INTEGRALI TRIPLI (E MULTIPLI)

In modo analogo si definiscono gli integrali tripli su $R \subseteq \mathbb{R}^3$ dove

$R = I \times J \times K$

$I = [a,b]$

$J = [c,d]$

$K = [p,q]$

costruendo i tassellamenti, le somme di Riemann inferiori e superiori, ecc.

Vale il teorema di Fubini

$$\iiint_{I\times J\times K} f(x,y,z)\,dx\,dy\,dz = \int_I \left(\iint_{J\times K} f(x,y,z)\,dy\,dz \right) dx$$

$$= \iint_{I\times J} \left(\int_K f(x,y,z)\,dz \right) dx\,dy$$

$$= \int_I \left(\int_J \left(\int_K f(x,y,z)\,dz \right) dy \right) dx$$

e tutte le altre combinazioni.

Più in generale si definisce, per $R \subseteq \mathbb{R}^n$, $R = [a_1,b_1] \times \dots \times [a_n,b_n]$

$$\iint_R \cdots \int f(x_1,x_2,\dots,x_n)\,dx_1\,dx_2\cdots dx_n$$

per cui ancora vale il teorema di Fubini.

INTEGRALI SU INSIEMI GENERALI NEL PIANO

Sia $\Omega \subseteq \mathbb{R}^2$ aperto e limitato e sia $f : \Omega \to \mathbb{R}$ limitata.

Vogliamo definire $\displaystyle\iint_\Omega f(x,y)\,dx\,dy$.

La definizione si dà così:

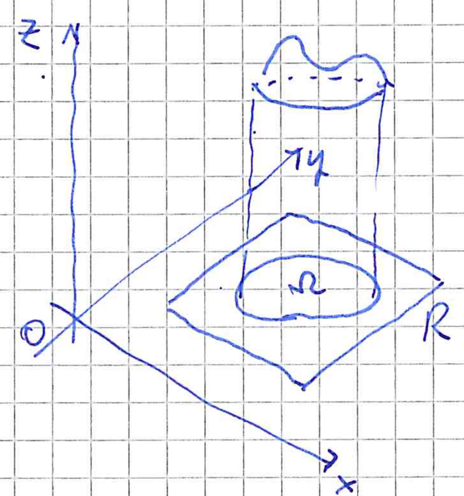
Sia $\tilde{f}$ l'estensione di f a tutto $\mathbb{R}^2$ così definita

$$\tilde{f}(x,y) := \begin{cases} f(x,y) & \text{se } (x,y) \in \Omega \\ 0 & \text{se } (x,y) \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Sia R un rettangolo tale che $\Omega \subseteq R$

Allora

$$\iint_\Omega f(x,y)\,dx\,dy := \iint_R \tilde{f}(x,y)\,dx\,dy.$$

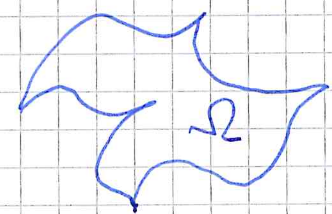


L'integrabilità di f su Ω dipende sia dalle proprietà di f, sia da quelle di Ω, in particolare di $\partial\Omega$.

PROPOSIZIONE

Sia $\Omega \subseteq \mathbb{R}^2$ aperto limitato, e supponiamo che $\partial\Omega$ sia l'unione ⊛ di archi di curva regolare. Sia f continua su $\overline{\Omega}$. Allora f è integrabile su Ω.

⊛ finita



Si definisce

$$\boxed{\text{Area}(\Omega) := \iint_{\Omega} dx\,dy}$$

Valgono le seguenti proprietà:

1) linearità:

$$\iint_{\Omega} (c_1 f_1(x,y) + c_2 f_2(x,y))\,dx\,dy = c_1 \iint_{\Omega} f_1(x,y)\,dx\,dy + c_2 \iint_{\Omega} f_2(x,y)\,dx\,dy$$

2) Monotonia

Se $f(x,y) \leq g(x,y)$ in Ω,

allora $\iint_\Omega f(x,y)\,dx\,dy \leq \iint_\Omega g(x,y)\,dx\,dy$.

In particolare $\left|\iint_\Omega f(x,y)\,dx\,dy\right| \leq \iint_\Omega |f(x,y)|\,dx\,dy$.

3) Additività:

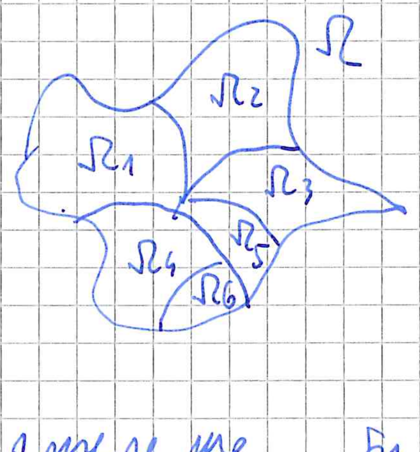
Se $\overline{\Omega} = D_1 \cup \dots \cup D_k$ tale che

$\forall i$, posto $\Omega_i = \mathring{D}_i$ (interno di Ω_i),

si abbia $\Omega_i \cap \Omega_j = \phi \; \forall i,j$

e $\partial\Omega_i$ regolare $\forall i$, allora

$$\iint_\Omega f(x,y)\,dx\,dy = \sum_{i=1}^{k} \iint_{\Omega_i} f(x,y)\,dx\,dy$$



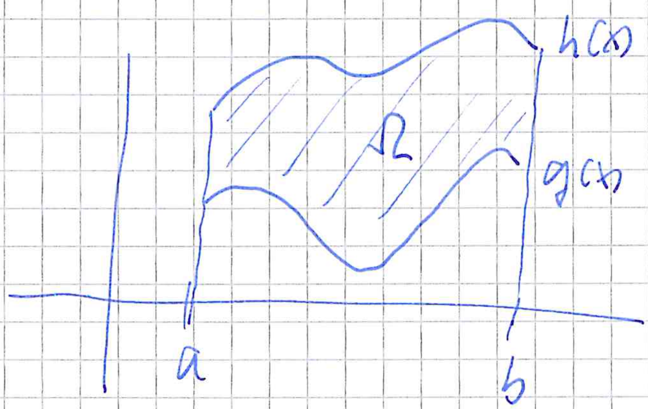
Ci soffermeremo su alcuni tipi speciali di domini.

Def

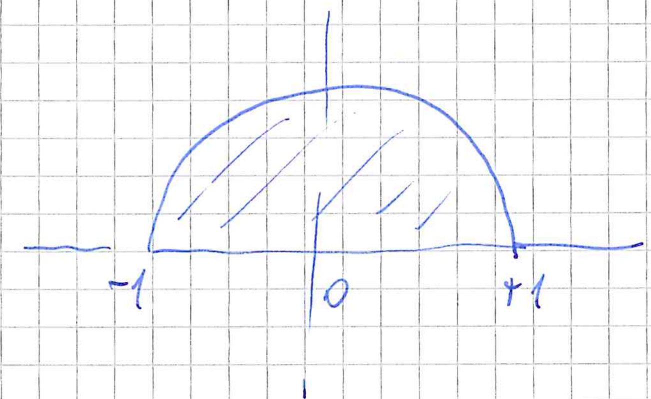
$\Omega \subseteq \mathbb{R}^2$ aperto si dice y-semplice

$\iff \bar{\Omega} = \{ (x,y) \in \mathbb{R}^2 \mid a \le x \le b ,\ g(x) \le y \le h(x) \}$

ovvero Ω è compreso tra i grafici di due funzioni continue $g(x)$ e $h(x)$



e anche



$g(x) \equiv 0$
$h(x) = \sqrt{1-x^2}$
(semicerchio).

In questo caso pongo $m = \min_{[a,b]} g(x)$

$M = \max_{[a,b]} h(x)$

Così $\Omega \subseteq [a,b] \times [m, M]$.

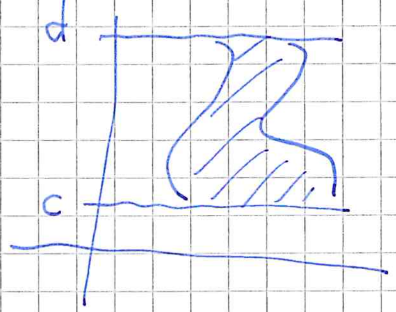
Vd il teorema di Fubini.

$$\iint_{\Omega} f(x,y)\,dx\,dy \overset{\text{def.}}{=} \iint_{R} \tilde{f}(x,y)\,dx\,dy = \int_a^b \left(\int_m^M \tilde{f}(x,y)\,dy \right) dx =$$

$$= \int_a^b \left(\int_m^{g(x)} \underbrace{\tilde{f}(x,y)}_{=0}\,dy + \int_{g(x)}^{h(x)} \underbrace{\tilde{f}(x,y)}_{=f(x,y)}\,dy + \int_{h(x)}^{M} \underbrace{\tilde{f}(x,y)}_{=0}\,dy \right) dx =$$

$$= \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

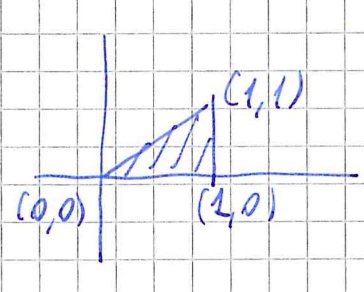
In modo analogo si definiscono gli insiemi X- semplici :



$\Omega = \{ (x,y) \mid y \in [c,d] \quad g(y) \le x \le h(y) \}$

ESEMPIO

$\iint_T xy\,dx\,dy$ dove $T =$ Triangolo di vertici $(0,0)$, $(1,0)$, $(1,1)$



In questo caso $[a,b] = [0, 1]$,
$g(x) = 0$, $h(x) = x$. È un dominio y-semplice

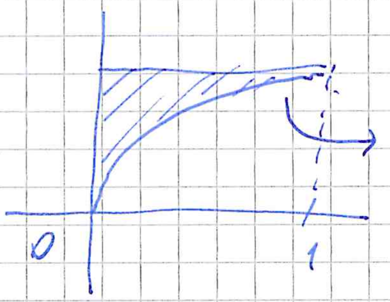
Si ha:

$$\iint_T xy\,dx\,dy = \int_0^1 \left(\int_0^x xy\,dy \right) dx = \int_0^1 x \left[\frac{1}{2} y^2 \right]_0^x dx$$

$$= \int_0^1 \frac{1}{2} x^3\,dx = \frac{1}{8} x^4 \Big|_0^1 = \frac{1}{8}$$

ESEMPIO

$$\iint_D e^{y^3}\,dx\,dy \quad , \quad D := \{ (x,y) \mid 0 \le x \le 1 ; \sqrt{x} \le y \le 1 \}$$



D è y-semplice, ma è anche x-semplice, infatti $D = \{ (x,y) \mid 0 \le y \le 1 ; 0 \le x \le y^2 \}$

Proviamo prima a fare il conto considerando D come y-semplice.

$$\iint_D e^{y^3}\,dx\,dy = \int_0^1 \Big(\underbrace{\int_0^{\sqrt{x}} e^{y^3}}_{\text{non sappiamo trovare la primitiva.}} \Big) dx$$

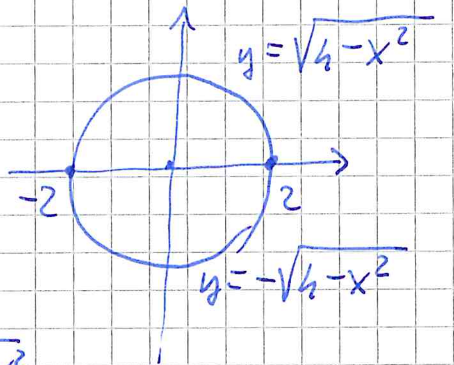
Allora proviamo a considerare D come x-semplice.

$$\iint_D e^{y^3}\,dx\,dy = \int_0^1 \left(\int_0^{y^2} e^{y^3}\,dx \right) dy = \int_0^1 e^{y^3} y^2\,dy =$$

$$= \frac{1}{3} e^{y^3} \Big|_0^1 = \frac{1}{3}(e-1)$$

ESEMPIO

$$\iint_D x\,dx\,dy \qquad D = \{ x^2+y^2 \leq 4 \}$$



$$\iint_D x\,dx\,dy = \int_{-2}^{2} \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x \quad dy \right) dx =$$

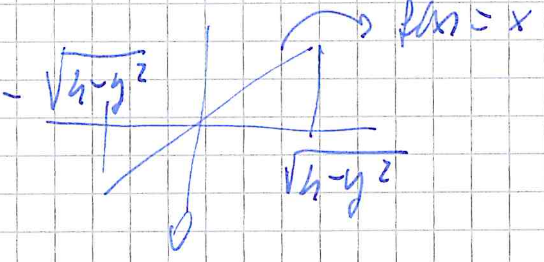
$$= \int_{-2}^{2} 2x\sqrt{4-x^2}\,dx$$

$$= -\frac{2}{3}(4-x^2)^{3/2} \Big|_{-2}^{2} = 0$$

È più semplice però

$$\iint_D x\,dx\,dy = \int_{-2}^{2} \left(\int_{-\sqrt{4-y^2}}^{\sqrt{4-x^2}} x\,dx \quad \right) dy = 0$$

perché $\displaystyle\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x\,dx$ è l'integrale di una funzione dispari su un intervallo simmetrico rispetto a 0.



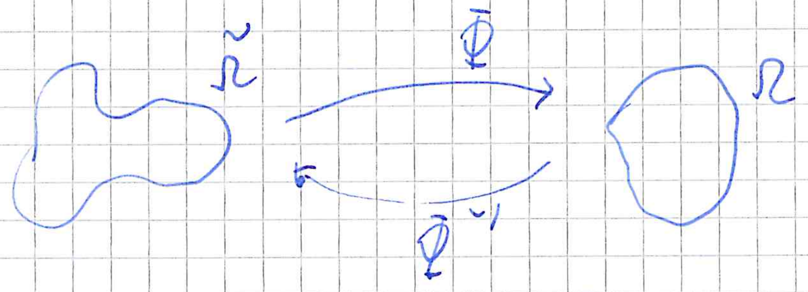
CAMBIO DI VARIABILI NEGLI INTEGRALI DOPPI

Sia $\tilde{\Omega} \subseteq \mathbb{R}^2$, $\Omega \subseteq \mathbb{R}^2$ e sia

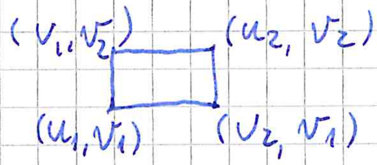
$\Phi : \tilde{\Omega} \longrightarrow \Omega$ una funzione C^1,

biiettiva, con $\det J\Phi(u,v) \neq 0$ $\forall (u,v) \in \tilde{\Omega}$

cosicché $\exists\ \Phi^{-1} : \Omega \to \tilde{\Omega}$ ed è a sua volta di classe C^1.



Sia ora $[u_1, u_2] \times [v_1, v_2]$ un piccolo rettangolo in $\tilde{\Omega}$

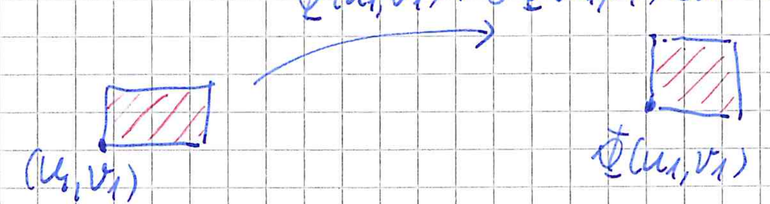


L' immagine di $[u_1, u_2] \times [v_1, v_2]$

è $\{ \Phi(u_1, v_1) + J\Phi(u_1, v_1)[(u-u_1, v-v_1)] + R(u-u_1, v-v_1)$

$| u \in [u_1, u_2], \ v \in [v_1, v_2] \}$

L' approssimazione $\boxed{\text{al primo ordine}}$ è



il parallelogramma di vertici

- $\Phi(u_1, v_1)$, • $\Phi(u_1, v_1) + J\Phi(u_1, v_1)[(u_2-u_1, 0)]$,
- $\Phi(u_1, v_1) + J\Phi(u_1, v_1)[(0, v_2-v_1)]$,
- $\Phi(u_1, v_1) + J\Phi(u_1, v_1)[(u_2-u_1, v_2-v_1)]$

che ha come area

$\left| \det \left(J\Phi(u_1, v_1)[(u_2-u_1), 0], \ J\Phi(u_1, v_1)[(0, v_2-v_1)] \right) \right|$

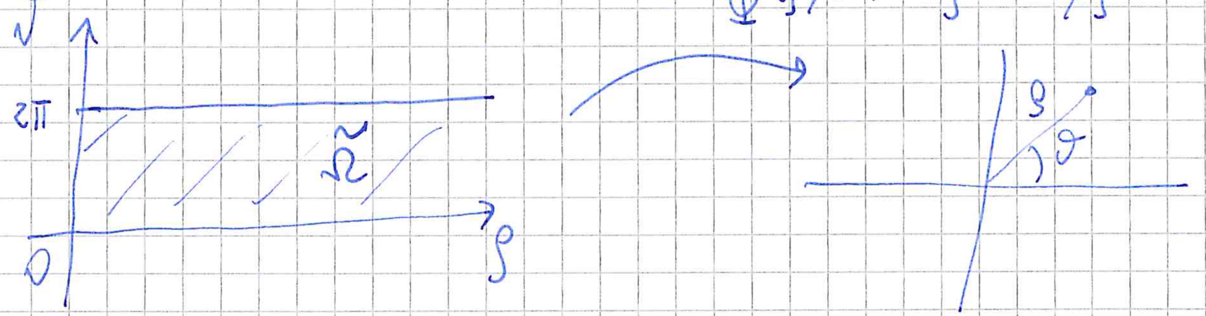
$= |\det J\Phi(u_1, v_1)| (u_2-u_1)(v_2-v_1).$

Si ha quindi che

$$\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\tilde{\Omega}} f(\Phi(u,v))\,|\det J\Phi(u,v)|\,du\,dv$$

COORDINATE POLARI

$$\Phi(\rho,\vartheta) = (\rho\cos\vartheta, \rho\sin\vartheta)$$



Calcoliamo $J\Phi(\rho,\vartheta) = \begin{pmatrix} \cos\vartheta & -\rho\sin\vartheta \\ \sin\vartheta & \rho\cos\vartheta \end{pmatrix}$

$$\Rightarrow \det J\Phi(\rho,\vartheta) = \rho\cos^2\vartheta + \rho\sin^2\vartheta = \rho$$

Allora $\displaystyle\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Phi^{-1}(\Omega)} f(\rho\cos\vartheta, \rho\sin\vartheta)\,\rho\,d\rho\,d\vartheta$

Lavorare in coordinate polari è utile quando ad esempio Ω è una corona circolare, o un settore circolare.

ESEMPIO

Calcolare l'area del cerchio di raggio R,

$D = \{(x,y) \mid x^2+y^2 \leq R^2\}$.

$$\iint_D dx\,dy = \iint_{\Phi^{-1}(D)} \rho\, d\rho\, d\theta \qquad \Phi^{-1}(D) = [0,R]\times[0,2\pi]$$

$$= \int_0^{2\pi}\left(\int_0^R \rho\, d\rho\right) d\theta = \int_0^{2\pi} \frac{1}{2}R^2\, d\theta = \pi R^2$$

ESEMPIO

$D = \{(x,y) \mid 1 \leq x^2+y^2 \leq 4\} \qquad \Phi^{-1}(D) = [1,2]\times[0,2\pi]$

$$\iint_D \frac{y}{x^2+y^2}\, dx\,dy = \iint_{\Phi^{-1}(D)} \frac{\rho \sin\theta}{\rho^2}\, \rho\, d\rho\, d\theta =$$

$$= \iint_{\Phi^{-1}(D)} \sin\theta\, d\rho\, d\theta = \int_0^{2\pi}\left(\int_1^2 \sin\theta\, d\rho\right) d\theta$$

$$= \int_0^{2\pi} \sin\theta\, [\rho]_1^2\, d\theta = \int_0^{2\pi} \sin\theta\, d\theta = 0$$

INTEGRALI TRIPLI

Se $D \subseteq \mathbb{R}^3$ non è un "rettangolo" e $f: D \to \mathbb{R}$, definiamo

$$\tilde{f}(x,y,z) := \begin{cases} f(x,y,z) & (x,y,z) \in D \\ 0 & (x,y,z) \notin D \end{cases}$$

Sia $D \subseteq R = I \times J \times K$

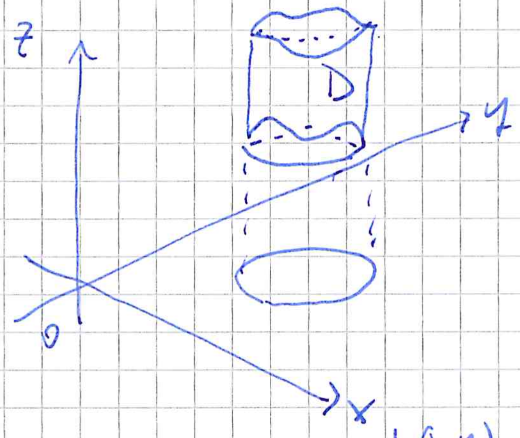
allora definiamo

$$\iiint_D f(x,y,z)\,dx\,dy\,dz \overset{\text{def.}}{=} \iiint_R \tilde{f}(x,y,z)\,dx\,dy\,dz$$

CASI SPECIFICI

Domini z-semplici (integrazione "per fili")

$$D = \{(x,y,z) \mid (x,y) \in G \subseteq \mathbb{R}^2,\ g(x,y) \leq z \leq h(x,y)\}$$

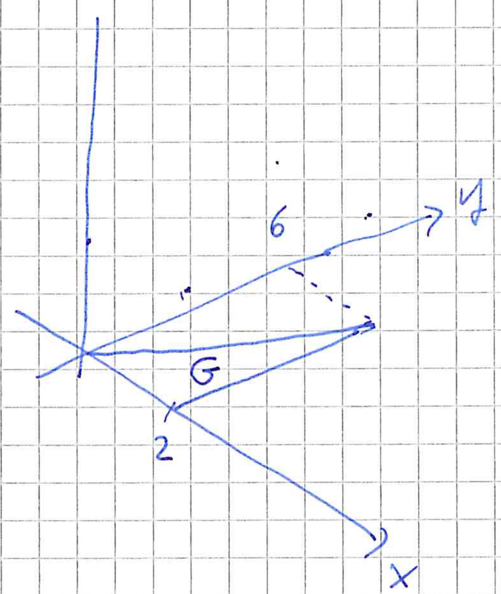


$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iint_G \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz \right) dx\,dy$$

ESEMPIO

$$\iiint_T x^2 y z \, dx\,dy\,dz$$

$$T = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 3x,\ x \le z \le x+y\}$$



$$\iiint_T x^2 y z \, dx\,dy\,dz = \iint_G \left(\int_x^{x+y} x^2 y z \, dz \right) dx\,dy$$

$$= \iint_G \left[\frac{1}{2} x^2 y z^2 \right]_x^{x+y} dx\,dy$$

$$= \iint_G \left(\frac{1}{2} x^2 y (x+y)^2 - \frac{1}{2} x^4 y \right) dx\,dy$$

$$= \frac{1}{2} \iint_G \left(x^2 y^3 + 2x^3 y^2 \right) dx\,dy \quad =$$

$$= \frac{1}{2}\int_0^2 \left(\int_0^{3x} (x^2 y^3 + 2x^3 y^2)\, dy \right) dx =$$

$$= \frac{1}{2} \int_0^2 \left(\frac{1}{4} x^2 (3x)^4 + \frac{2}{3} x^3 (3x)^3 \right) dx$$

$$= \frac{81}{8} \frac{1}{7} 2^7 + 9 \frac{1}{7} 2^7 = \frac{153 \cdot 16}{7}$$

Integrazione per strati

$$D = \{ (x, y, z) \mid z \in [a, b],\ (x,y) \in D_z \}$$

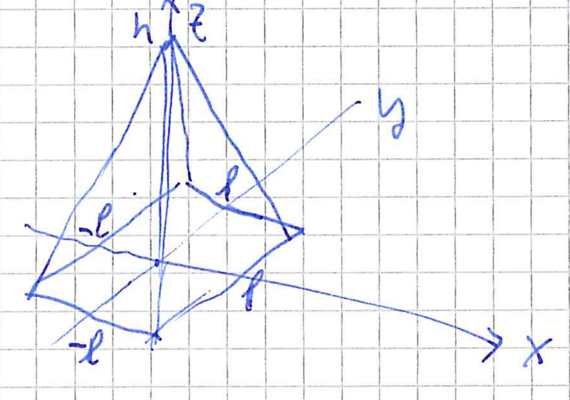
dove $\forall z$ $D_z \subseteq \mathbb{R}^2$ è "sensato".

allora

$$\iiint_D f(x,y,z)\, dx\, dy\, dz = \int_a^b \left(\int_{\Omega_z} f(x,y,z)\, dx\, dy \right) dz$$

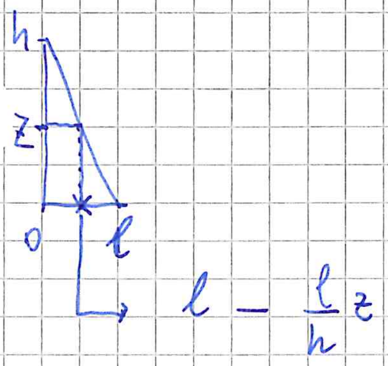
ESEMPIO

Volume della piramide



D

$$D_z = \left[-\ell + \frac{\ell}{h}z, \ \ell - \frac{\ell}{h}z\right] \times \left[-\ell + \frac{\ell}{h}z, \ \ell - \frac{\ell}{h}z\right]$$



$$\iiint_D dx\,dy\,dz = \int_0^h \left(\iint_{D_z} dx\,dy \right) dz$$

$$= \int_0^h \left[2\left(\ell - \frac{\ell}{h}z\right)\right]^2 dz$$

$$= 4\int_0^{\ell} \zeta^2 \frac{h}{\ell}\, d\zeta = \frac{4h}{\ell}\,\frac{1}{3}\zeta^3 \Big|_0^{\ell} =$$

$$= 4\,\frac{h}{\ell}\,\frac{1}{3}\ell^3 = \frac{1}{3}h(2\ell)^2$$

$$\left(\frac{\text{Area base} \times \text{altezza}}{3} \right)$$

CAMBIO DI COORDINATE IN INTEGRALI TRIPLI

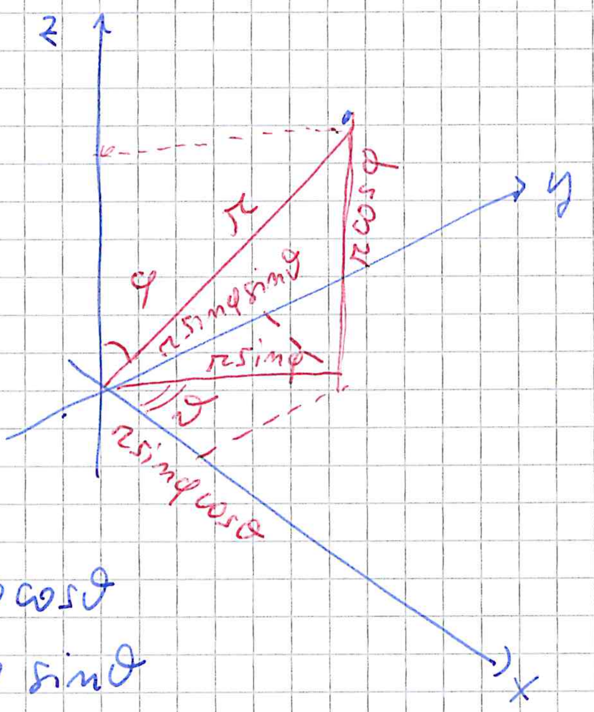
$\Phi : \tilde{\Omega} \subseteq \mathbb{R}^3 \longrightarrow \Omega \subseteq \mathbb{R}^3$ invertibile C^1

con $\det J\Phi(u,v,w) \neq 0 \quad \forall\ (u,v,w)$.

Allora

$$\iiint_{\Omega} f(x,y,z)\, dx\,dy\,dz = \iiint_{\tilde{\Omega}} f(\Phi(u,v,w))\, |\det J\Phi(u,v,w)|\, du\,dv\,dw$$

COORDINATE SFERICHE



$$\begin{cases} x = r \sin\varphi \cos\vartheta \\ y = r \sin\varphi \sin\vartheta \\ z = r \cos\varphi \end{cases}$$

$r \in [0, +\infty[$

$\varphi \in]0, \pi[$

$\vartheta \in]0, 2\pi[$

$$J\Phi(r,\vartheta,\varphi) = \begin{pmatrix} \sin\varphi\cos\vartheta & r\cos\varphi\cos\vartheta & -r\sin\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta & r\cos\varphi\sin\vartheta & r\sin\varphi\cos\vartheta \\ \cos\varphi & -r\sin\varphi & 0 \end{pmatrix}$$

$$\det J\Phi(r,\vartheta,\varphi) = r^2\cos^2\varphi\sin\varphi + r^2\sin^2\varphi\sin\varphi$$
$$= r^2\sin\varphi$$

cioè $|\det J\Phi(r,\vartheta,\varphi)| = r^2\sin\varphi$

Quindi

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \iiint_{\Phi^{-1}(\Omega)} f(r\sin\varphi\cos\vartheta, r\sin\varphi\sin\vartheta, r\cos\varphi)\cdot r^2\sin\varphi\,d\varphi\,d\vartheta\,dr$$

ESEMPIO

Il volume della palla di raggio R.

$$\mathrm{Vol}(B_R) = \iiint_{B_R} dx\,dy\,dz =$$

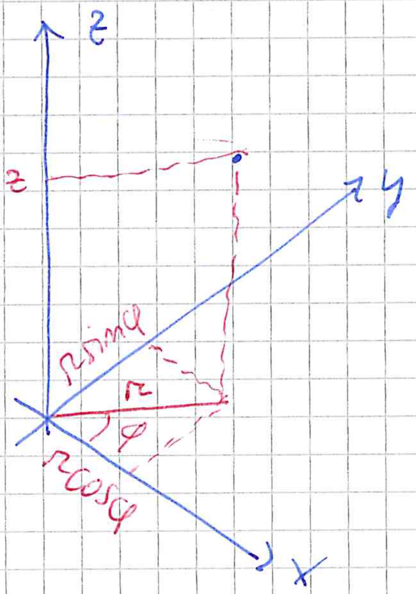
$$= \iiint_{\Phi^{-1}(B_R)} r^2\sin\varphi\,dr\,d\vartheta\,d\varphi$$

$$= \int_0^R \left(\int_0^{2\pi} \left(\int_0^\pi r^2 \sin\varphi \, d\varphi \right) d\vartheta \right) dr$$

$$= \int_0^R \left(\int_0^{2\pi} r^2 \, [-\cos\varphi]_0^{\pi} \, d\vartheta \right) dr$$

$$= \int_0^R \left(\int_0^{2\pi} 2r^2 \, d\vartheta \right) dr = 4\pi \int_0^R r^2 \, dr = \frac{4}{3}\pi R^3 .$$

COORDINATE CILINDRICHE



$$\begin{cases} x = r\cos\varphi \\ y = \sin\varphi \\ z = z \end{cases} \qquad \begin{aligned} & r \in [0, +\infty[\\ & \varphi \in]0, 2\pi[\\ & z \in \mathbb{R} \end{aligned}$$

$$\Phi(r, \varphi, z) = (r\cos\varphi, r\sin\varphi, z)$$

$$J\Phi(r, \varphi, z) = \begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ r\cos\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad |\det J\Phi(r,\varphi,z)| = r$$

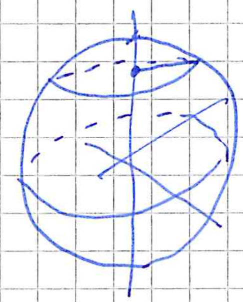
$$\Rightarrow$$

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Phi^{-1}(\Omega)} f(r\cos\varphi, r\sin\varphi, z)\, r\, dr\, d\varphi\, dz$$

ESEMPIO

Momento d'inerzia della palla $B_R(0)$ di densità costante, rispetto all'asse z

$$I = \iiint_{B_R} (x^2+y^2)\,dx\,dy\,dz =$$



$$= \iiint_{\Phi^{-1}(B_R)} r^2\, r\, dr\, d\varphi\, dz$$

$$= \int_{-R}^{R} \left(\int_0^{\sqrt{R^2-z^2}} r^3\,dr \int_0^{2\pi} d\varphi \right) dz$$

$$= \int_{-R}^{R} \frac{1}{4} r^4 \Big|_0^{\sqrt{R^2-z^2}} 2\pi \; dz$$

$$= \frac{\pi}{2} \int_{-R}^{R} (R^2-z^2)^2\,dz = \frac{\pi}{2} \int_{-R}^{R} (R^4 - 2R^2z^2 + z^4)\,dz$$

$$= \frac{\pi}{2} \left(R^4 z - \frac{1}{3} R^2 z^3 + \frac{1}{5} z^5 \right) \Bigg|_{-R}^{R} = \ldots = \frac{8}{15}\pi R^5$$