24 Nov

$$\begin{cases} i\partial_t u = -\Delta u + \lambda |u|^{p-1} u \\ u|_{t=0} = u_0 \end{cases}$$

$$\varphi \in C^\infty((-1,1), [0,1])$$

$$Q_n = \varphi\left(\frac{\sqrt{-\Delta}}{n}\right)$$

$$P_n = \chi_{[0,1]}\left(\frac{\sqrt{-\Delta}}{n}\right)$$

$$\begin{cases} i\partial_t u_n = -P_n \Delta u_n + \lambda Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right) \\ u_n|_0 = Q_n u_0 \end{cases}$$

$$E_n(v) \doteq \frac{1}{2}|P_n \nabla v|^2_{L^2} + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n v|^{p+1}\,dx$$

$$dE_n \in C^0(L^2(\mathbb{R}^d), \mathcal{L}(L^2(\mathbb{R}^d), \mathbb{R}))$$

$$dE_n(v) = \langle \nabla E_n(v), \cdot \rangle$$

$$Q(v) = \frac{1}{2}|v|^2_{L^2}$$

$$P_j(v) = \frac{1}{2} \langle i\partial_j v, v \rangle$$

Maximal forward solution

$u_n \qquad [0, T(n))$

If $\quad T(n) < +\infty$ then

$$\lim_{t \to T(n)} |u_n|_{H^1} = +\infty$$

$$|u_n|_{H^1} = |P_n u_n|_{H^1} = |\langle \xi \rangle \widehat{P_n u_n}|_{L^2}$$

$$\leq \langle n \rangle \ |u_n|_{L^2} \leq \langle n \rangle |Q_n u_0|_{L^2}$$

$$\leq \langle n \rangle \ |u_0|_{L^2}$$

Fix $\quad M > |u_0|_{H^1} \geq |Q_n u_0|_{H^1}$

$$\vartheta_n \doteq \sup \{ \tau > 0 : |u_n(t)|_{H^1} < 2M \text{ for } 0 \leq t < \tau \}$$

We want to show $\exists \ T(M) > 0$ s.t.

$\vartheta_n \geq T(M)$.

$u_n \in C^{0,\frac{1}{2}}([0, \vartheta_n), L^2)$ with ~~are~~

Hölder norm bounded by a $C(M)$

$$|u_n(t)-u_n(s)|_{L^2} \leq |u_n(t)-u_n(s)|^{\frac12}_{H^1} \, |u_n(t)-u_n(s)|^{\frac12}_{H^{-1}} \leq$$

$$\int |\hat{u}_n(t)-\hat{u}_n(s)|^2 \langle\xi\rangle\langle\xi\rangle^{-1} d\xi \leq \left(\int |\hat{u}_n(t)-\hat{u}_n(s)|^2 \langle\xi\rangle^2 d\xi\right)^{\frac12} \left(\int |\hat{u}_n - \hat{u}_n|^2 \langle\xi\rangle^{-2} d\xi\right)^{\frac12}$$

$$\lesssim |u_n|^{\frac12}_{L^\infty([0,\vartheta_n), H^1)} \, |t-s|^{\frac12} \, |\partial_t u_n|^{\frac12}_{L^\infty([0,\vartheta_n); H^{-1})}$$

$$|i\partial_t u_n|_{H^{-1}} \leq |\cancel{P_n}\Delta u_n|_{H^{-1}} + \cancel{\pm}\left|Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right)\right|_{H^{-1}}$$

$$\leq C_1(M)$$

$$|u_n(t)-u_n(s)|_{L^2} \leq C(M)\,|t-s|^{\frac12} \qquad t,s \text{ in } [0,\vartheta_n)$$

We show

$$|u_n(t)|^2_{H^1} \leq |u_0|^2_{H^1} + C(M)\,t^b \qquad \text{for any } t\in[0,\vartheta_n)$$

for a fixed b

$$E_n(u_n(t)) = E_n(Q_n u_0)$$

$$Q(u_n(t)) = Q(Q_n u_0)$$

$$|\nabla u_n(t)|_{L^2}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n u_n(t)|^{p+1} dx =$$

$$= |\nabla Q_n u_0|_{L^2}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n^2 u_0|^{p+1} dx$$

$$|u_n(t)|_{L^2}^2 = |Q_n u_0)|_{L^2}^2$$

$$|u_n(t)|_{H^1}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n u_n(t)|^{p+1} dx$$

$$= |Q_n u_0|_{H^1}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |Q_n^2 u_0|^{p+1} dx$$

$0 \le \cancel{t} \le \vartheta_n$

$$|u_n(t)|_{H^1}^2 \le |Q_n u_0|_{H^1}^2 + \frac{2}{p+1}\int_{\mathbb{R}^d} \left| |Q_n u_n(t)|^{p+1} - |Q_n^2 u_0|^{p+1} \right| dx$$

$$\le |Q_n u_0|_{H^1}^2 + C\int_{\mathbb{R}^d} \left(|Q_n u_n|^p + |Q_n^2 u_0|^p \right) |Q_n u_n - Q_n^2 u_0| dx$$

$$\le |Q_n u_0|_{H^1}^2 + C\left(|Q_n u_n|_{L^{p+1}}^p + |Q_n^2 u_0|_{L^{p+1}}^p \right) |u_n(t) - Q_n u_0|_{L^{p+1}}$$

$$|u_n(t)|_{H^1}^2 \le |u_0|_{H^1}^2 + C(M)\, |u_n(t) - Q_n u_0|_{\dot{H}^1}^{\vartheta} |u_n(t) - u_n(0)|_{L^2}^{1-\vartheta}$$

$$2 < p+1 < d^* + 1$$

$$\dot{H}^{\vartheta}(\mathbb{R}^d) \hookrightarrow L^{p+1} \qquad \alpha < \vartheta < 1$$

$$|u_n(t)|_{H^1}^2 \le |u_0|_{H^1}^2 + C(M)\; t^b \qquad b > 0$$

Now $\quad C(M)T^b(M) = 2M^2$

Then in $[0, T(M)]$

$$|u_n(t)|^2_{H^1} \leq M^2 + C(M)\,T^b(M) =$$
$$= 3M^2$$

$\Rightarrow$ in $[0, T(M)] \Rightarrow |u_n(t)|_{H^1} \leq \sqrt{3}\,M < 2M$

$\Rightarrow 0 < T(M) \leq \vartheta_n$

$[0, T(M))$ $\quad u_n \longrightarrow u$

Let us prove the preservation of $\mathcal{Q}$ and P_j by the truncated systems

$$\Omega(X, Y) = \langle iX, Y \rangle$$
$$\langle X, Y \rangle = \mathrm{Re} \int X\, \bar{Y}\, dx$$

Def Let E be a Banach space E' the dual. A strong symplectic form Ω_p on E is a 2-differential form s.t. $d\Omega = 0$ and the map

$$\begin{aligned} E &\longrightarrow E' \\ x &\longrightarrow \Omega_p(x, \cdot) \end{aligned}$$

is an isomorphism

<u>Def</u> Let $p \to \Omega_p$ be a strong symplectic form on E and let $F \in C^1(E, \mathbb{R})$. Then the hamiltonian vector field $X_F(p)$ is defined by

$$\Omega_p(X_F(p), Y) = dF(p)Y \quad \forall p \in E \text{ and } \forall Y \in E$$

$$\Omega(X_F, Y) = \langle iX_F, Y \rangle$$

$$= dF\, Y = \langle \nabla F, Y \rangle$$

$$X_F = -i\nabla F$$

<u>Def</u> Ω_p in E $F, G \in C^1(E, \mathbb{R})$

$$\{F, G\}(p) := \Omega_p(X_F(p), X_G(p)) =$$

$$= dF(p)\, X_G(p)$$

$$\{F,G\} = dF\, X_G = \langle \nabla F, X_G \rangle =$$

$$= \langle \nabla F, -i\nabla G \rangle = \langle i\nabla F, \cancel{\nabla} G \rangle$$

$$H^1(\mathbb{R}^d) \qquad F \in C^1(H^1(\mathbb{R}^d), \mathbb{R})$$

$$\frac{d}{dt} F(u_n(t)) = dF(u_n(t))\, \dot{u}_n(t) =$$

$$= \langle \nabla F(u_n), \dot{u}_n(t) \rangle =$$

$$= \langle \nabla F(u_n), X_{E_n}(u_n) \rangle$$

$$= \langle \nabla F(u_n), -i\nabla E_n(u_n) \rangle$$

$$= \{F, E_n\}(u_n)$$

$$H^1 \qquad u \to e^{i\vartheta} u$$

$$S^1 = \{ e^{i\vartheta} : \vartheta \in \mathbb{R} \}$$

$$0 = \frac{d}{d\vartheta} E_n(e^{i\vartheta} u)\Big|_{\vartheta=0} = dE_n(u)\, \partial_\vartheta e^{i\vartheta} u \Big|_{\vartheta=0}$$

$$= dE_n(u)\, iu = \langle \nabla E_n(u), iu \rangle =$$

$$= \langle \nabla E_n(u), i\nabla Q(u) \rangle = \{Q, E_n\}$$

$$Q(u) = \frac{1}{2} \langle u, u \rangle$$

$$\frac{d}{dt} Q(u+tX)\Big|_{t=0} = \langle u, X \rangle = dQ(u) \frac{d}{dt}(u+tX)\Big|_{t=0}$$

$$= dQ(u)X = \langle \nabla Q(u), X \rangle$$

$$\nabla Q(u) = u$$

$$\frac{d}{dt} Q(u_n(t)) = \{Q, E_n\}(u_n(t)) \equiv 0$$

$$\frac{\partial}{\partial \vartheta} e^{i\vartheta} u \Big|_{\vartheta=0} = iu =$$

$$u = \nabla Q(u)$$

$x_0 \in \mathbb{R}^d$ acts by Translations on $H^1(\mathbb{R}^d)$

$$\tau_{x_0} u = u(\cdot - x_0)$$

$$\tau_{\lambda e_1} u = u(\cdot - \lambda e_1)$$

$$\frac{\partial}{\partial \lambda} \tau_{\lambda e_1} u \Big|_{\lambda=0} = -\partial_1 u = \nabla P_J$$

$$0 = \frac{d}{d\lambda} E_n(\tau_{\lambda e_1} u) = dE_n(u)\, \partial_\lambda \tau_{\lambda e_1} u \Big|_{\lambda=0}$$

$$= -dE_n\, \partial_1 u = \langle \nabla E_n, -\partial_1 u \rangle = \langle \nabla E_n, i\nabla P_1 \rangle$$

$$= \{P_1, E_n\} \equiv 0$$

$$P_1(u) = \frac{1}{2}\langle i\partial_1 u, u \rangle$$

$$\frac{d}{dt} P_1(u + tX)\Big|_{t=0} = \langle i\partial_1 u, X \rangle = \langle \nabla P_1, X \rangle$$

$$i\partial_1 u = \nabla P_1$$

$$-\partial_1 u = i\nabla P_1$$

$$\frac{d}{dt} P_1(u_n(t)) = \{P_1, E_n\}(u_n(t)) \equiv 0$$

$$u_n \longrightarrow u \quad \text{in} \quad [0, T(M)]$$

$$u_n(t) = e^{it\Delta \cancel{P_n}} Q_n u_0 - i\lambda \int_0^t ds\, e^{i(t-s)\Delta \cancel{P_n}} Q_n(|Q_n u_n|^{p-1} Q_n u_n)$$

$$e^{it\Delta P_n} Q_n u_0 = \sum_{j=0}^{\infty} \frac{(it\Delta)^j}{j!} Q_n u_0$$

$$= e^{it\Delta} Q_n u_0$$

$$e^{-it|\xi|^2} \chi_{D(0,n)} \varphi\left(\frac{\cdot}{n}\right) = \sum_{j=0}^{\infty} (-it|\xi|^2) \quad \underbrace{\chi_{D(0,n)} \varphi\left(\frac{\cdot}{n}\right)}_{\varphi\left(\frac{\cdot}{n}\right)}$$

$$= e^{-it|\xi|^2} \varphi\left(\frac{\cdot}{n}\right)$$

$$u_n \longrightarrow u \quad \text{in } C^0([0,T(M)], H^1)$$

$$\begin{cases} u_n(t) = e^{it\Delta} Q_n u_0 - i\lambda \int_0^t ds\, e^{i(t-s)\Delta} Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right) \\ u(t) = e^{it\Delta} u - i\lambda \int_0^t ds\, e^{i(t-s)\Delta} \left(| u|^{p-1} u_n\right) \end{cases}$$

$$Q(u_n) \longrightarrow Q(u) \quad \text{in } \underline{C^0([0,T(M)], \mathbb{R})}$$

$$P_j(u_n) \longrightarrow P_j(u)$$

$$E_n(u_n) \longrightarrow E(u)$$