10 November

Cazenave

Linares – Ponce

Dodson defocusing

u^3

$\boxed{f(|u|^2)\, u}$

Morawetz inequality

$$i\partial_t u = -\Delta u + \lambda |u|^{p-1} u$$

$\lambda = 1$ defocusing

$\lambda = -1$ focusing

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} dx$$

$$\langle u, v \rangle = \mathrm{Re} \int_{\mathbb{R}^d} u(x)\, \overline{v(x)}\, dx$$

$$\omega(u,v) := \langle i u, v \rangle = -\langle u, i v \rangle$$

$$d^* = \begin{cases} +\infty & d = 1,2 \\ \frac{d+2}{d-2} & d \geq 3 \end{cases}$$

$p < d^*$ sub-(energy) critical

(mass) critical

Lemma $1 < p < d^*$

1) Gagliardo - Nirenberg

$$|u|_{L^{p+1}(\mathbb{R}^d)} \leq C_p \, |\nabla u|_{L^2}^{\alpha} \, |u|_{L^2}^{1-\alpha}$$

$$\frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}$$

$$(\dot{H}^{\alpha} \hookrightarrow L^{p+1} \qquad \mathbb{R}^d)$$

2) $u \longrightarrow |u|^{p-1} u$ is Locally Lipschitz from $H^1(\mathbb{R}^d) \longrightarrow H^{-1}(\mathbb{R}^d)$

$$\left(\langle \xi \rangle = \sqrt{1+|\xi|^2} \right)$$

3) $u \in W^{1,p+1}(\mathbb{R}^d, \mathbb{C})$

$$\nabla (|u|^{p-1} u) = p \, |u|^{p-1} \nabla u + $$
$$+ (p-1) \, |u|^{p-1} \left(\frac{u}{|u|} \right)^2 \nabla \bar{u}$$

and is in $L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C})$ $(\subseteq H^{-1}(\mathbb{R}^d)$

$$H^1(\mathbb{R}) \hookrightarrow L^{p+1}(\mathbb{R}^d, \mathbb{C}) \qquad p+1 < d^*$$

$$L^{\frac{p+1}{p}} \hookrightarrow H^{-1}$$

<u>Dim</u> $\quad u \in H^1(\mathbb{R}^d) \Longrightarrow u \in L^{p+1}(\mathbb{R}^d)$

$$\Longrightarrow \quad |u|^{p-1}u \in L^{\frac{p+1}{p}}(\mathbb{R}^d)$$

$$\| \, |u|^{p-1}u \|_{L^{\frac{p+1}{p}}} = \| \, |u|^p \, \|_{L^{\frac{p+1}{p}}}$$

$$= \| u \|^p_{L^{p+1}}$$

$$\| \, |u|^{p-1}u - |v|^{p-1}v \|_{L^{\frac{p+1}{p}}}$$

$$\leq C_p \, \| (|u|^{p-1} + |v|^{p-1}) (u-v) \|_{L^{\frac{p+1}{p}}}$$

$$\leq C_p \, \| \, |(u,v)|^{p-1} (u-v) \|_{L^{\frac{p+1}{p}}}$$

$$\frac{p}{p+1} = \frac{1}{p+1} + \frac{p-1}{p+1}$$

$$\leq C_p \, \| \, |(u,v)|^{p-1} \|_{L^{\frac{p+1}{p-1}}} \, \|u-v\|_{L^{p+1}}$$

$$\leq C_p \, \| (u,v) \|_{L^{p+1}}^{p-1} \, \|u-v\|_{L^{p+1}}$$

$$\leq C_p \left(\|u\|_{L^{p+1}}^{p-1} + \|v\|_{L^{p+1}}^{p-1} \right) \|u-v\|_{L^{p+1}}$$

$$\left| |u|^{p-1} u - |v|^{p-1} v \right| \leq$$

$$\leq C_p \left(|u|^{p-1} + |v|^{p-1} \right) |u-v|$$

$$w = u - v$$

$$|u|^{p-1} u - |v|^{p-1} v =$$

$$= \int_0^1 \frac{d}{dt} |v+tw|^{p-1} (v+tw)$$

$$= \int_0^1 w \, |v+tw|^{p-1} +$$

$$+ \int_0^1 (v+tw) \frac{d}{dt} \left((v_1+tw_1)^2 + (v_2+tw_2)^2 \right)^{\frac{p-1}{2}}$$

$$= w \int_0^1 |v+tw|^{p-1} dt +$$

$$+ \sum_{j=1}^{2} \int_0^1 (v+tw)\ 2(v_j+tw_j)w_j \ \times$$

$$\times \frac{p-1}{2} \left((v_1+tw_1)^2+(v_2+tw_2)^2 \right)^{\frac{p-3}{2}}$$

$$|\cancel{u} - v| \quad |(1-t)v+tu|^{p-1}$$

$$\leq |u-v| \quad (|u|+|v|)^{p-1}$$

$$\leq |u-v| \quad 2^{p-1}\left(|u|^{p-1}+|v|^{p-1}\right)$$

$$|u| \geq |v|$$

$$(|u|+|v|)^{p-1} \leq (2|u|)^{p-1}$$

$$\leq 2^{p-1}\left(|u|^{p-1}+|v|^{p-1}\right)$$

$$|w|\ |v+tw|\ |v+tw| \quad |v+tw|^{p-3}$$

$$|u-v| \quad |v+tw|^{p-1}$$

$$G(0)=0$$

$$G \in C^1(\mathbb{C}, \mathbb{C})$$

$$u \in W^{1,p+1}(\mathbb{R}^n)$$

$$|\nabla G| \leq M < +\infty$$

$$\nabla(G(u)) = \underline{\partial_u G(u)\,\nabla u + \partial_{\bar u} G(u)\,\nabla \bar u}$$

∂_u

∂_u

$\frac{\partial}{\partial(x+iy)}$ $\quad \frac{\partial}{\partial iy}$

$$\partial_z = \frac{1}{2}(\partial_x - i\,\partial_y)$$

$$\partial_{\bar z} = \frac{1}{2}(\partial_x + i\,\partial_y)$$

$$u_n \longrightarrow u$$

$$G(u_n) \longrightarrow G(u)$$

$$\nabla G(u_n) \longrightarrow \nabla G(u)$$

$$G(u) = |u|^{p-1} u$$

$G_m(u)$ $\qquad g \in C^\infty(\mathbb{R}_+, \mathbb{R})$

$$g(s) = \begin{cases} s^{\frac{p-1}{2}} & 0 \le s \le 1 \\ 2^{\frac{p-1}{2}} & s \ge 2 \end{cases}$$

$$G_m(u) = m^{p-1} g\left(\frac{|u|^2}{m^2}\right) u \qquad m \in \mathbb{N}$$

So for $\quad |u| \leq m \iff \dfrac{|u|^2}{m^2} \leq 1$

$$\Rightarrow G_m(u) = \cancel{m^{p-1}} \frac{|u|^{p-1}}{\cancel{m^{p-1}}} u$$

$$G_m(u) = |u|^{p-1} u \quad \text{for} \quad |u| \leq m$$

$$\underbrace{-\int G_m(u) \nabla \varphi} = \int \left(\partial_u G_m(u) \nabla u + \partial_{\bar{u}} G_m(u) \nabla \bar{u}\right) \varphi$$

$$-\int G(u) \nabla u = \int \underbrace{\left(\partial_u G(u) \nabla u + \partial_u G(u) \nabla \bar{u}\right)} \varphi$$

$$\int G_m(u) \nabla \varphi = \int_{|u| \leq m} |u|^{p-1} u \nabla \varphi + \int_{|u| \geq m} G_m(u) \nabla \varphi$$

$$= \int |u|^{p-1} u \nabla u \; - \int_{|u| \geq m} |u|^{p-1} u \nabla \varphi \to 0$$

$$+ \int_{|u| \geq m} G_m(u) \nabla \varphi \to 0$$

$$\int_{|u|\geq m} \underline{|u|^{p-1}u \nabla\varphi} =$$

$$= \int |u|^{p-1} u \nabla\varphi \, \chi_{\{|u|\geq m\}} \, dx$$

$$\chi_{\{|u|\geq m\}}(x) \longrightarrow 0 \qquad a.e.$$

$$\int_{|u|\geq m} |G_m(u) \nabla\varphi|$$

$$|G_m(u)| = m^{p-1} \underbrace{g\left(\frac{|u|^2}{m^2}\right)}_{\leq 2^{p-1}} |u|$$

$$\frac{|u|^2}{m^2} \geq 1$$

$$\leq \int_{|u|\geq m} m^{p-1} |u| \, |\nabla\varphi|$$

$$\leq \int_{|u|\geq m} |u|^{p-1} |\nabla\varphi| \longrightarrow 0$$

$$\nabla(|u|^{p-1}u) = p|u|^{p-1}\nabla u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \nabla\bar{u}$$

$$|u|^{p-1}u = u^{\frac{p-1}{2}+1}\,\bar{u}^{\frac{p-1}{2}}$$

$$= u^{\frac{p+1}{2}}\,\bar{u}^{\frac{p-1}{2}}$$

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\,dx + \frac{\lambda}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}\,dx$$

$\langle\ ,\ \rangle$

$\langle i\ ,\ \rangle$

X_E $\qquad dE$

$$\langle \nabla E, X\rangle = dE\ X$$

$$\omega(X_E, Y) = dE\ Y.$$

$$\langle i X_E, y \rangle = \langle \nabla E, y \rangle$$

$$X_E = -i \nabla E$$

$$\dot{u} = X_E(u)$$

$$\partial_t u = -i \nabla E$$

$$i \partial_t u = \nabla E = -\Delta u + \lambda |u|^{p-1} u$$

$$i \partial_t u = -\Delta u + \lambda |u|^{p-1} u$$

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \lambda \int \frac{|u|^{p+1}}{p+1} du$$

$$E(e^{i\vartheta_0} u) = E(u) \qquad S^1 \times \mathbb{R}^d$$

$$E(u(\cdot + c e_j)) = E(u)$$

$$Q(u) = \frac{1}{2} \|u\|_{L^2}^2$$

$$P_j(u) = \frac{1}{2} \operatorname{Im} \int \partial_j u \quad \bar{u} \quad dx$$

$$\langle i \partial_j u, u \rangle$$