15 Nov

$$p < d^* \begin{cases} \infty & d = 1,2 \\ \frac{d+2}{d-2} & d \geq 3 \end{cases}$$

$$\begin{cases} i \partial_t u = -\Delta u + \lambda |u|^{p-1} u \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^d) \end{cases}$$

$u(t,x)$

$\overline{u(-t,x)}$

$u(t,x)$ $\qquad \tau^{\frac{2}{p}} u(\tau^2 t, \tau x)$

$s_p = \frac{d}{2} - \frac{2}{p}$

$$| \tau^{\frac{2}{p}} u_0(\tau \cdot) |_{\dot{H}^{s_p}} = |u_0|_{\dot{H}^{s_p}}$$

$$u(t) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds$$

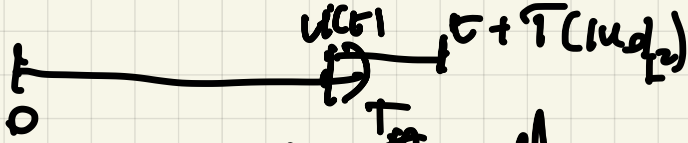
Prop (l.w.p. in $L^2(\mathbb{R}^d)$) $\quad p \in (1, 1 + \frac{4}{d})$

Then $\forall\ u_0 \in L^2(\mathbb{R}^d)$ $\exists\ T > 0$ and a unique

$$u \in C^0([0,T], L^2) \cap L^q([0,T], L^{p+1})$$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$$

$\exists \quad T(\cdot): [0, +\infty) \to (0, +\infty]$ s.t.

$T \geq T(|u_0|_{L^2})$ 

$\forall\ T' \in (0,T)$ $\exists$ a neighborhood $V \ni u_0$ s.t. the map

$V \ni v_0 \longrightarrow v(t)$

is well defined in

$$C^0([0,T'], L^2) \cap L^q([0,T'], L^{p+1})$$

and it is Lipschitz

$u \in L^\alpha([0,T], L^\beta)$ $\forall\ (\alpha,\beta)$ admissible

<u>Pf</u>

$$i\dot{u} = -\Delta u \pm |u|^{p-1}u$$
$$\dot{u} = i\Delta u \mp i|u|^{p-1}u$$

$$\frac{d}{dt}\frac{1}{2}|u(t)|^2_{L^2} = \langle \dot{u}(t), u(t)\rangle$$

$$= \langle i\Delta u, u\rangle + \langle i|u|^{p-1}u, u\rangle$$

$$= -\langle i\nabla u, \nabla u\rangle + \langle i, |u|^{p+1}\rangle = 0$$

$$\langle f, g\rangle = \mathrm{Re}\int f\bar{g}\,dx$$

Fix $a > 0$

$$E(T,a) = \{ v \in C^0([0,T], L^2) \cap L^q([0,T], L^{p+1}) \,\vert\, \|v\|_T \le a \}$$

$$\|v\|_T = \underbrace{|v|_{L^\infty([0,T],L^2)}}_{} + |v|_{L^q([0,T],L^{p+1})}$$

$$u(t) = e^{it\Delta}u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\, ds$$

$$\Phi(v) = e^{it\Delta}u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |v(s)|^{p-1} v(s)\, ds$$

$\Phi : E(T,a) \circlearrowleft$ and is a contraction

$$|\Phi(u)|_T \le |e^{it\Delta}u_0|_T + \left| \int_0^t e^{i(t-s)\Delta} |u|^{p-1} u \, ds \right|_T$$

$$\le c_0 |u_0|_{L^2} + c_0 \, \big| \, |u|^p \, \big|_{L^{q'}([0,T], L^{\frac{p+1}{p}})}$$

$$= c_0 |u_0|_{L^2} + c_0 \, |u|^p_{L^{q'p}([0,T], L^{p+1})}$$

We will see that

$$p \in \left(1, 1 + \tfrac{4}{d}\right) \Longleftrightarrow p q' < q \qquad \vartheta > 0$$

$$\le c_0 |u_0|_{L^2} + c_0 \, T^{\vartheta} \, |u|^p_{L^q([0,T], L^{p+1})}$$

$$\leq c_0 \, |u_0|_{L^2} + c_0 \, T^{\vartheta} \, |u|_T^{p}$$

$$\leq C_0 \, |u_0|_{L^2} + C_0 \, T^{\vartheta} \, a^{p-1} \, a$$

Choose T so that

$$c_0 \, T^{\vartheta} a^{p-1} \leq \frac{1}{2}$$

$$\leq c_0 \, |u_0|_{L^2} + \frac{a}{2} \quad < \frac{a}{2} + \frac{a}{2} = a$$

$$a \geq 2 \, c_0 |u_0|_{L^2}$$

$$|\phi(u) - \phi(v)|_T = \left| \int_0^t e^{i(t-s)\Delta} \left(|u|^{p-1} u - |v|^{p-1} v \right) ds \right|_T$$

$$\leq c_0 \quad \left| |u|^{p-1} u - |v|^{p-1} v \right|_{L^{q'}([0,T], L^{\frac{p+1}{p}})}$$

$$\leq C \, C_p \left| \left(|u|^{p-1} + |v|^{p-1} \right) (u-v) \right|_{L^{q'}([0,T], L^{\frac{p+1}{p}})}$$

$$\frac{p-1}{q} + \frac{1}{S} = \frac{1}{q'}$$

$$\leq C \, C_0 \left(|u|^{p-1}_{L^q([0,T], L^{p+1})} + |v|^{p-1}_{L^q([0,T], L^{p+1})} \right) |u-v|_{L^S([0,T], L^{p+1})}$$

$$\frac{p}{p+1} = \frac{p-1}{p+1} + \frac{1}{p+1}$$

$$\left| \, |u|^{p-1} \, |u-v| \, \right|_{L^{\frac{p+1}{p}}}$$

$$\leq \left\| |u|^{p-1} \right\|_{L^{\frac{p+1}{p-1}}} |u-v|_{L^{p+1}}$$

$$= |u|^{p-1}_{L^{p+1}} |u-v|_{L^{p+1}}$$

$$|\phi(u)-\phi(v)|_T \leq$$

$$\leq C\, c_0 \left(|u|^{p-1}_{L^q([0,T],L^{p+1})} + |v|^{p-1}_{L^q([0,T],L^{p+1})} \right) |u-v|_{L^\rho(0,T,L^{p+1})}$$

$$\leq C\, c_0\, 2\, a^{p-1} |u-v|_{L^\rho([0,T],L^{p+1})}$$

$$\frac{p-1}{q} + \frac{1}{\rho} = \frac{1}{q'}$$

here $\rho < q$ since if

$\rho \geq q$ $\qquad \frac{1}{\rho} \leq \frac{1}{q}$

$$\boxed{\frac{1}{q'} \leq \frac{p}{q}} \qquad q \leq p\, q'$$

but this contradicts

$$p \in (1, 1+\tfrac{4}{d}) \Leftrightarrow \quad p q' < q$$

$$|\phi(u) - \phi(v)|_T \leq \qquad\qquad \rho < q$$

$$\leq C c_0 2 a^{p-1} |u-v|_{L^\rho([0,T], L^{p+1})}$$

$$\leq C c_0 2 a^{p-1} T^{\vartheta} |u-v|_{L^q([0,T], L^{p+1})}$$

$$\leq \underbrace{C c_0 2 a^{p-1} T^{\vartheta}}_{\leq \frac{1}{2}} |u-v|_T$$

$$\Rightarrow \exists! \quad u \in E(a,T)$$

$$\text{st} \qquad u = \phi(u)$$

$$p \in (1, 1+\tfrac{4}{d}) \Leftrightarrow \quad p q' < q$$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2} \qquad\qquad \frac{1}{q} = \frac{d}{4} - \frac{1}{2}\frac{d}{p+1}$$

$$q = \infty \qquad p+1 = 2$$

$$p q' < q \Leftrightarrow \quad \frac{p}{q} < 1 - \frac{1}{q}$$

$$\Longleftrightarrow \quad \frac{p+1}{q} < 1$$

$$\Longleftrightarrow \quad \frac{1}{q} < \frac{1}{p+1}$$

$$\frac{d}{4} - \frac{d}{2} \quad \frac{1}{p+1} < \frac{1}{p+1}$$

$$\frac{d}{4} < \frac{1}{p+1}\left(\frac{d}{2} + 1\right) = \frac{1}{p+1} \quad \frac{d+2}{2}$$

$$\frac{d}{2} \leq \frac{1}{p+1} \quad (d+2)$$

$$\frac{d}{2d+4} \leq \frac{1}{p+1}$$

$$\frac{2d+4}{d} \geq p+1$$

$$2 + \frac{4}{d} \geq p+1$$

$$p \leq 1 + \frac{4}{d}$$

$$a > 2c_0 |u_0|_{L^2}$$

If I take an appropriate $V \ni u_0$

$$\Rightarrow \quad a > 2c_0 |v_0|_{L^2}$$

$$\Rightarrow \quad \forall \; v_0 \in V$$

$$\Rightarrow \quad v(t) \quad \in C^0([0,T], L^2) \cap L^q([0,T], L^{p+1})$$

$$|u-v|_T \leq c_0 |u_0 - v_0|_{L^2} +$$

$$+ c_0 C \, T^{\vartheta} \Big(\underbrace{|u|_T^{p-1}}_{\leq a^{p-1}} + \underbrace{|v|_T^{p-1}}_{a^{p-1}} \Big) |u-v|_T$$

$$|u-v|_T \leq c_0 |u_0 - v_0|_{L^2} + \underbrace{2 c_0 C \, T^{\vartheta}}_{\leq \frac{1}{2}} a^{p-1} |u-v|_T$$

$$\leq c_0 |u_0 - v_0| + \frac{1}{2} |u-v|_T$$

$$|u-v|_T \leq 2c_0 |u_0 - v_0|$$

Prop (l.w. H^1) $\forall$ $1 < p < d^*$

and any $u_0 \in H^1(\mathbb{R}^d)$ $\exists$ $T > 0$

and a unique solution

$$u \in C^0([0,T], H^1) \cap L^q([0,T], W^{1,p+1})$$

and $\exists$ $T(\cdot) : [0, +\infty) \to (0, +\infty]$

s.t. $T \geq T(|u_0|_{H^1})$

Moreover $\forall$ $0 < T' \leq T$

$\exists$ $V \ni u_0$

~~the~~ the dat. map is Lip.

$$V \ni v_0 \longrightarrow v(t) \in C^0([0,T], H^1) \cap L^q([0,T], W^{1,p+1})$$

$$u \in L^{\alpha}([0,T], W^{1,\beta}) \quad \forall\ \alpha,\beta \text{ admissible}$$

$$E(u(t)) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} dx$$

$$P_j(u(t)) = \frac{1}{2} \langle i \partial_j u, u \rangle$$

$$Q(u(t)) = \frac{1}{2} \|u(t)\|_{L^2}^2$$

are all preserved.

$\Longrightarrow$ In the previous proposition

$$Q(u(t)) = Q(u_0)$$

$\boxed{1}$ Suppose in the L^2 proposition

$$u \in C^0([0,T^*), L^2)$$

$$u_0^{(n)} \xrightarrow{n \to +\infty} u_0 \quad \text{in } L^2$$

$\cap$
H^1 $\qquad Q(u(t))$ is locally constant

il $T' < T^*$

$p < 1 + \frac{4}{d}$

$u_0^{(n)} \in C^0([0, T^*], L^2)$

$$Q_n(u^{(n)}(t)) = Q(u_0^{(n)})$$

$$\downarrow \qquad\qquad \downarrow$$

$$Q(u(t)) = Q(u_0)$$