23 Novembre

<u>Proposizione</u> Sia $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$ ed esista il polinomio di Taylor di ordine n ~~di~~ f in x_0 $P_n(x)$

(in altre parole, esistono $f^{(j)}(x_0)$ per $0\le j\le n$)

Allora, se $p(x)$ è un polinomio di grado $\le n$ t.c. $f(x)=p(x)+o((x-x_0)^n)$

risulta che $p(x)=P_n(x)$. $\square$

Ricordiamo la dimostrazione

$$f(x)=p(x)+o((x-x_0)^n)$$

$$f(x)=P_n(x)+o((x-x_0)^n)$$

~~f~~ facendo la differenza si ottiene

$$0=p(x)-P_n(x)+o((x-x_0)^n)$$

$$\underbrace{p(x)}_{\deg\le n}-\underbrace{P_n(x)}_{\deg\le n}=o((x-x_0)^n).$$

$$\Rightarrow \deg(p-P_n)\le n$$

$$\Rightarrow p-P_n\equiv 0$$

$$(1+x)^a = \sum_{j=0}^{n} \binom{a}{j} x^j + o(x^n)$$

$$\binom{a}{0} = 1 \qquad \binom{a}{j} = \frac{\prod_{k=1}^{j} (a-(k-1))}{j!}$$

$$\boxed{(1-x)^a = \sum_{j=0}^{n} \binom{a}{j} (-1)^j x^j + o(x^n)}$$

$c \neq 0$

$$o(c x^n) = o(x^n) \quad . \quad \text{Infatti,}$$

$$\lim_{x\to 0} \frac{o(cx^n)}{x^n} = \lim_{x\to 0} \frac{o(cx^n)}{c x^n} \, c = 0 \cdot c = 0$$

$$o((-x)^n) = o((-1)^n x^n) = o(x^n)$$

$$(1-x)^{-1} = \sum_{j=0}^{n} \binom{-1}{j} (-1)^j x^j + o(x^n)$$

All'inizio del corso abbiamo discusso la somma

$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

$$\frac{1}{1-x} = \sum_{j=0}^{n} x^j + \frac{x^{n+1}}{1-x}$$

$o(x^n)$

Notiamo che $\frac{x^{n+1}}{1-x} = o(x^n)$, infatti

$$\lim_{x\to 0} \frac{\frac{x^{n+1}}{1-x}}{x^n} = \lim_{x\to 0} \frac{x^{\cancel{n}+1}}{\cancel{x^n}(1-x)}$$

$$= \lim_{x\to 0} \frac{x}{1-x} = 0$$

$$\frac{1}{1-x} = \sum_{j=0}^{n} x^j + o(x^n)$$

$$\frac{1}{1+x} = \sum_{j=0}^{n} (-1)^j x^j + o(x^n)$$

$$\frac{1}{1+x^2} = \sum_{j=0}^{n} (-1)^j (x^2)^j + o((x^2)^n)$$

$$\frac{1}{1+x^2} = \sum_{j=0}^{n} (-1)^j x^{2j} + o(x^{2n}) \qquad o(x^n)$$

Qui in realtà ho

$$\frac{1}{1+x^2} = \sum_{j=0}^{n} (-1)^j x^{2j} + o(x^{2n+1})$$

Infatti $$\frac{1}{1+x^2} = \sum_{j=0}^{n+1} (-1)^j x^{2j} + o(x^{2n+2})$$

$$= \sum_{j=0}^{n} (-1)^j x^{2j} + \underbrace{\underbrace{(-1)^{n+1} x^{2n+2}}_{o(x^{2n+1})} + \underbrace{o(x^{2n+2})}_{o(x^{2n+1})}}_{o(x^{2n+1})}$$

$$o(x^{2n+2}) = o(x^{2n+1})$$

$$\lim_{x\to 0} \frac{o(x^{2n+2})}{x^{2n+1}} = \lim_{x\to 0} \frac{o(x^{2n+2})}{x^{2n+2}} \, x = 0$$

Pertanto

$$\cos(x) = \sum_{j=0}^{n} (-1)^j \frac{x^{2j}}{(2j)!} + o(x^{2n+1})$$

$$\sin(x) = \sum_{j=0}^{n} (-1)^j \frac{x^{2j+1}}{(2j+1)!} + o(x^{2n+2})$$

$$sh(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(e^x - e^{-x}\right) =$$

$$e^x = \sum_{j=0}^{n} \frac{x^j}{j!} + o(x^n)$$

$$e^{-x} = \sum_{j=0}^{n} (-1)^j \frac{x^j}{j!} + o(x^n)$$

$$sh(x) = \frac{1}{2}\left(\sum_{j=0}^{n} \frac{x^j}{j!} - \sum_{j=0}^{n} (-1)^j \frac{x^j}{j!} + o(x^n) \right)$$

$$= \sum_{\substack{j=0 \\ j \text{ dispari}}}^{n} \frac{x^j}{j!} \frac{\left(1-(-1)^j\right)}{2} + o(x^n)$$

$$sh(x) = \sum_{\substack{j=1 \\ j \text{ dispari}}}^{2n+1} \frac{x^j}{j!} \underbrace{\frac{\left(1-(-1)^j\right)}{2}}_{=1} + o(x^{2n+1})$$

$$= \sum_{\substack{j=1 \\ j \text{ dispari}}}^{2n+1} \frac{x^j}{j!} + o(x^{2n+1})$$

$j = 2k+1$

$0 \le k \le n$

$$sh(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$$

$$\sin(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$$

$$sh(x) = \sum_{k=0}^{n+1} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+3})$$

$$= \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + \underbrace{\frac{x^{2n+3}}{(2n+3)!} + o(x^{2n+3})}_{o(x^{2n+2})}$$

$2n$

$$ch(x) = \frac{1}{2}\left(e^x + e^{-x}\right) =$$

$$= \frac{1}{2}\left(\sum_{j=0}^{2n} \frac{x^j}{j!} + o(x^{2n}) + \sum_{j=0}^{2n} \frac{(-1)^j x^j}{j!} + o(x^{2n}) \right)$$

$$= \sum_{\substack{j=0 \\ j \text{ pari}}}^{2n} \frac{x^j}{j!} \underbrace{\frac{1+(-1)^j}{2}}_{=1} + o(x^{2n})$$

$j = 2k$

$0 \le k \le n$

$$ch(x) = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+1})$$

$f(x) = \sin(\operatorname{sh}(x)) \qquad P_4(x) = ?$

$$P_4(x) = \sum_{j=0}^{4} \frac{f^{(j)}(0)}{j!} x^j = f'(0)\,x + \frac{f'''(0)}{6} x^3$$

$$f(-x) = \sin(\operatorname{sh}(-x)) = \sin(-\operatorname{sh}(x)) = -\sin(\operatorname{sh}(x)) = -f(x)$$

$$\sin(y) = y - \frac{y^3}{6} + o(y^4)$$

$$\operatorname{sh}(x) = x + \frac{x^3}{6} + o(x^4)$$

$$f(x) = \sin(\operatorname{sh}(x)) = \operatorname{sh}(x) - \frac{\operatorname{sh}^3(x)}{6} + o(\operatorname{sh}^4(x))$$

$$= \left(x + \frac{x^3}{6} + o(x^4)\right) - \frac{\left(x + \frac{x^3}{6} + o(x^4)\right)^3}{6} + \underbrace{o\left(\left(x + \frac{x^3}{6} + o(x^4)\right)^4\right)}$$

$$o\left(\left(x + \frac{x^3}{6} + o(x^4)\right)^4\right) = o\left(x^4\left(1 + \frac{x^2}{6} + o(x^3)\right)^4\right)$$

$$= o(x^4)$$

$$\lim_{x\to 0} \frac{o\left(x^4\left(1+\frac{x^2}{6}+o(x^4)\right)^4\right)}{x^4\left(1+\frac{x^2}{6}+o(x^4)\right)^4} \to 0 \qquad \left(1+\frac{x^2}{6}+o(x^3)\right)^4 \to 1 \qquad = 0$$

$$\sin(\operatorname{sh}(x)) = x + \frac{x^3}{6} - \frac{\left(x + \frac{x^3}{6} + o(x^3)\right)^3}{6} + o(x^4)$$

$$= x + \frac{x^3}{6} - \frac{x^3 + \overbrace{3x^2\left(\frac{x^3}{6} + o(x^4)\right) + 3x\left(\frac{x^3}{6} + o(x^4)\right)^2}^{o(x^4)} + \left(\frac{x^3}{6} + o(x^4)\right)^3}{6}$$

$$+ o(x^4)$$

$$\sin(\operatorname{sh}(x)) = x + \cancel{\frac{x^3}{6}} - \cancel{\frac{x^3}{6}} + o(x^4) = x + o(x^4)$$

$$x^7 = o(x^4) \qquad \lim_{x\to 0} \frac{x^7}{x^4} = 0 \Rightarrow x^7 = o(x^4)$$

$$f(x) = \text{sen}(\text{sh}(x)) \qquad f' \quad f'' \quad f'''(0)$$

$$f'(x) = \cos(\text{sh}(x))\ \text{ch}(x)\Big|_{x=0} = \cos(0)\ \text{ch}(0) = 1$$

$$f''(x) = -\text{sen}(\text{sh}(x))\ \text{ch}^2(x) + \cos(\text{sh}(x))\ \text{sh}(x)$$

$$f'''(0) = -\cos(\text{sh}(x))\ \text{ch}^3(x)\Big|_{x=0} + \cos(\text{sh}(x))\ \text{ch}(x)\Big|_{x=0}$$

$$= -\cos(0) + \cos(0) = 0$$