20 Novembre

Lezione del 18 Dic in Aula Ciamician ed B
19 Dic E 1 C1

<u>Lemma</u> Consideriamo x^a $\quad a \in \mathbb{R} \quad x > 0$

Risulta $(x^a)^{(n)} = a(a-1)\cdots(a-(n-1))\, x^{a-n}$

$$= \prod_{j=1}^{n} (a-(j-1))\; x^{a-n} \qquad n \geq 1$$

<u>Dim</u> Per induzione

$(x^a)' = a\, x^{a-1}$ per la regola della potenza

e si ha anche $\prod_{j=1}^{1} (a-(j-1))\, x^{a-1} = a\, x^{a-1}$

Supponendo la formula vera per n, la si dimostra per $n+1$, cioè si tratta di dimostrare

$$(x^a)^{(n+1)} = \prod_{j=1}^{n+1} (a-(j-1))\; x^{a-n-1}$$

$$(x^a)^{(n+1)} = \left((x^a)^{(n)}\right)' \overset{\text{per induzione}}{=} \left(\prod_{j=1}^{n} (a-(j-1))\, x^{a-n}\right)'$$

$$= \prod_{j=1}^{n} (a-(j-1))\; (x^{a-n})' =$$

$$= \prod_{j=1}^{n} (a-(j-1))\; (a-n)\; x^{a-n-1}$$

$$= \prod_{j=1}^{n+1} (a-(j-1))\; x^{a-n-1}$$

<u>Corollario</u> $\left((x-x_0)^a\right)^{(n)} = \prod_{j=1}^{n} (a-(j-1))(x-x_0)^{a-n}$ $\quad x_0 \in \mathbb{R}$, $x > x_0$

$x_0 = -1$

<u>Esempio</u> $\left((1+x)^a\right)^{(n)} = \prod_{j=1}^{n} (a-(j-1))\; (1+x)^{a-n}$

$\left((1+x)^a\right)^{(n)}(0) = \prod_{j=1}^{n} (a-(j-1))$

$x > -1$

<u>Esempio</u> $\left(\lg(1+x)\right)^{(n)} = ?$

$\left(\lg(1+x)\right)' = (1+x)^{-1}$ $\qquad n \geq 2$

$$\left(\lg(1+x)\right)^{(n)} = \left(\left(\lg(1+x)\right)'\right)^{(n-1)} =$$

$$= \left((1+x)^{-1}\right)^{(n-1)} = \prod_{j=1}^{n-1} (\cancel{-1}-(j-\cancel{1}))\; (1+x)^{\overbrace{-1-(n-1)}^{-n}}$$

$\prod_{j=1}^{n} a_j b_j = \prod_{j=1}^{n} a_j \prod_{j=1}^{n} b_j$

$$= \prod_{j=1}^{n-1} (-j) \quad (1+x)^{-n}$$

$$= \prod_{j=1}^{n-1} (-1) \prod_{j=1}^{n-1} j \quad (1+x)^{-n}$$

$n \geq 2$

$$= (-1)^{n-1} \; (j-1)! \; (1+x)^{-n}$$

$$\left(\lg(1+x)\right)^{(n)}(0) = (-1)^{n-1} \; (j-1)!$$

Def (Polinomi di Taylor) Sia $f: (a,b) \to \mathbb{R}$ e $x_0 \in (a,b)$

Supponiamo per $n \in \mathbb{N}$ esistano $\underset{\overset{\shortparallel}{f(x_0)}}{f^{(0)}(x_0)}, \dots, f^{(n)}(x_0)$

Allora il ~~polinomio~~ di Taylor di ordine n di f in x_0 è il polinomio

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j$$

Se $x_0 = 0$ si parla di polinomio di McLaurin

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j$$

Osservazione Consideriamo $P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j$

Allora $P_n(x_0) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x_0-x_0)^j = \frac{f^{(0)}(x_0)}{\underset{\overset{\shortparallel}{1}}{0!}} \underbrace{(x_0-x_0)^0}_{1}$

$= f(x_0)$

$$P_1(x) = \sum_{j=0}^{1} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j = f(x_0) + f'(x_0)(x-x_0)$$

Ricordiamoci che se $f'(x_0)$ esiste allora

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{P_1(x)} + o(x-x_0)$$

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j =$$

$$= \underbrace{\sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j}_{P_{n-1}(x_0)} + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

<u>Lemma</u> Fissiamo $n+1$ numeri reali $a_0, \dots, a_n$ e fissiamo $x_0 \in \mathbb{R}$. Allora esiste un unico polinomio $P(x)$ di grado $\leq n$ t.c. $p^{(j)}(x_0) = a_j \quad j = 0, \dots, n$

e si ha esattamente

$$p(x) = \sum_{j=0}^{n} \frac{a_j}{j!} (x - x_0)^j \qquad (1)$$

<u>Dim</u> Il polinomio $p(x)$ ha grado $\leq n$

$$p(x_0) = \left(a_0 + \sum_{j=1}^{n} \frac{a_j}{j!} (x - x_0)^j \right) \Big|_{x = x_0}$$

$$= a_0 + \sum_{j=1}^{n} \frac{a_j}{j!} \underbrace{(x_0 - x_0)^j}_{0^j} = a_0$$

$1 \leq m \leq n$

$$p^{(m)}(x) = \left(\sum_{j=0}^{n} \frac{a_j}{j!} (x - x_0)^j \right)^{(m)} =$$

$$= \sum_{j=0}^{n} \frac{a_j}{j!} \left((x - x_0)^j \right)^{(m)}$$

$$= \sum_{j=0}^{n} \frac{a_j}{j!} \prod_{l=1}^{m} (j - (l-1)) \, (x - x_0)^{j-m}$$

$$p^{(m)}(x) = \sum_{j=m}^{n} \frac{a_j}{j!} \prod_{l=1}^{m} (j - (l-1)) \, (x - x_0)^{j-m}$$

$$\boxed{p^{(m)}(x_0) = \frac{a_m}{m!} \prod_{l=1}^{m} (m - (l-1)) \underbrace{(x_0 - x_0)^{m-m}}_{1}}$$

$$+ \underbrace{\sum_{j=m+1}^{n} \frac{a_j}{j!} \prod_{l=1}^{m} (j - (l-1)) \underbrace{(x_0 - x_0)^{j-m}}_{0}}_{0}$$

$$p^{(m)}(x_0) = \frac{a_m}{m!} \prod_{l=1}^{m} (m - (l-1)) = \frac{a_m}{\cancel{m!}} \cancel{m!}$$

Unicità di p consideriamo il caso $x_0 = 0$

$\deg q(x) \leq n$

$q(x) = b_n x^n + \dots + b_0$

Ricordo $p(x) = \frac{a_n}{n!} x^n + \frac{a_{n-1}}{(n-1)!} x^{n-1} + \dots + \frac{a_0}{0!} = \sum_{j=0}^{n} \frac{a_j}{j!} x^j$

$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0 = \sum_{j=0}^{n} b_j x^j$$

$$= \sum_{j=0}^{n} \frac{b_j \, j!}{j!} x^j \qquad = \sum_{j=0}^{n} \frac{a_j}{j!} x^j$$

$$\Rightarrow q^{(j)}(0) = \boxed{b_j \, j! = a_j} \Rightarrow b_j = \frac{a_j}{j!}$$

$0 \leq j \leq n$

$f(x)$ x_0

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j$$

$$P_n^{(j)}(x_0) = f^{(j)}(x_0) \quad \forall$$

$\forall$

$$0 \le j \le n$$

Esempi di polinomi di Mc Laurin $(x_0 = 0)$

$f(x) = e^x \qquad f^{(n)}(0) = 1 \quad \forall n \qquad x_0 = 0$

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j = \sum_{j=0}^{n} \frac{x^j}{j!}$$

$(x-0)^j$

$f(x) = (1+x)^a$

$$f^{(n)}(0) = \prod_{j=1}^{n} (a-(j-1))\,(1+0)^{a-n}$$

$$f^{(n)}(0) = \prod_{j=1}^{n} (a-(j-1)) \qquad n \geq 1$$

$$f^{(0)}(0) = (1+0)^a = 1 \qquad f(x) = (1+x)^a$$

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j =$$

$$= 1 + \sum_{j=1}^{n} \frac{f^{(j)}(0)}{j!} x^j = 1 + \sum_{j=1}^{n} \frac{\prod_{k=1}^{j}(a-(k-1))}{j!} x^j$$

$$P_n(x) = 1 + \sum_{j=1}^{n} \frac{\prod_{k=1}^{j}(a-(k-1))}{j!} x^j$$

$= \binom{a}{j}$

$n \geq j \geq 0$

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} = \frac{\prod_{l=1}^{j}(n-(l-1))}{j!} \qquad \binom{a}{0} = 1$$

$$n! = \prod_{l=1}^{j}(n-(l-1))\,(n-j)!$$

$$P_n(x) = \binom{a}{0} + \sum_{j=1}^{n} \binom{a}{j} x^j = \sum_{j=0}^{n} \binom{a}{j} x^j$$