24 Nov

Teor (Lagrange) Sia $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$ — esistono continue $\forall j \le n$ (a,b)

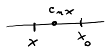
supponiamo su (a,b) esistono le derivate $f^{(j)}(x)$ $j\le n$

e che in $(a,b)\setminus\{x_0\}$ esista $f^{(n+1)}(x)$.

$$f(x)=P_n(x)+R_n(x)$$

$$P_n(x)=\sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

Vale la seguente formula: ~~esiste un~~ per ogni $x\neq x_0$ $\exists\ c_{nx}$ intermedio tra x e x_0



t.c.

$$R_n(x)=\frac{f^{(n+1)}(c_{nx})}{(n+1)!}(x-x_0)^{n+1}$$

$\square$

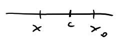
Osservazione. Questo teorema generalizza il teorema di Lagrange che garantisce quanto segue

$n=0$

$$f(x)=f(x_0)+R_0(x)\ .\ \text{Se } x\neq x_0$$

il teor di Lagrange garantisce che $\exists\, c$ tra x e x_0

t.c. $f'(c)=\dfrac{f(x)-f(x_0)}{x-x_0}$



$$f(x)=f(x_0)+\underbrace{f'(c)(x-x_0)}_{R_0(x)}$$

Dim (per $n=2$)

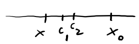
$$f(x)=P_2(x)+R_2(x)$$

$$R_2(x)=f(x)-P_2(x) \qquad f\in C^2((a,b))$$

$R_2\in C^2((a,b))$

$$f(x_0)=P_2(x_0),\quad f'(x_0)=P_2'(x_0),\quad f''(x_0)=P_2''(x_0)$$

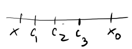
$$=\frac{R_2(x)-R_2(x_0)}{(x-x_0)^3-(x_0-x_0)^3}=\frac{R_2(x)}{(x-x_0)^3}$$



$$=\frac{R_2'(c_1)}{3(c_1-x_0)^2}=\frac{R_2'(c_1)-R_2'(x_0)}{3(c_1-x_0)^2-3(x_0-x_0)^2}$$

$$=\frac{R_2^{(2)}(c_2)}{6(c_1-x_0)}$$

$$=\frac{R_2^{(2)}(c_2)-R_2^{(2)}(x_0)}{6(c_2-x_0)-6(x_0-x_0)}$$



$$=\frac{R_2^{(3)}(c_3)}{6}=\frac{f^{(3)}(c_3)-\cancel{P_2^{(3)}(c_3)}}{3!}$$

$$=\frac{R_2(x)}{(x-x_0)^3}$$

$$R_2(x)=\frac{f^{(3)}(c_3)}{3!}(x-x_0)^3$$

$$R_n(x)=\frac{f^{(n+1)}(c_{nx})}{(n+1)!}(x-x_0)^{n+1}$$

e non è un numero razionale.

Esercizio. Approssimare e con un numero razionale con un errore $< 10^{-3}$

$$e^x = \sum_{j=0}^{n} \frac{x^j}{j!} + R_n(x)$$

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1} \quad \text{con } c \text{ tra } x \text{ e } 0$$

Per $x = 1$

$$e = \underbrace{\sum_{j=0}^{n} \frac{1}{j!}}_{p_n(1) \in \mathbb{Q}} + R_n(1)$$

$$R_n(1) = \frac{e^c}{(n+1)!} \quad \text{con } c \text{ tra } 0 \text{ e } 1.$$

$$0 < R_n(1) < \frac{e}{(n+1)!} \leq \frac{3}{(n+1)!} < \frac{1}{1000}$$

$$(n+1)! > 3000$$

$n = 6$

n	1	2	3	4	5	6
$n+1$	2	3	4	5	6	7
$(n+1)!$	2	6	24	120	720	5040

Esempio e è irrazionale.

Supponiamo per assurdo che $e \in \mathbb{Q}$

$$e = \frac{a}{b} \qquad a, b \in \mathbb{N}.$$

$$\frac{a}{b} = e = P_n(1) + R_n(1) = P_n(1) + \frac{e^{c_n}}{(n+1)!} \qquad 0 < c_n < 1$$

$$\frac{a}{b} = \sum_{j=0}^{n} \frac{1}{j!} + \frac{e^{c_n}}{(n+1)!} \qquad \cdot n!$$

$$\frac{a\, n!}{b} = \sum_{j=0}^{n} \underbrace{\frac{n!}{j!}}_{\in \mathbb{N}} + \frac{e^{c_n}}{n+1}$$

$$a\, n! = \sum_{j=0}^{n} \frac{n!}{j!} b + \frac{e^{c_n}\, b}{n+1}$$

$$a\, n! - \sum_{j=0}^{n} \frac{n!}{j!} b = \frac{e^{c_n}\, b}{n+1} > 0$$

$$\Rightarrow \quad 1 \le a\, n! - \sum_{j=0}^{n} \frac{n!}{j!} b = \frac{e^{c_n}\, b}{n+1} \qquad 0 < c_n < 1$$

$$1 \le \frac{e^{c_n}\, b}{n+1} < \underbrace{\frac{e\, b}{n+1}}_{\substack{\to 0 \\ n \to +\infty}} \qquad \forall n$$

Assurdo!

$$\lim_{n\to+\infty} \frac{(2n)!}{n^n \; n!}$$

$$\frac{(2n)!}{n^n \; n!} = \frac{\overbrace{1\cdot\cdots\cdot n}^{\cancel{n!}}\;(n+1)\cdots(n+n)}{\cancel{n!}\;\; n^n}$$

$$= \frac{(n+1)}{n}\cdots\frac{n+n}{n} = \prod_{j=1}^{n}\left(\frac{n+j}{n}\right)$$

$$= \prod_{j=1}^{n}\left(1+\frac{j}{n}\right) = \prod_{j=1}^{[\frac{n}{2}]}\underbrace{\left(1+\frac{j}{n}\right)}_{>1} \;\; \prod_{j=[\frac{n}{2}]+1}^{n}\left(1+\frac{j}{n}\right)$$

$$\geq \prod_{j=[\frac{n}{2}]+1}^{n}\left(1+\frac{[\frac{n}{2}]+1}{n}\right)$$

$$> \prod_{j=[\frac{n}{2}]+1}^{n}\left(1+\frac{\frac{n}{2}}{n}\right) = \prod_{j=[\frac{n}{2}]+1}^{n}\frac{3}{2}$$

$$= \left(\frac{3}{2}\right)^{n-[\frac{n}{2}]-1}$$

$$\frac{n}{2} \geq \left[\frac{n}{2}\right]$$

$$-\frac{n}{2} \leq -\left[\frac{n}{2}\right]$$

$$\geq \left(\frac{3}{2}\right)^{n-\frac{n}{2}-1} = \left(\frac{3}{2}\right)^{\frac{n}{2}-1} \xrightarrow[n\to+\infty]{} +\infty$$

Formula di Stirling

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\left(1+o(1)\right)$$

$$\frac{(2n)!}{n^{n}\,n!} = \frac{\sqrt{4\cancel{\pi}\cancel{n}}\left(\frac{2n}{e}\right)^{2n}\cancel{(1+o(1))}}{n^{n}\sqrt{2\cancel{\pi}\cancel{n}}\left(\frac{n}{e}\right)^{n}\cancel{(1+o(1))}}$$

$$= \sqrt{2}\,\frac{\left(\frac{2}{e}\right)^{2n}\cancel{n^{2n}}}{\cancel{n^{2n}}\,\frac{1}{e}^{n}}$$

$$= \sqrt{2}\,\frac{2^{2n}}{e^{2n}}\,e^{n} = \sqrt{2}\,\frac{2^{2n}}{(\sqrt{e})^{2n}} =$$

$$= \sqrt{2}\left(\frac{2}{\sqrt{e}}\right)^{2n} \longrightarrow +\infty \qquad \frac{2}{\sqrt{e}} > 1$$