22 Nov Linares - Ponce

$$\begin{cases} i\partial_t u = -\Delta u + \lambda |u|^{p-1} u \\ u|_{t=0} = u_0 \end{cases} \quad 1 < p < d^* = \begin{cases} +\infty & d = 1,2 \\ \frac{d+2}{d-2} & d \geq 3 \end{cases}$$

<u>Prop</u> $\forall\, u_0 \in H^1(\mathbb{R}^d)$ $\exists\, T > 0$ and a unique solution

$$u \in C^0([0,T], H^1(\mathbb{R}^d)) \cap L^q([0,T], W^{1,p+1}(\mathbb{R}^d))$$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2} \qquad (q, p+1) \text{ admissible}$$

$\exists\, T(\cdot) : [0,+\infty) \to (0,+\infty]$ s.t.

$$T \geq T(\|u_0\|_{H^1}) > 0$$

$\forall\, T' \in (0,T)$ $\exists$ a neighbourhood V of u_0 st the map $v_0 \to v(t)$ unique

$$V : \to C^0([0,T'], H^1(\mathbb{R}^d)) \cap L^q([0,T'], W^{1,p+1}(\mathbb{R}^d))$$

$u \in L^a([0,T], W^{1,b})$ $\forall$ (a,b) admissible.

Pf We define

$$E(T,a) = \{ v \in C^0([0,T], H^1(\mathbb{R}^d)) \cap L^q([0,T], W^{1,p+1}(\mathbb{R}^d))$$

$$|v|_T + |\nabla v|_T : \quad |v|_T^{(1)} \leq a \}$$

NLS

$$|v|_T^{(1)} = |v|_{L^\infty([0,T],H^1)} + |v|_{L^q([0,T],W^{1,p+1})}$$

$$\nabla \phi(v) = e^{it\Delta}\nabla u_0 - i\int_0^t e^{i(t-s)\Delta}\nabla(|v|^{p-1}v)\,ds$$

We prove $\phi : E(T,a) \circlearrowleft$

and is a contraction.

$$|\nabla\phi(u)|_T \leq c_0|\nabla u_0|_{L^2} + c_0\,\big|\,|u|^{p-1}\nabla u\,\big|_{L^{q'}([0,T],L^{\frac{p+1}{p}})}$$

$$\leq c_0|\nabla u_0|_{L^2} + c_0\,|u|^{p-1}_{L^\beta([0,T],L^{p+1})}\,|\nabla u|_{L^q([0,T],L^{p+1})}$$

$$\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'}$$

$$p < 1 + \frac{4}{d} < d^* \Rightarrow \beta < q$$

Now $p < d^*$

$$H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}$$

$d \geq 3$ $\qquad H^1(\mathbb{R}^d) \hookrightarrow L^{d^*+1}(\mathbb{R}^d)$

$$L^{p+1}$$

$$L^2(\mathbb{R}^d)$$

$$d^* = \frac{d+2}{d-2}$$

$$d^*+1 = \frac{d+\cancel{2}}{d-2} + \frac{d-\cancel{2}}{d-2} = \frac{2d}{d-2}$$

$$\frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d} = \frac{d-2}{2d}$$

$$|\nabla\Phi(u)|_T \leq c_0 |\nabla u_0|_{L^2} + c_0 \left| |u|^{p-1} \nabla u \right|_{L^{q'}([0,T], L^{\frac{p+1}{p}})}$$

$$\leq c_0 |\nabla u_0|_{L^2} + c_0 |u|^{p-1}_{L^\beta([0,T], H^1)} |\nabla u|_{L^q([0,T], L^{p+1})}$$

$$\leq c_0 |\nabla u_0|_{L^2} + c_0 \, T^{\frac{p-1}{\beta}} |u|^{p-1}_{L^\infty([0,T], H^1)} |\nabla u|_{L^q([0,T],}$$

$$\leq c_0 |\nabla u_0|_{L^2} + c_0 T^{\vartheta} a^p \leq$$

$$a > 2 c_0 |u_0|_{H^1}$$

$$\leq \left(\frac{1}{2} + c_0 T^{\vartheta} a^{p-1} \right) a < a$$

$$c_0 T^{\vartheta} a^{p-1} \le \frac{1}{2}$$

$T(\|u_0\|_{H^1})$ $\qquad\qquad$ $[0,T]$

$$u - v = e^{it\Delta}(u_0 - v_0)$$

$$- i\int_0^t e^{i(t-s)\Delta}\left(|u|^{p-1}u - |v|^{p-1}v\right) ds$$

$$\|u-v\|_{L^q([0,T],L^{p+1})} \le c_0 \|u_0 - v_0\|_{L^2}$$

$$+ c_0 T^{\vartheta}\left(\|u\|^{p-1}_{L^\infty([0,T],H^1)} + \|v\|^{p-1}_{L^\infty([0,T],H^1)}\right)$$

$$\|u-v\|_{L^q([0,T],L^{p+1})}$$

or $$\|u-v\|_{L^q([0,T],L^{p+1})} \le c_0 \|u_0 - v_0\|_{L^2}$$

$$+ \frac{1}{2}\|u-v\|_{L^q([0,T],L^{p+1})}$$

$$\|u-v\|_{L^q([0,T],L^{p+1})} \le 2c_0 \underbrace{\|u_0 - v_0\|_{L^2}}_{0} \Rightarrow$$

$\Rightarrow$ if $u(0) = v(0) \Rightarrow u(t) = v(t)$

in some small interval $[0,T]$ H^2

$u, v \in C^0_t H^1$

from I know $X = \{ t : u(t) = v(t) \}$ (H^1)

is closed in $[0, +\infty)$

but the above argument shows that X is also open

Prop Let u be a maximal solution

Then $E(u(t)) = E(u_0)$

$Q(u(t)) = Q(u_0)$

$P_j(u(t)) = \frac{1}{2} \langle i \partial_j u(t), u(t) \rangle = P_j(u_0)$

Pf $\quad i \partial_t u = -\Delta u + \lambda |u|^{p-1} u$

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s) \, ds$$

$\varphi \in C^\infty_c(\mathbb{R}, [0,1])$ $\quad \varphi = 1$ near 0

$\operatorname{supp} \varphi \subseteq [-1, 1]$

$$Q_n = \varphi\left(\frac{\sqrt{-\Delta}}{n}\right) \qquad P_n = \chi_{[0,1]}\left(\frac{\sqrt{-\Delta}}{n}\right)$$

$$= \chi_{\overline{D(0,n)}}(\sqrt{-\Delta})$$

$$\widehat{Q_n f}(\xi) = \varphi\left(\frac{|\xi|}{n}\right)\hat{f}(\xi)$$

$$i\partial_t u = -\Delta u + \lambda |u|^{p-1} u$$

$$\begin{cases} i\partial_t u_n = -P_n \Delta u_n + \lambda Q_n\left(|Q_n u|^{p-1} Q_n u\right) \\ u_n(0) = Q_n u_0 \end{cases}$$

$$P_n : L^2(\mathbb{R}^d) \circlearrowleft$$

but for $d > 1$

$$\cancel{P_n : L^\alpha(\mathbb{R}^d) \circlearrowleft} \quad \text{if} \quad \alpha \neq 2$$

$$\widehat{Q_n f} = \varphi\left(\frac{\xi}{n}\right)\hat{f}$$

$$|Q_n f(x)|_{L^\alpha} = |n^d \check{\varphi}(n\,\cdot) * f|_{L^\alpha} \leq$$

$$\leq \underbrace{\left(n^d |\check{\varphi}(n\,\cdot)|_{L^1}\right)}_{C = |\check{\varphi}|_{L^1}} |f|_{L^\alpha}$$

$$\begin{cases} i\partial_t u_n = -P_n \Delta u_n + \lambda Q_n\left(|Q_n u|^{p-1} Q_n u\right) \\ u_n(0) = Q_n u_0 \end{cases}$$

this is a family of ODE's

$$H^1(\mathbb{R}^d) \longrightarrow H^{-1}(\mathbb{R}^d) \xrightarrow{Q_n^*} H^1$$

$$u \longrightarrow |u|^{p-1}u$$

$$\begin{tikzcd} H^1(\mathbb{R}^d) \arrow[r] \arrow[d, hook] & L^{\frac{p+1}{p}}(\mathbb{R}^d) \arrow[r, hook] & H^{-1}(\mathbb{R}^d) \\ L^{p+1} \arrow[ur] & & \end{tikzcd}$$

$$\left| |u|^{p-1}u - |v|^{p-1}v \right|_{L^{\frac{p+1}{p}}} \leq$$

$$\lesssim \left| \left(|u|^{p-1} + |v|^{p-1}\right) |u-v| \right|_{L^{\frac{p+1}{p}}}$$

$$\lesssim \left(|u|_{L^{p+1}}^{p-1} + |v|_{L^{p+1}}^{p-1} \right) |u-v|_{L^{p+1}}$$

$\forall R > 0 \quad \exists\ C_R$ s.t.

$$\left| |u|^{p-1}u - |v|^{p-1}v \right| \leq C_R\, |u-v|_{H^1}$$

if $u, v \in \overline{D_{H^1}(0,R)}$

$$\begin{cases} i\partial_t u_n = -P_n \Delta u_n + \lambda Q_n\left(|Q_n u|^{p-1} Q_n u\right) \\ u_n(0) = Q_n u_0 \end{cases}$$

$\forall\ n\ \exists\ u_n \in C^1([0, T(n)), H^1)$

~~it~~ is since $(1-P_n)Q_n = 0$

$$\left(1-\cancel{\chi_{D(0,n)}(\xi)}\right)\varphi\left(\frac{\xi}{n}\right) \equiv 0$$

$$\chi_{|\xi|\geq n}$$

$(1-P_n)u_n = 0$

$$\begin{cases} i\partial_t (1-P_n)u_n = -\underbrace{(1-P_n)P_n}_{0}\Delta u_n + \lambda \underbrace{(1-P_n)Q_n}_{0}(\dots) \Rightarrow \\ (1-P_n)Q_n u_0 = 0 \end{cases}$$

$$T(n) < +\infty \Rightarrow \lim_{t\to T(n)} |u_n(t)|_{H^1} = +\infty$$

Lemma $E_n(v) : \frac{1}{2}|P_n \nabla v|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n v|^{p+1} dx$

$P_j(v)$

$Q(v)$

are invariants of motion

$$|u_n(t)|_{H^1} = |P_n u_n(t)|_{H^1} = |\langle \xi \rangle \chi_{D(0,n)} \hat{u}_n(t)|_{L^2}$$

$$\leq \langle n \rangle \, |\chi_{D(0,n)} \hat{u}_n(t)|_{L^2}$$

$$\leq \langle n \rangle \, |u_n(t)|_{L^2} = \langle n \rangle \, |Q_n u_0|_{L^2}$$

$C^1(H^1, \mathbb{R})$

$$E_n(v) = \frac{1}{2}\langle P_n \nabla v, P_n \nabla v \rangle + \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |Q_n v|^{p+1} \, dx$$

$$\frac{d}{dt} E_n(v + tX)\Big|_{t=0} =$$

$$= \langle P_n \nabla v, P_n \nabla X \rangle + \frac{\lambda}{p+1} \frac{d}{dt}\Big|_{t=0} \int_{\mathbb{R}^d} |Q_n(v+tX)|^{p+1} dx$$

$$\langle P_n \nabla (v+tX), P_n \nabla (v+tX) \rangle =$$

$$= \langle P_n \nabla v, P_n \nabla v \rangle + 2t \langle P_n \nabla v, P_n \nabla X \rangle$$

$$+ t^2 \langle P_n \nabla X, P_n \nabla X \rangle$$

$$\frac{d}{dt}\Big|_{t=0} |Q_n(v+tX)|^{p+1} = (p+1) \, |Q_n(v+tX)|^{p-1}$$

$$\left\langle \frac{Q_n(v+tX)}{|Q_n(\cancel{v+tX})|}, Q_n \cancel{X} \right\rangle_{\mathbb{C}}$$

$$\langle z, w \rangle_{\mathbb{C}} = \operatorname{Re} z \bar{w}$$

$$\frac{d}{dt} E_n(v + tX) \Big|_{t=0} =$$

$$= \langle P_n \nabla v, P_n \nabla X \rangle + \frac{\lambda}{p+1} \frac{d}{dt}\Big|_{t=0} \int_{\mathbb{R}^d} |Q_n(v+tX)|^{p+1} dx$$

$$= \cancel{\langle} \quad '' \quad + \lambda \operatorname{Re} \int_{\mathbb{R}^d} |Q_n(v)|^{p-1} Q(v) \overline{Q_n X}$$

$$= \langle -P_n \Delta v, X \rangle + \lambda \langle Q_n (|Q_n(v)|^{p-1} Q_n v), X \rangle$$

$$d E_n(v) X =$$

$$= \langle \underbrace{-P_n \Delta v + \lambda Q_n |Q(v)|^{p-1} Q_n v}_{\nabla E(v)}, X \rangle$$