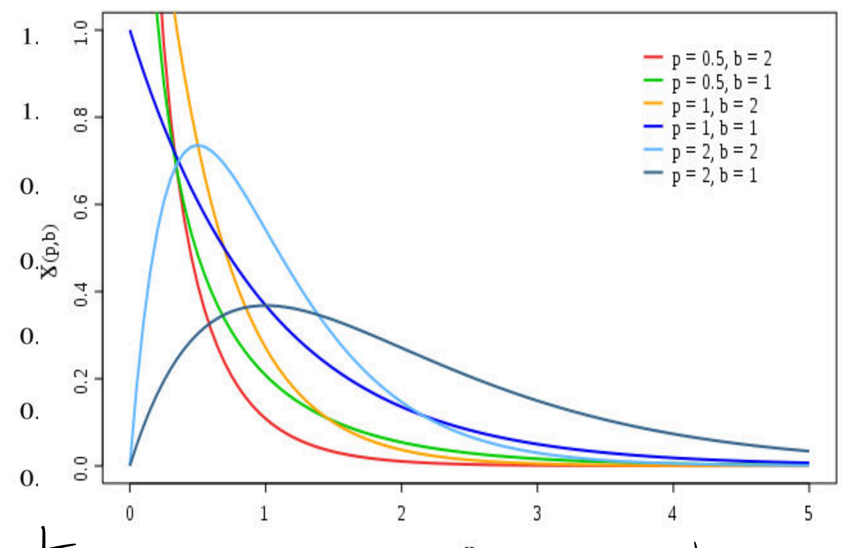


radial component of gaussian distribution

Lorentz distribution



Gamma distribution



"long tail" distributions   decay as power law

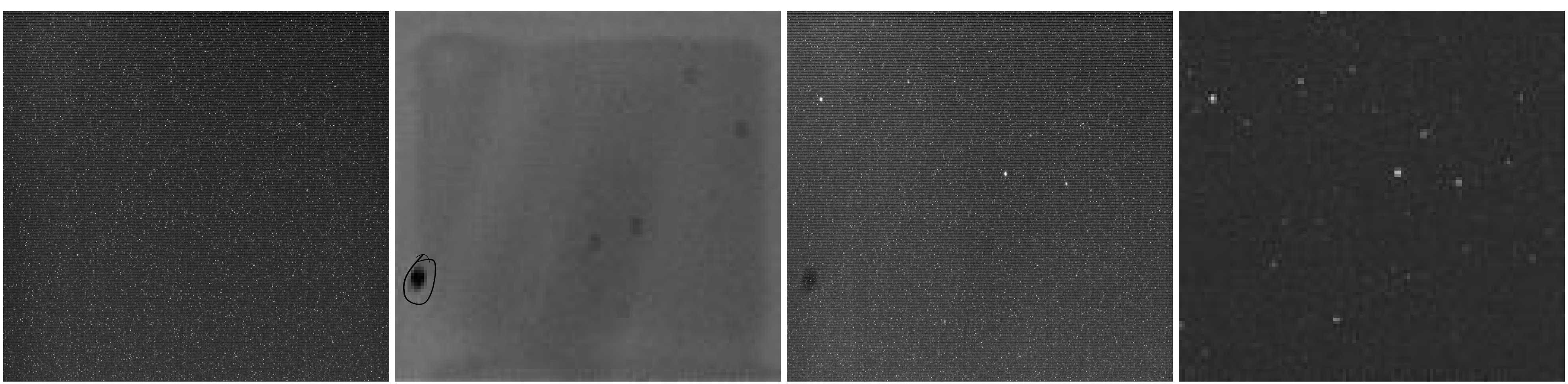
Detector noise (CCD)

- Various sources:
  - shot noise (photon statistics, Poisson)
  - dark current (thermal electronic fluctuations in semiconductor, Poisson)
  - readout noise (fluctuations during amplification and digitization, Gauss)
  - many other imperfections ...
- dark frame measures detector noise, hot pixels, dead pixels
- bright frame measures gain differences and imperfections (dust, etc)

"flat"

uniform illumination

dark frame *bright frame* *raw image* *calibrated image*



source: H. Raab, Johannes-Kepler-Observatory, Linz

Correlation & Convolution

* Convolution: $f * g = \int_{-\infty}^{\infty} f(x') \, g(x - x') \, dx'$

* Convolution theorem $\mathcal{F}\{f * g\} = F \cdot G$

* Correlation $f \circledast g = \int_{-\infty}^{\infty} f^*(x') \, g(x + x') \, dx'$

$$f \circledast g = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} F^*(u) \, e^{-2\pi i u x'} du \int_{-\infty}^{\infty} G(u') \, e^{2\pi i u'(x + x')} du'$$

$$= \iint_{-\infty}^{\infty} du \, du' \, F^*(u) \, G(u') \, e^{2\pi i u' x} \underbrace{\int dx' \, e^{2\pi i x'(u' - u)}}_{\delta(u' - u)}$$

$$= \int_{-\infty}^{\infty} du \, F^*(u) \, G(u) \, e^{2\pi i u x} \Rightarrow \mathcal{F}\{f \circledast g\} = F^* G$$

Noise power spectrum

NPS: "noise power spectrum"

- power spectrum of pure noise image

$$NPS = \left\langle \left| \mathcal{F}\{n(x,y)\} \right|^2 \right\rangle$$

ensemble average

multivariate random variable

$$\mathcal{F}\{n(x,y)\} = N(u,v)$$

- connection to auto-correlation

$$|N(u,v)|^2 = N^*(u,v)\, N(u,v)$$

$$\mathcal{F}^{-1}\left\{\left\langle |N(u,v)|^2 \right\rangle\right\} = \langle n \circledast n \rangle = \text{auto-correlation of } n$$

Procedure for noise characterization:

1) measure multiple realizations of the random variable $n(x,y)$

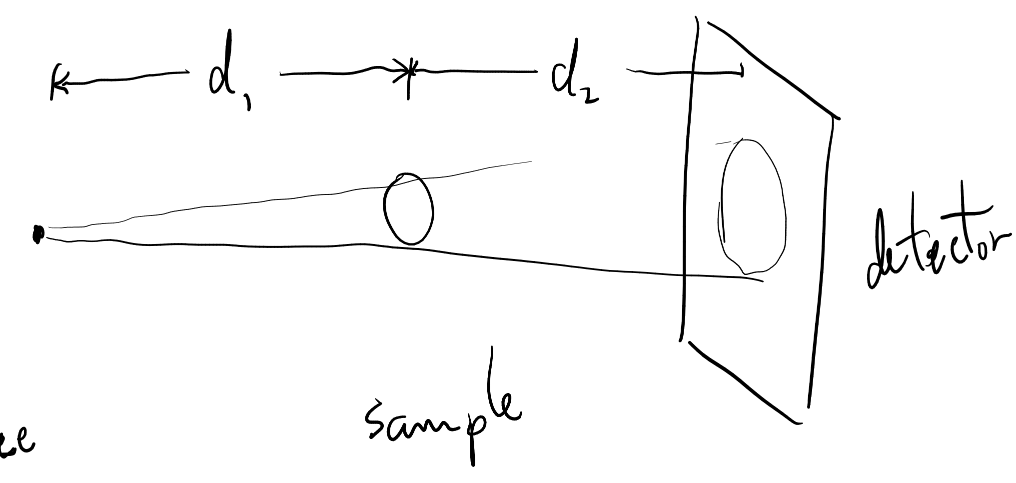
   ↳ take many dark frames $n_i(x,y)$

   $0 < i \leq M$

2) $N_i(u,v) = \mathcal{F}\{n_i(x,y)\}$

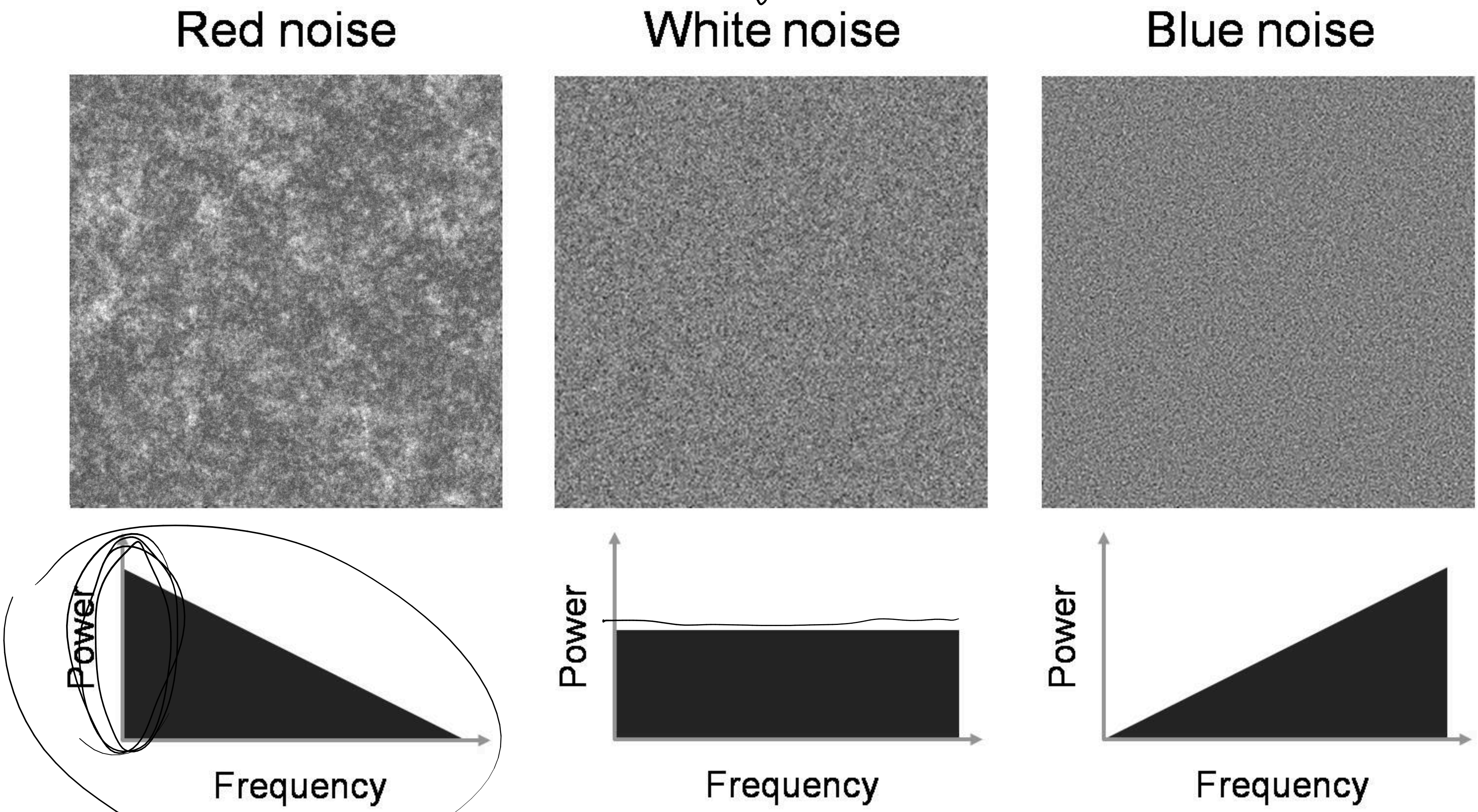
3) $\langle |N(u,v)|^2 \rangle \approx \frac{1}{M} \sum_i |N_i(u,v)|^2 = NPS$

4) $\mathcal{F}^{-1}\{NPS\} \longrightarrow$ estimate of noise autocorrelation



Noise power spectrum

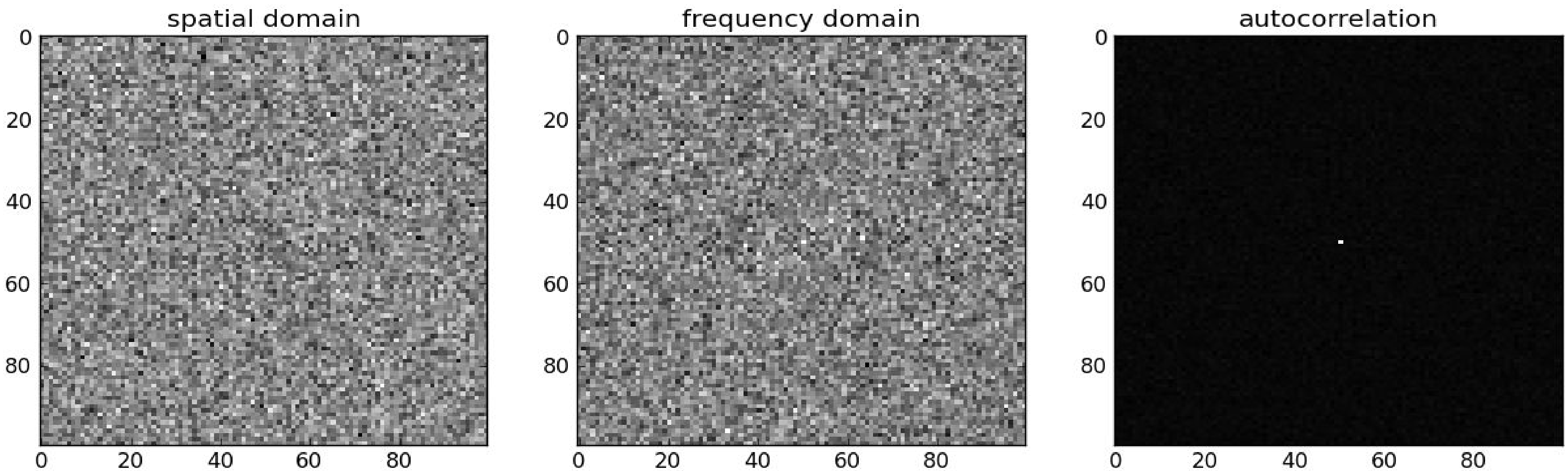
no spatial correlation



NPS

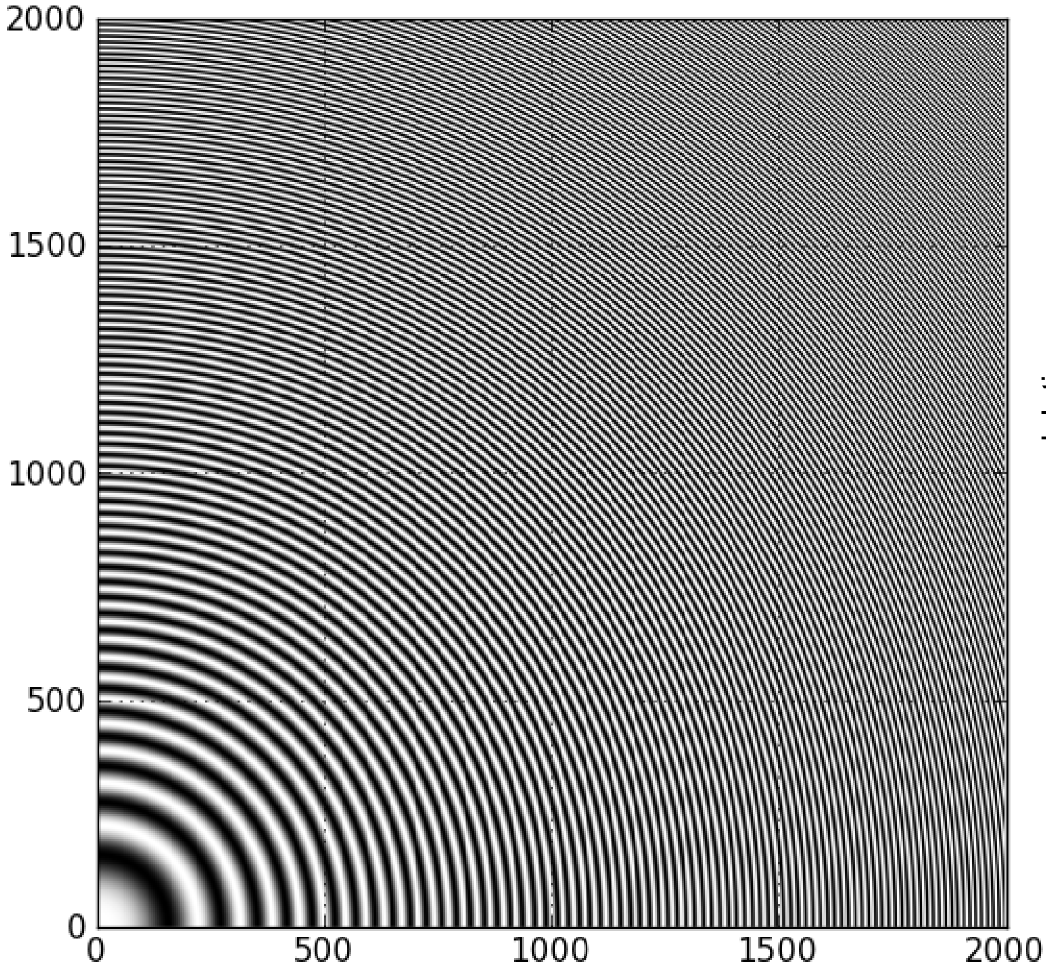
source: http://scien.stanford.edu/pages/labsite/2008/psych221/projects/08/AdamWang/project_report.htm

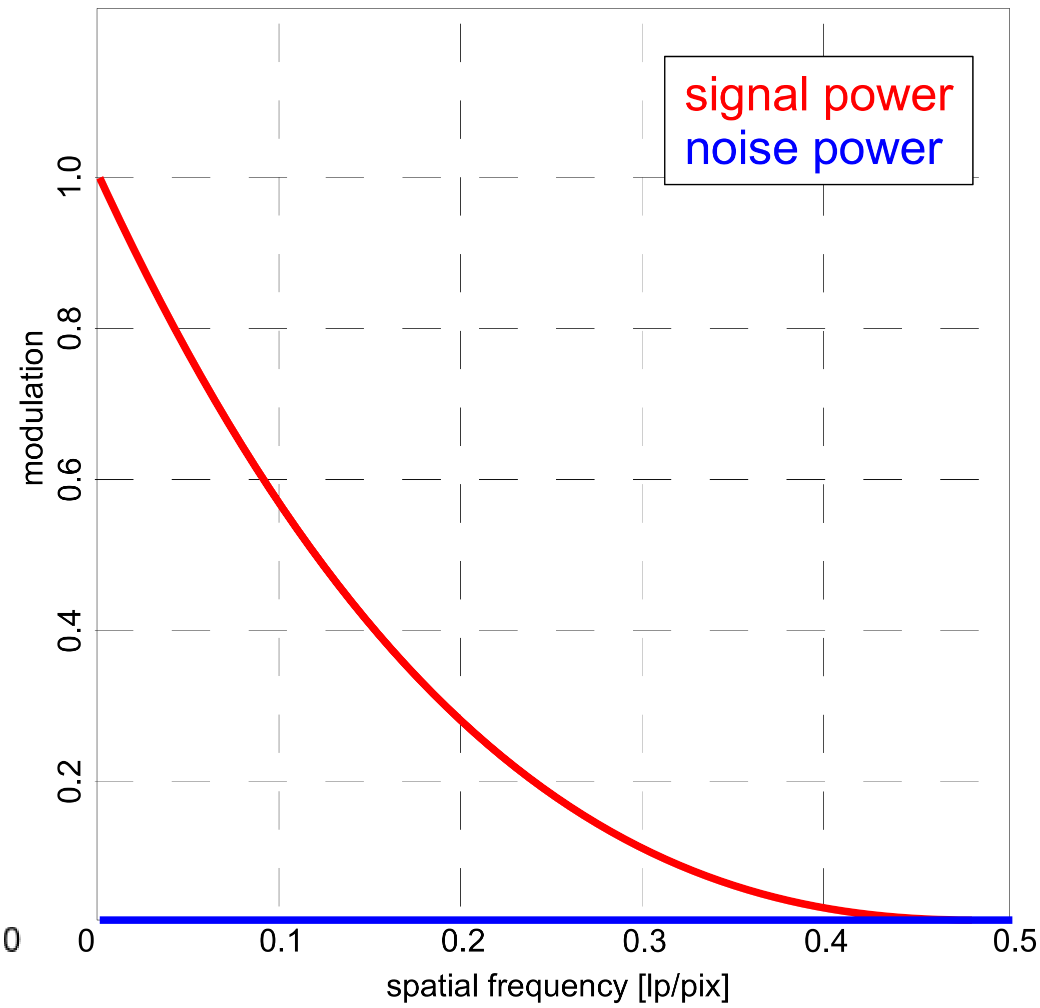
White noise



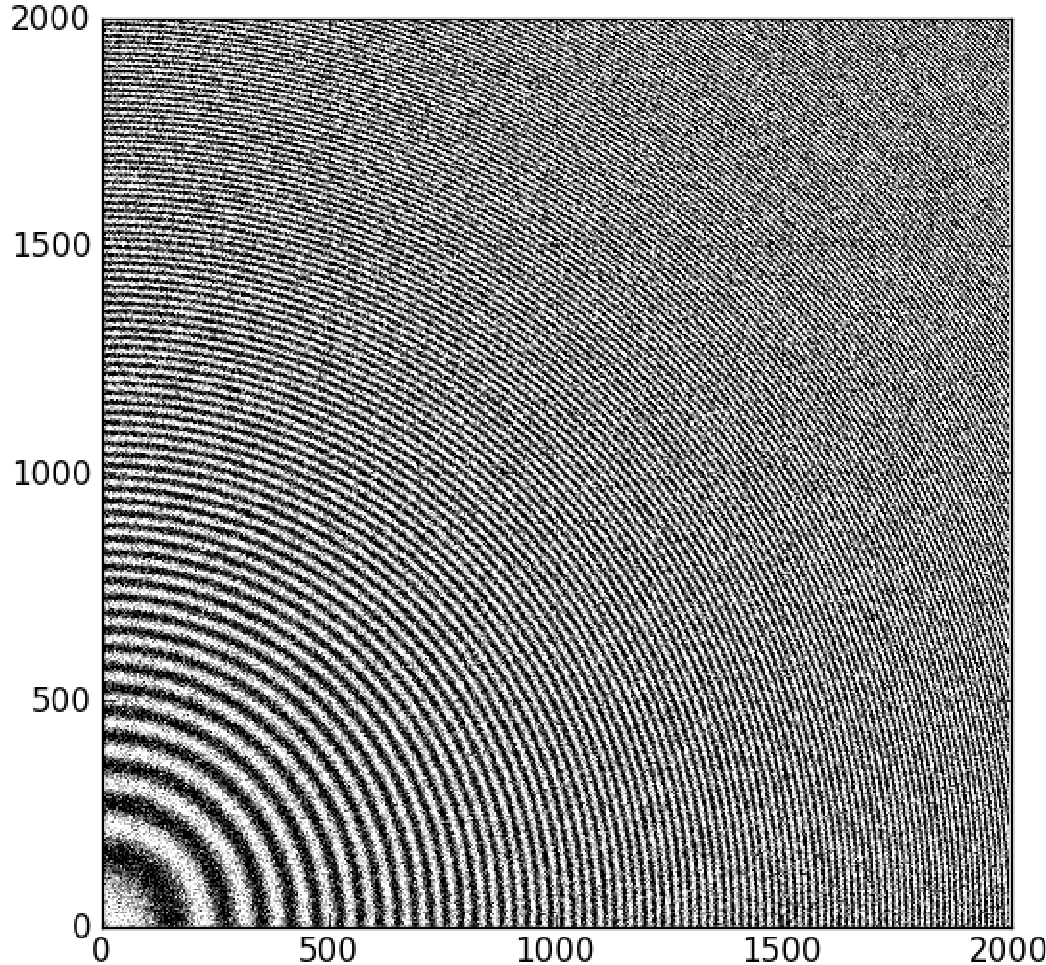
- white noise in spatial domain equals white noise in frequency domain
- white noise is perfectly uncorrelated
- all other types of noise are correlated to some degree
- white noise is an idealization

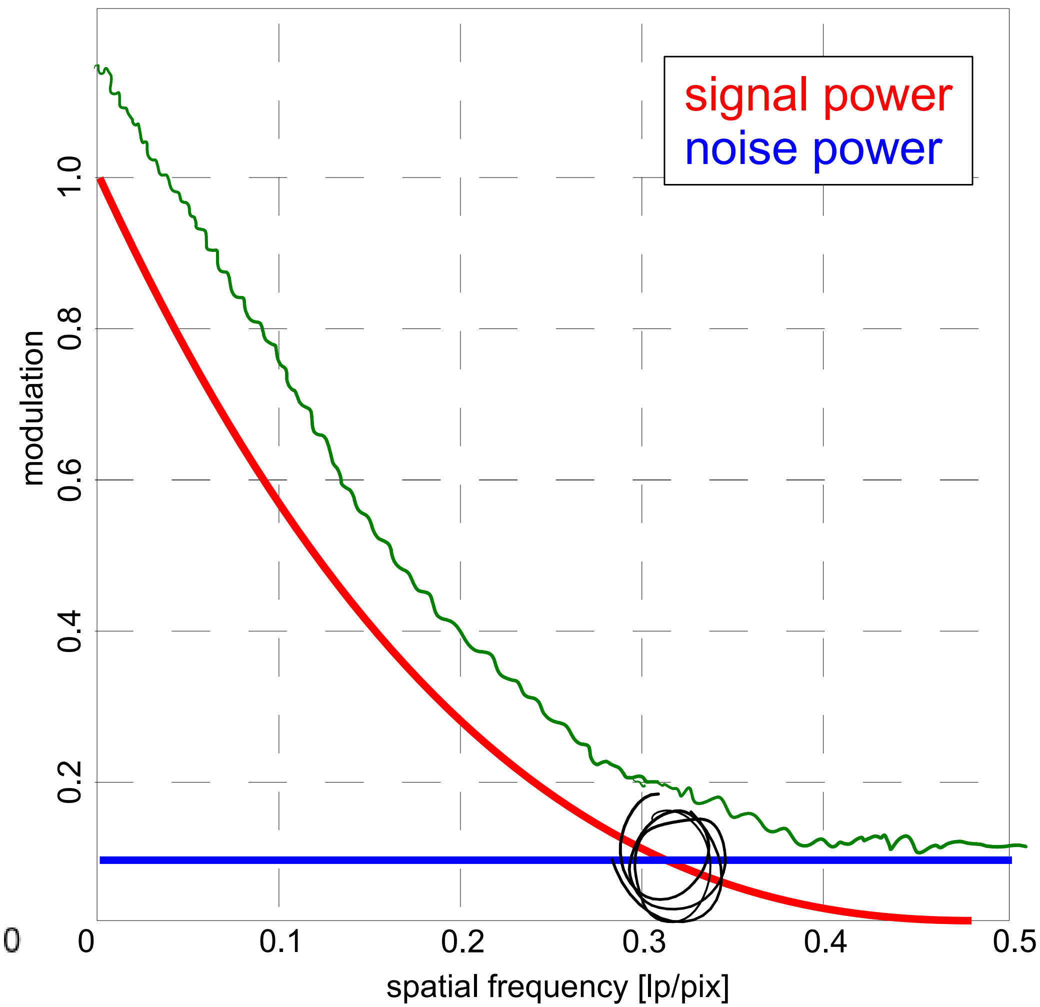
Signal power vs. noise power



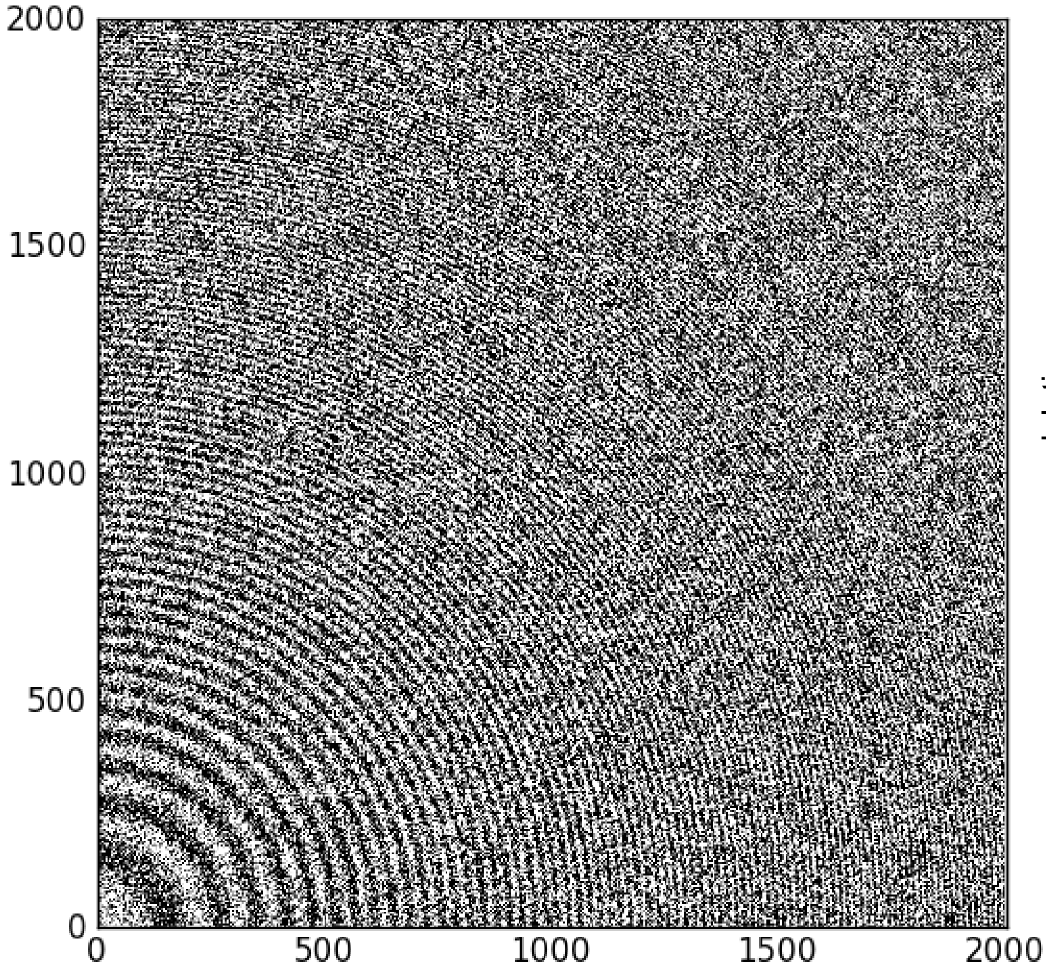


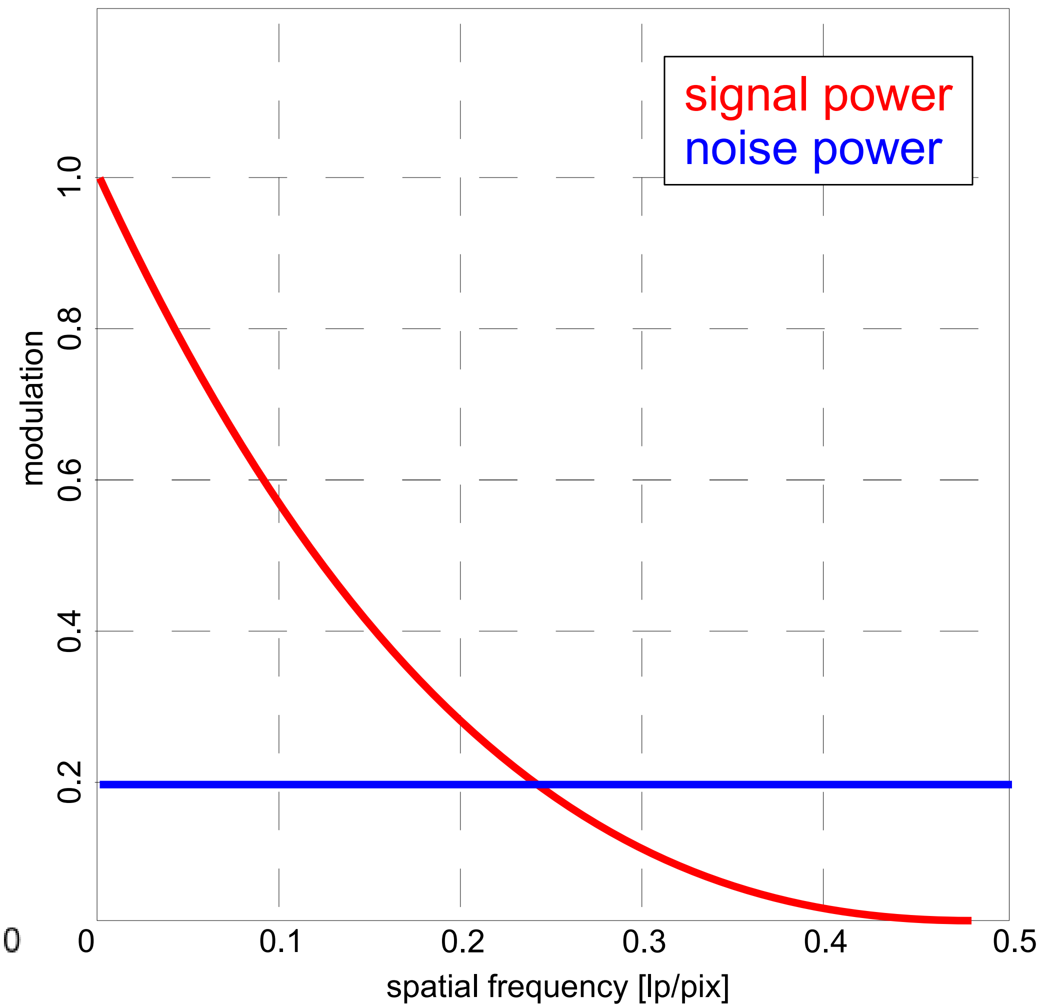
Signal power vs. noise power



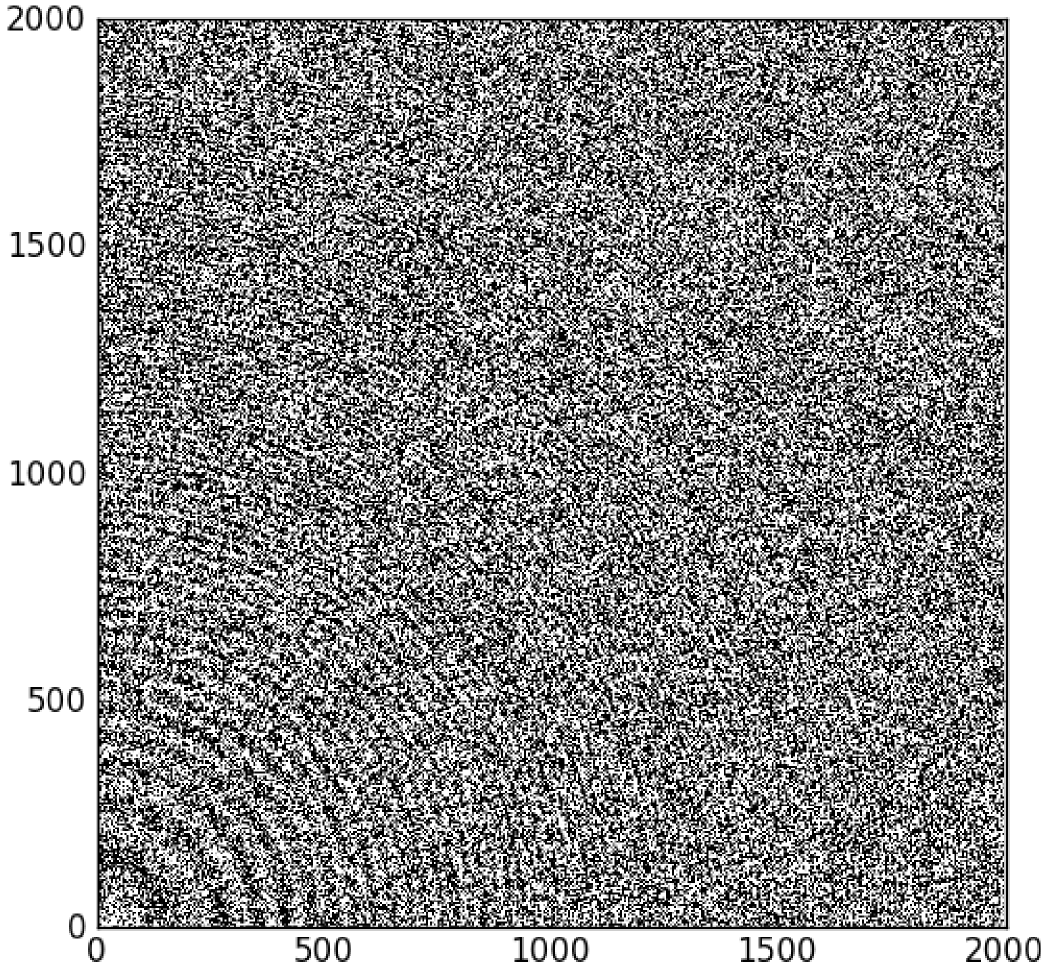


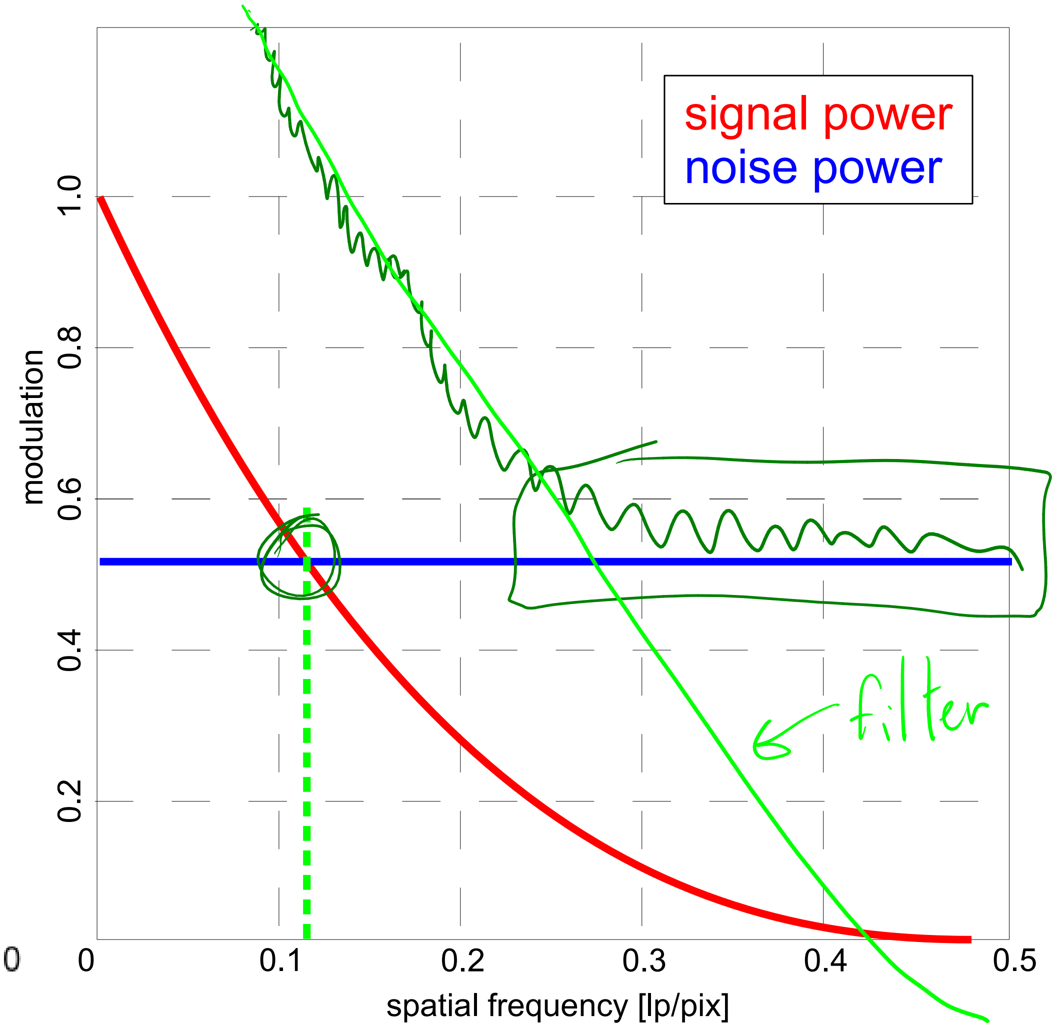
Signal power vs. noise power





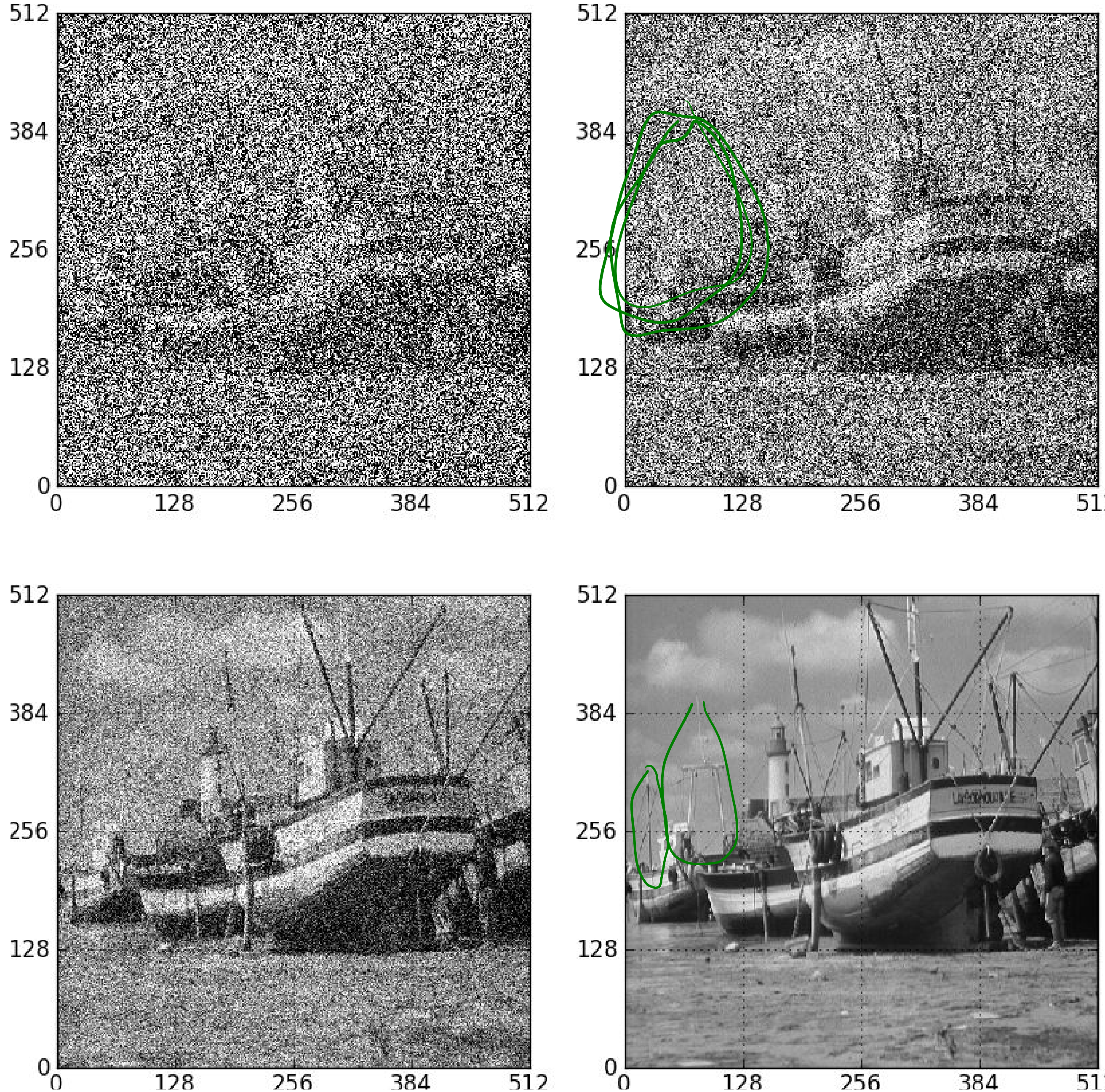
Signal power vs. noise power





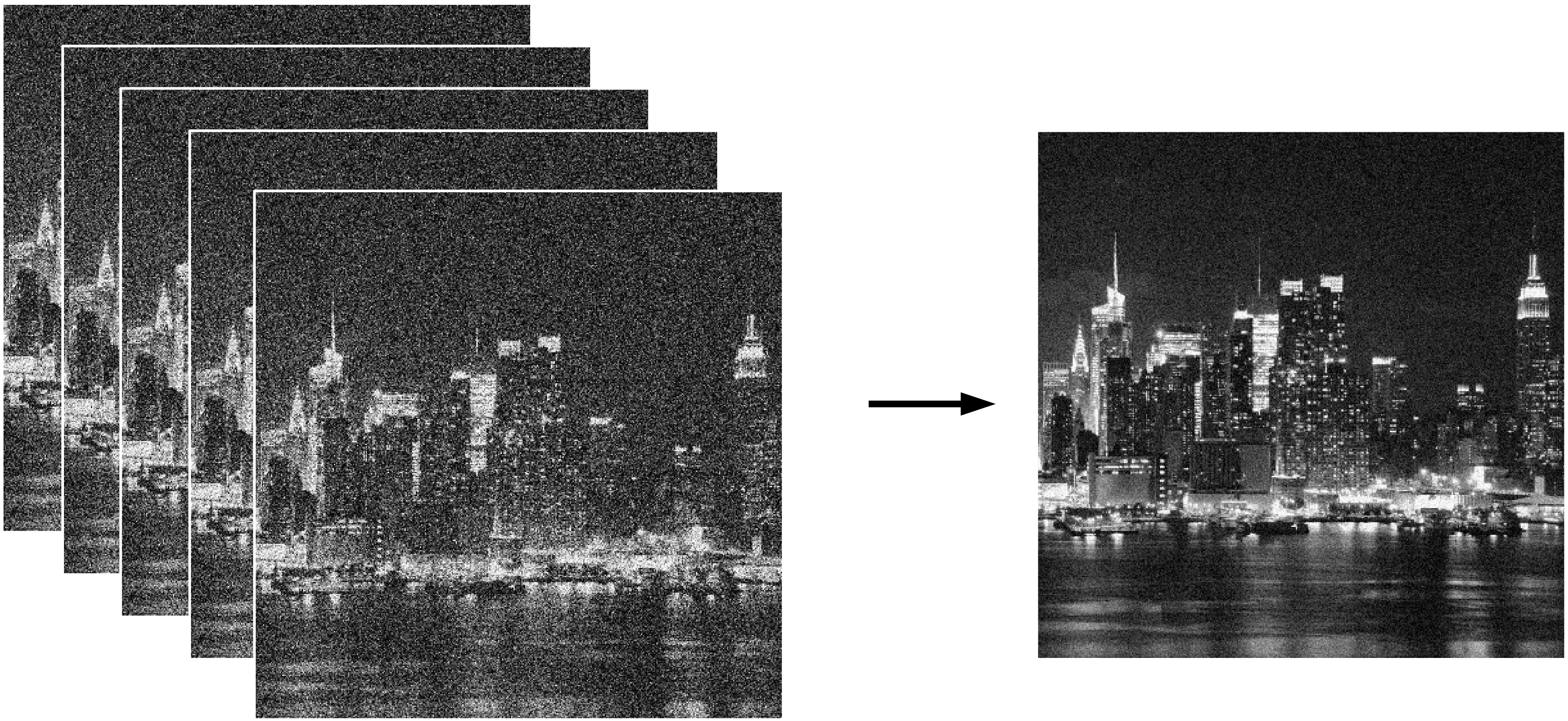
- Noise power exceeds signal power for high frequencies
- Small scale image details are lost in noise first

Signal power vs. noise power



Noise reduction by averaging

- Average multiple images



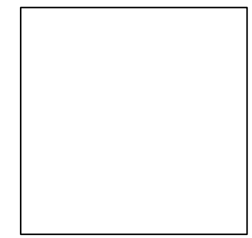
- requirement: additive noise, zero mean

Denoising by linear filtering

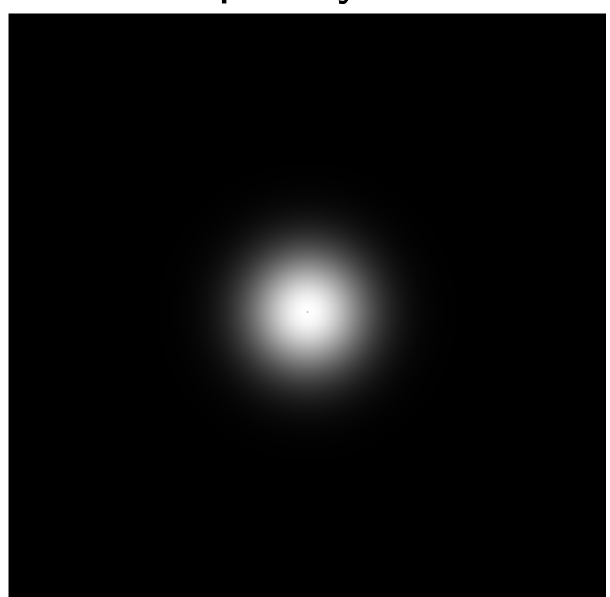
- use spatial convolution or frequency filtering to reduce noise
- noise reduction possible, but at cost of sharpness
- trade-off between noise reduction and resolution
- need fancier methods

original

convolution kernel



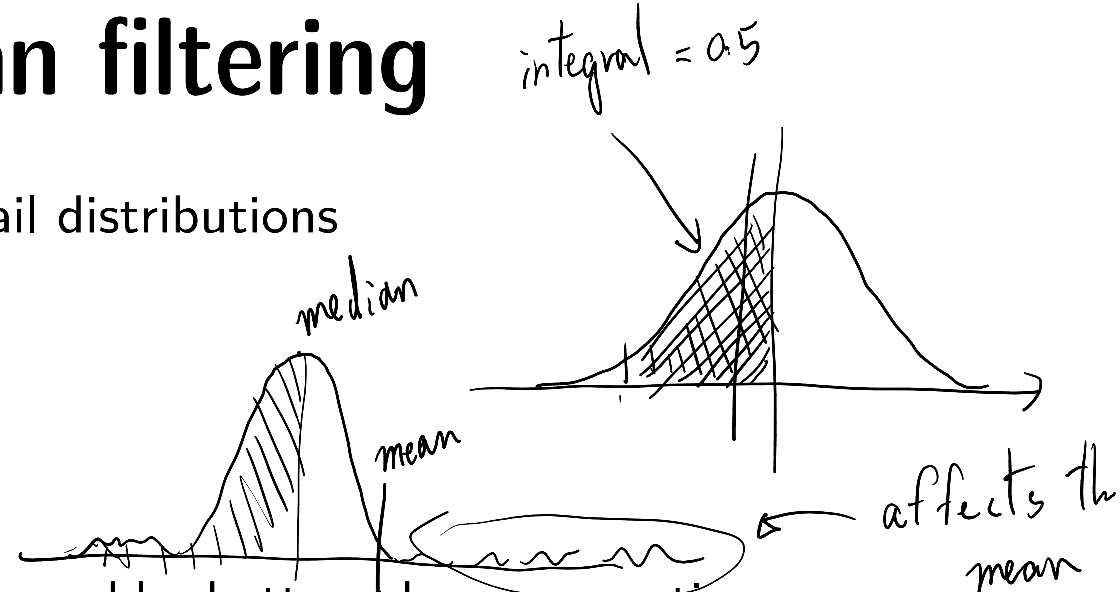
frequency filter



Resulting image

Median filtering

- Use median as estimator for fat tail distributions



* applied on pixel neighborhood

* non-linear filter!

- less sensitive to outliers in pixel ensemble, better edge preservation

Salt and pepper noise

Gauss sigma=1 pixel

Median 1 pixel

Median filtering


1x Gauss


2x Gauss


5x Gauss


1x Median


2x Median


5x Median

Common abbreviations

Abbreviation	Name	Definition
IRF	Impulse response function	Linear operator map of delta function
PSF	Point spread function	Image of point object (optical IRF)
OTF	Optical transfer function	Fourier transform of PSF
PTF	Phase transfer function	Phase part of OTF
MTF	Modulation transfer function	Amplitude of OTF
CTF	Contrast transfer function	MTF for non-sinusoidal objects
PDF	Probability density function	Probability distribution for a given random variable
SPS	Signal power spectrum	Amplitude squared of signal F.T.
NPS	Noise power spectrum	Amplitude squared of noise F.T.
SNR	Signal to noise ratio	Mean signal / mean noise
CNR	Contrast to noise ratio	Mean contrast / mean noise