14 Novembre

Teor (2° regola Hôpital $\frac{0}{0}$) I intervallo, $x_0 \in \overline{\mathbb{R}}$ punto di accumulazione di x_0. $f, g : I \setminus \{x_0\} \to \mathbb{R}$

$$\lim_{x \to x_0} f(x) = 0 = \lim_{x \to x_0} g(x)$$

Supponiamo esistano $g'(x)$ ed $f'(x)$ in $I \setminus \{x_0\}$ con

$$g(x) \neq 0 \quad e \quad g'(x) \neq 0 \quad \forall \quad x \in I \setminus \{x_0\}$$

Allora se esiste $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}}$ si ha

anche $\lim_{x \to x_0} \frac{f(x)}{g(x)} = L$

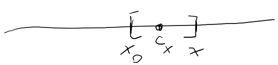
Dim Consideriamo solo il caso $x_0 \in \mathbb{R}$.

Possiamo estendere $f(x)$ e $g(x)$ anche nel punto x_0 ponendo $f(x_0) = g(x_0) = 0$. In questo modo f e g sono continue in x_0

Dimostreremo che $\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = L$

$x > x_0$

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \underset{\exists}{\overset{\text{per Cauchy}}{=}}$$



$$= \frac{f'(c_x)}{g'(c_x)}$$

$$x_0 < c_x < x$$

$$\downarrow \quad \downarrow_{x \to x_0^+} \quad \downarrow$$

$$x_0 \quad x_0 \quad x_0$$

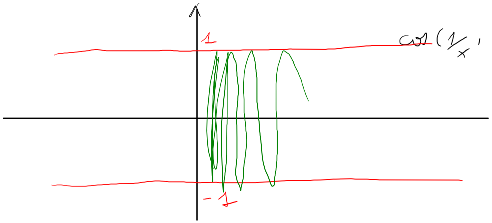
$$\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = \lim_{x \to x_0^+} \frac{f'(c_x)}{g'(c_x)}$$

$$= \lim_{c \to x_0^+} \frac{f'(c)}{g'(c)} = L$$

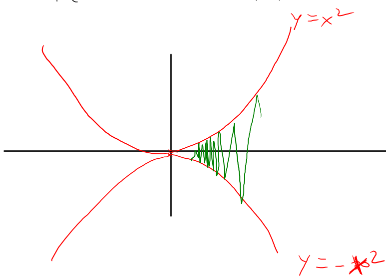
$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{6}$$

$$\lim_{x \to 0} \frac{1}{6} \underbrace{\frac{\sin x}{x}}_{1} = \frac{1}{6}$$

Esempio $\quad f(x) = x^2 \cos\left(\frac{1}{x}\right) \qquad g(x) = x$



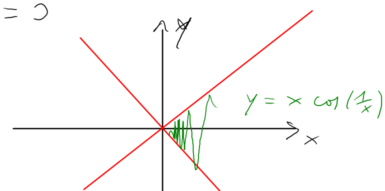
$f(x) = x^2 \cos\left(\frac{1}{x}\right) \qquad g(x) = x$



$$\frac{f(x)}{g(x)} = \frac{x^2 \cos\left(\frac{1}{x}\right)}{x} = x \cos\left(\frac{1}{x}\right)$$

$$\underbrace{-|x|}_{\xrightarrow{x\to 0}\, 0} \le \frac{f(x)}{g(x)} = x \cos\left(\frac{1}{x}\right) \le \underbrace{|x|}_{\xrightarrow{x\to 0}\, 0}$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x)}{g(x)} = 0$$



$$f(x) = x^2 \cos\left(\frac{1}{x}\right)$$

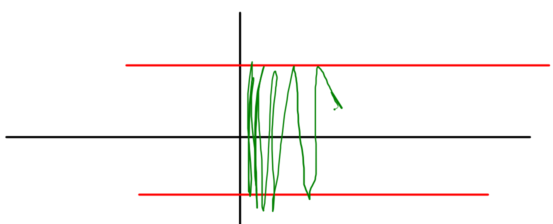
$$f'(x) = 2x\cos\left(\frac{1}{x}\right) + x^2\left(\cos\left(\frac{1}{x}\right)\right)'$$

$$= 2x\cos\left(\frac{1}{x}\right) + \cancel{x^2}\left(\cancel{-}\sin\left(\frac{1}{x}\right)\right)\left(\cancel{-}\frac{1}{\cancel{x^2}}\right)$$

$$f'(x) = 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) \qquad \left(x^{-1}\right)' = -1\, x^{-2}$$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\left(2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)\right) =$$

$$= \lim_{x\to 0}\sin\frac{1}{x}$$ non esiste



Teorema (3° regola $\frac{\infty}{\infty}$) Sia I un intervallo, $x_0 \in \overline{\mathbb{R}}$ punto di accumulazione di I.

$f, g : I \setminus \{x_0\} \longrightarrow \mathbb{R}$ $\quad \lim_{x \to x_0} f(x) = \pm\infty$

e $\lim_{x \to x_0} g(x) = \pm\infty$. f e g derivabili in $I \setminus \{x_0\}$

con $g'(x) \neq 0$ $\forall x \in I \setminus \{x_0\}$.

Allora se $\quad \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}}$

si ha $\quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = L$

$n \geq 1$

$$\lim_{x \to +\infty} \frac{x^n}{e^x} = 0 \qquad \frac{+\infty}{+\infty}$$

$$= \lim_{x \to +\infty} \frac{n x^{n-1}}{e^x} = \lim_{x \to +\infty} \frac{n(n-1) x^{n-2}}{e^x}$$

$$\ldots = \lim_{x \to +\infty} \frac{(x^n)^{(n)}}{e^x} = \lim_{x \to +\infty} \frac{n!}{e^x} = \frac{n!}{+\infty} = 0$$

$\Longrightarrow \quad x^n << e^x \quad$ per $\quad x >> 1$

per qualsiasi scelta di $n \in \mathbb{N}$

x^{ε} $\varepsilon > 0$

$\lim_{x \to +\infty} x^{\varepsilon} = +\infty$

$\lim_{x \to +\infty} \lg x = +\infty$

$$\lim_{x \to +\infty} \frac{x^{\varepsilon}}{\lg x}$$

$$= \lim_{x \to +\infty} \frac{\varepsilon x^{\varepsilon - 1}}{x^{-1}} = \lim_{x \to +\infty} \frac{\varepsilon x^{\varepsilon} \cancel{x^{-1}}}{\cancel{x^{-1}}}$$

$$= \lim_{x \to +\infty} \varepsilon x^{\varepsilon} = +\infty$$

Per ogni fissato $\varepsilon > 0$ si ha $x^{\varepsilon} >> \lg x$

se $x >> 1$.

Studiare $f(x) = x^{\varepsilon} - \lg x$

$$\lim_{x \to 0^+} x \lg x \qquad\qquad \lim_{x \to 0^+} \lg x = -\infty$$

$$= \lim_{x \to 0^+} \frac{\lg x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{\cancel{x}}}{-\frac{1}{x^{\cancel{2}\,1}}}$$

$$= \lim_{x \to 0^+} \frac{1}{-\frac{1}{x}} = \lim_{x \to 0^+} -x = 0$$

$$\lim_{x \to +\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to +\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} =$$

$$= \lim_{x \to +\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{1+\frac{1}{x^2}}} = \lim_{x \to +\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}}$$

$$1 \qquad = 1$$

$$\lim_{x \to +\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to +\infty} \frac{1}{\left((x^2+1)^{\frac{1}{2}}\right)'} =$$

$$= \lim_{x \to +\infty} \frac{1}{\frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}\cancel{2}x} = \lim_{x \to +\infty} \frac{\sqrt{x^2+1}}{x}$$

$(x^x)'$ $\qquad x > 0$

$x^x \qquad \lg x$

$$x^x = e^{\lg(x^x)} = e^{x \lg x}$$

$$(x^x)' = \left(e^{x \lg x}\right)' = e^{x \lg x} \left(x \lg x\right)'$$

$$= x^x \left(\lg x + x \frac{1}{x}\right) = x^x (\lg x + 1)$$

$$f(x)^{g(x)} = e^{g(x) \lg f(x)}$$