Image Processing for Physicists

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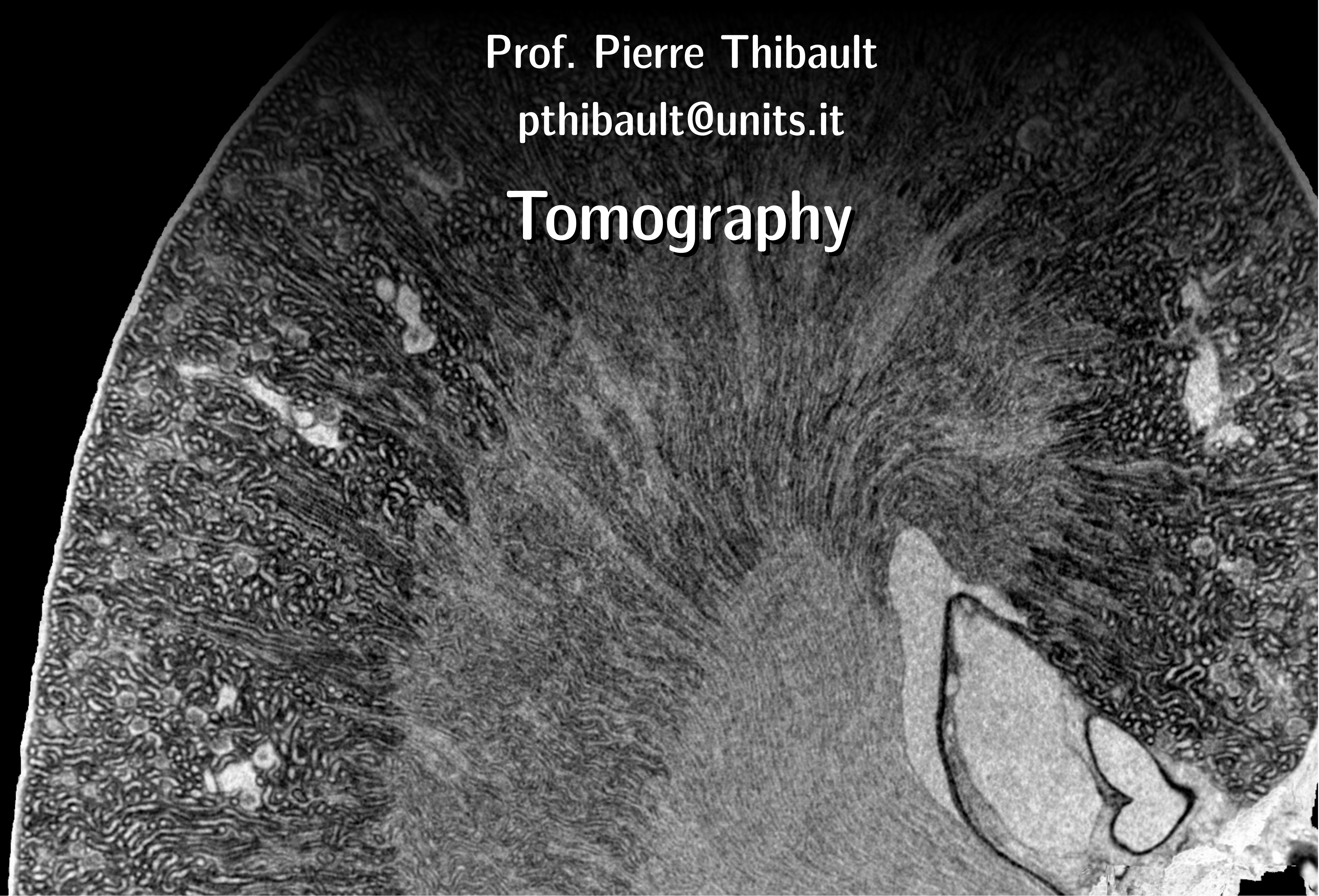
Tomography

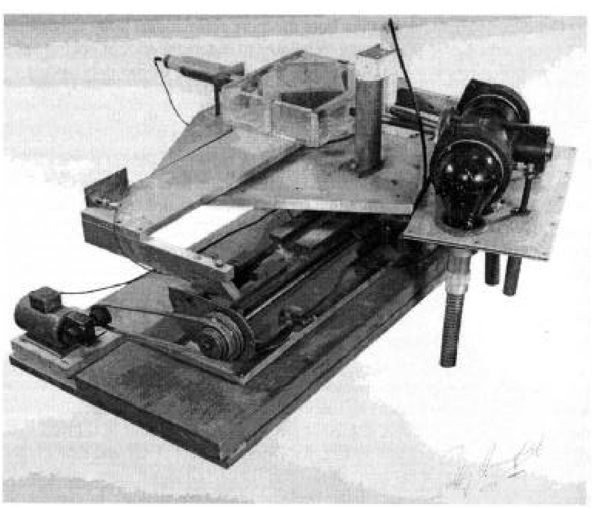


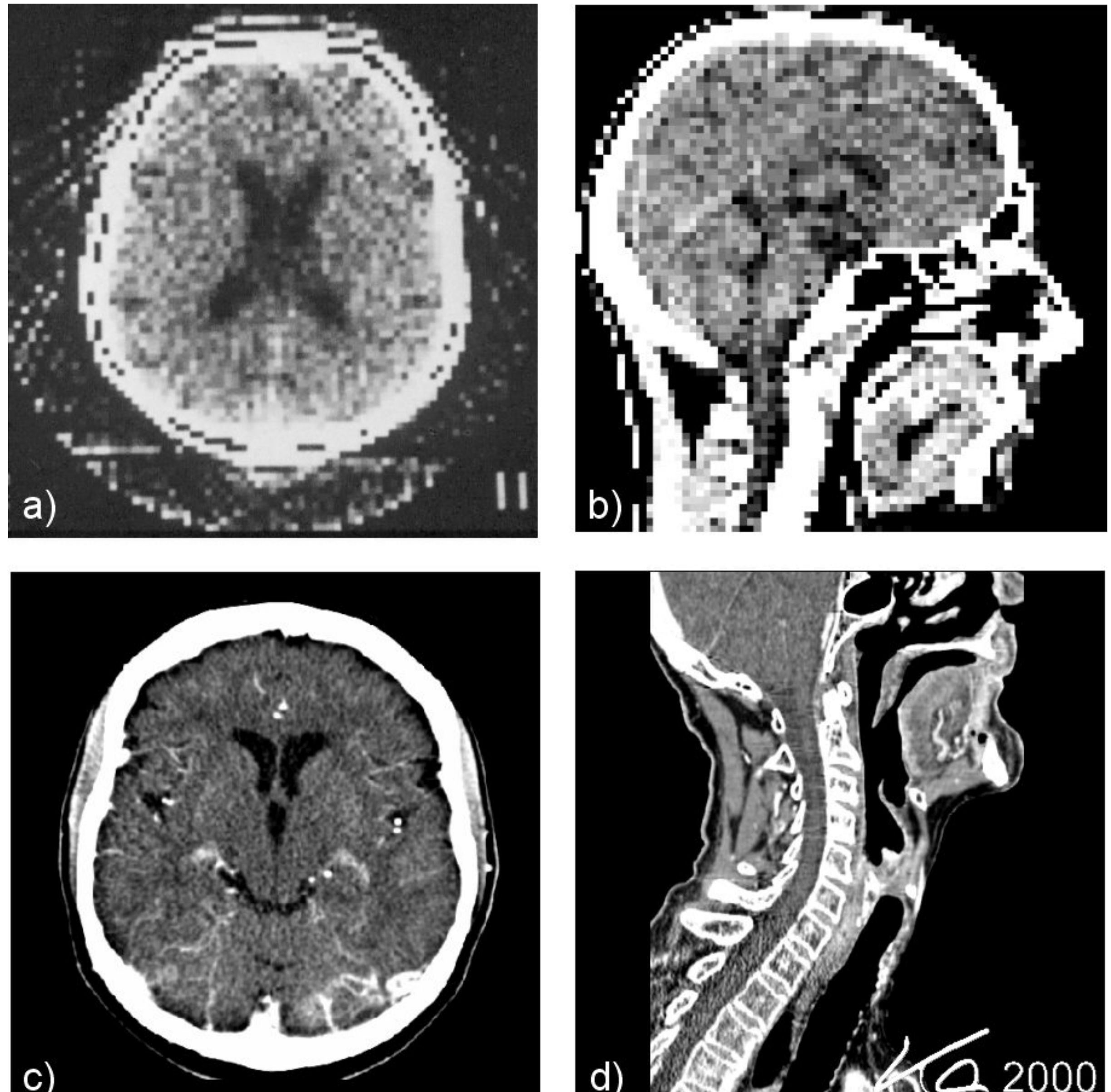
Overview

- Fundamentals of tomography
  - Physics & geometry
- Analytic formulation
  - Radon transform
  - Filtered back-projection
- Algebraic formulation

Examples of tomographic imaging

Computed (X-ray) Tomography (CT)





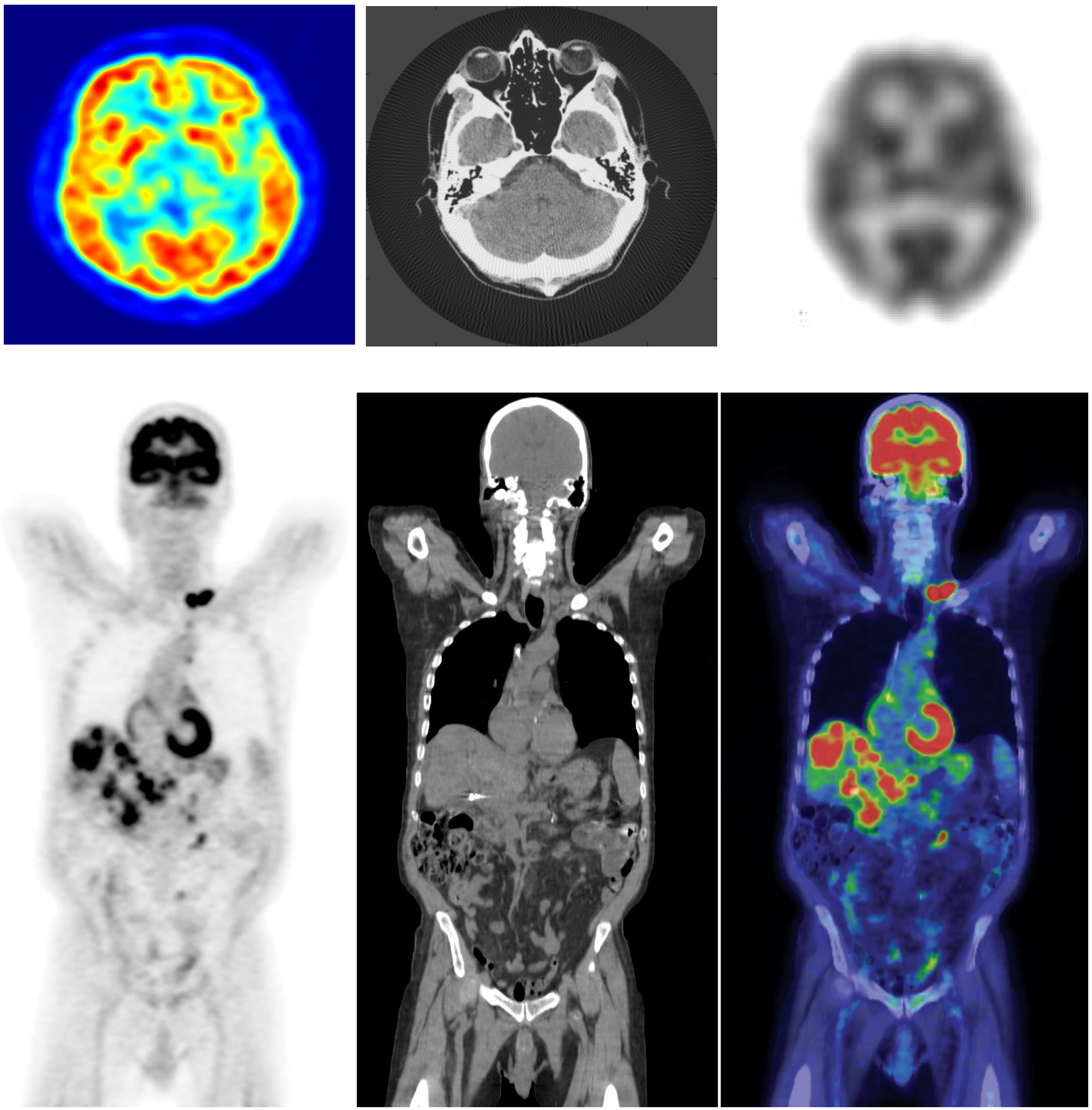
1974, 80x80 pixels

2000, 512x512 pixels, spiral CT

source: W. Kalender, Publicis, 3rd ed. 2011

Examples of tomographic imaging

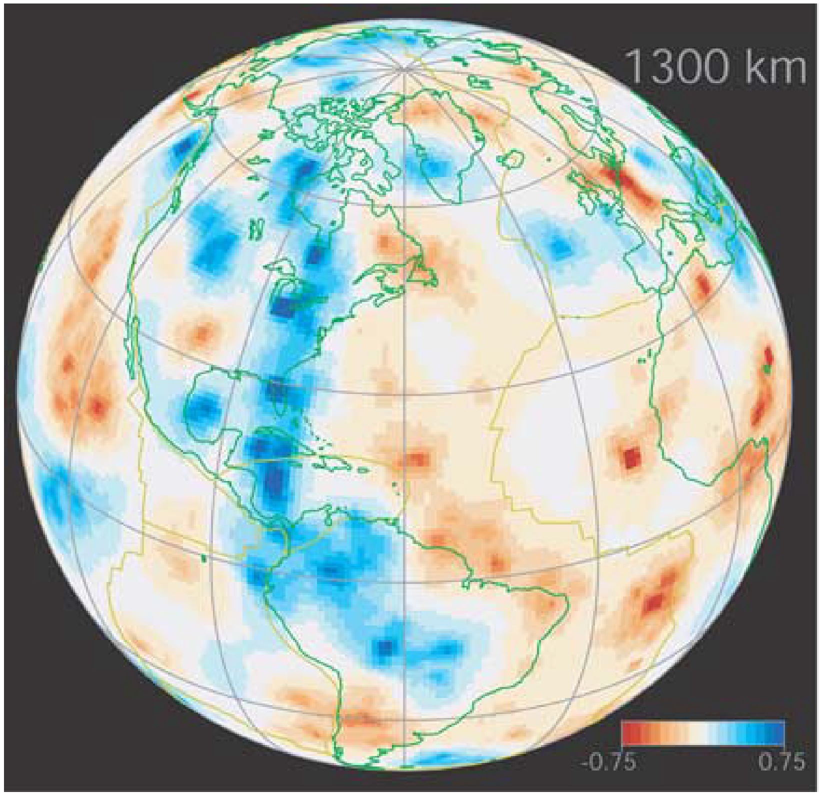
Positron emission tomography (PET) + CT

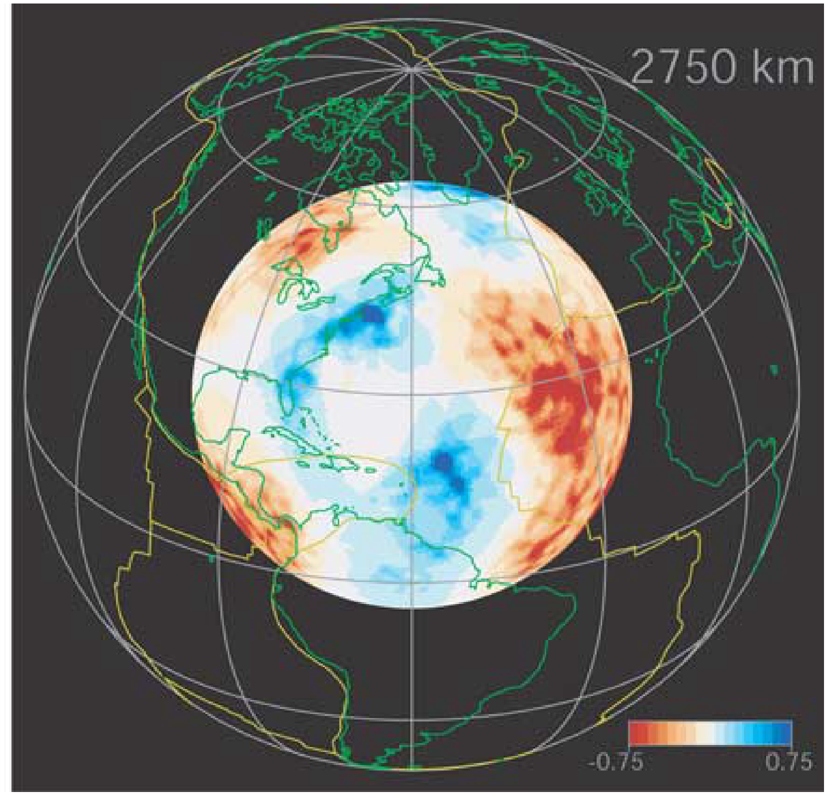


Single-Photon Emission Computed Tomography (SPECT)

Examples of tomographic imaging

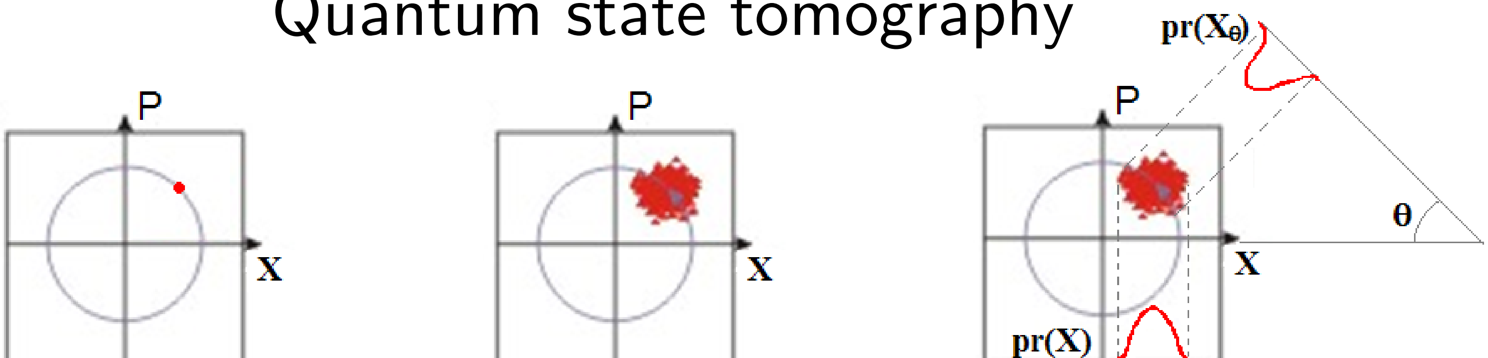
Seismic tomography





source: Sambridge et al. G3 Vol.4 Nr.3 (2003)

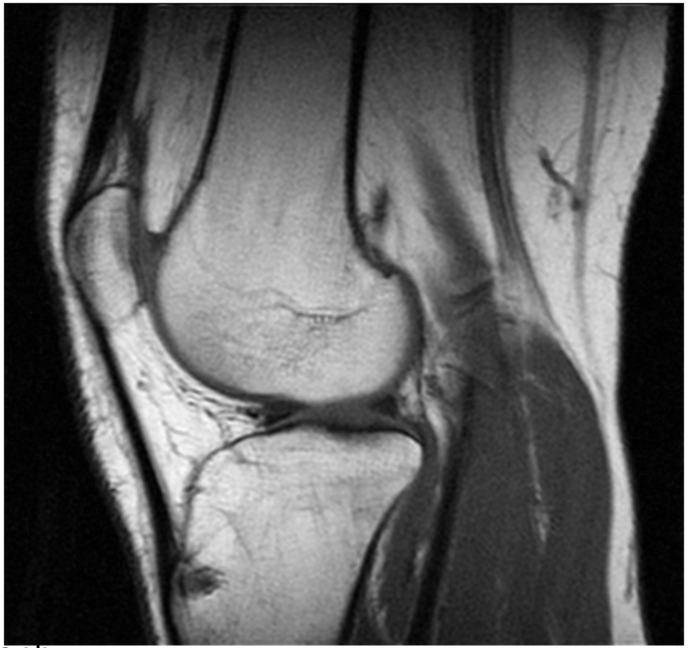
Quantum state tomography

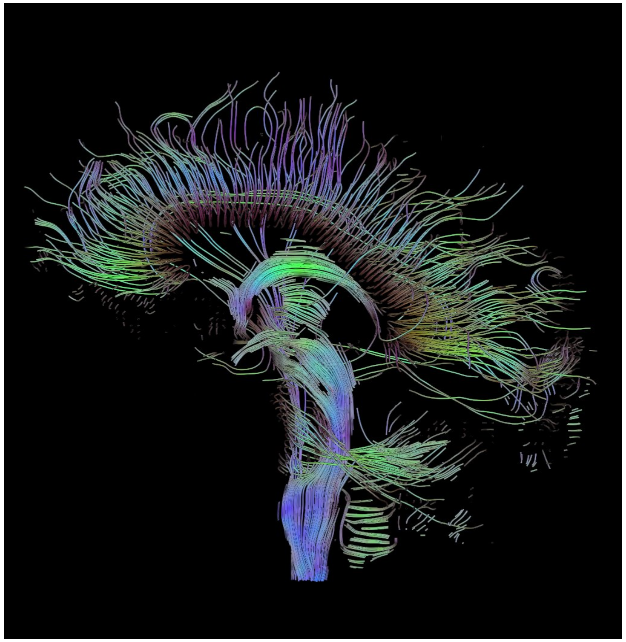


Examples of tomographic imaging

Ultrasonography/tomography (US/UST)

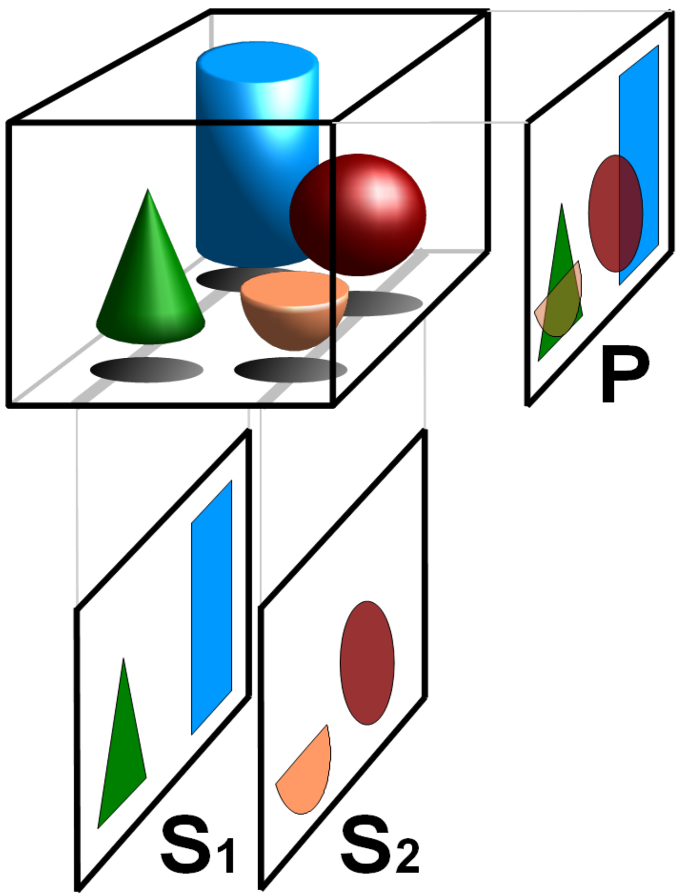
Magnetic resonance imaging/tomography (MRI/MRT)





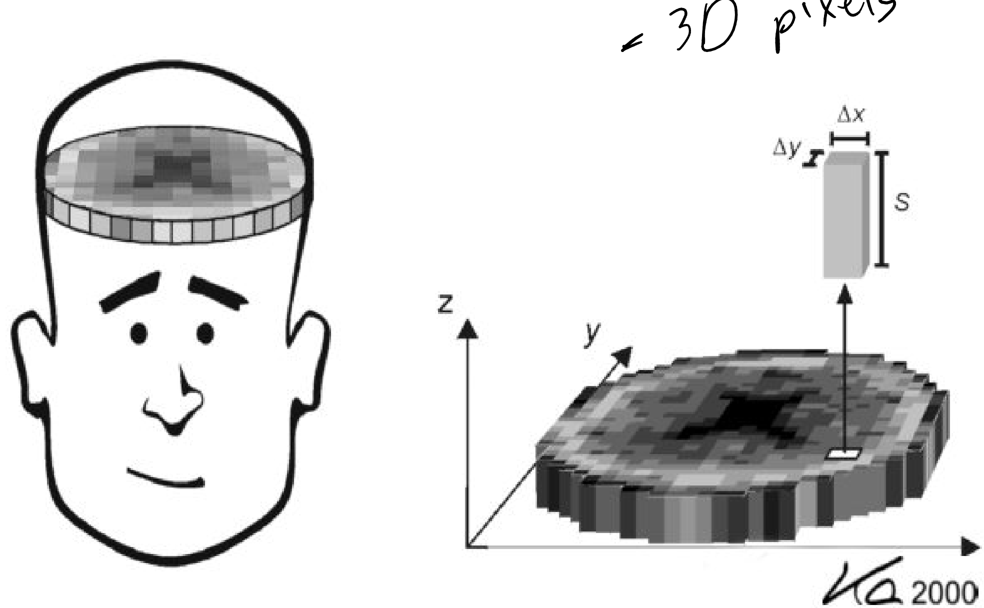
Reconstructions from projections

Reconstruction of volume from projections



Digitization into voxels

volume elements
= 3D pixels



source: W. Kalender, Publicis, 3rd ed. 2011

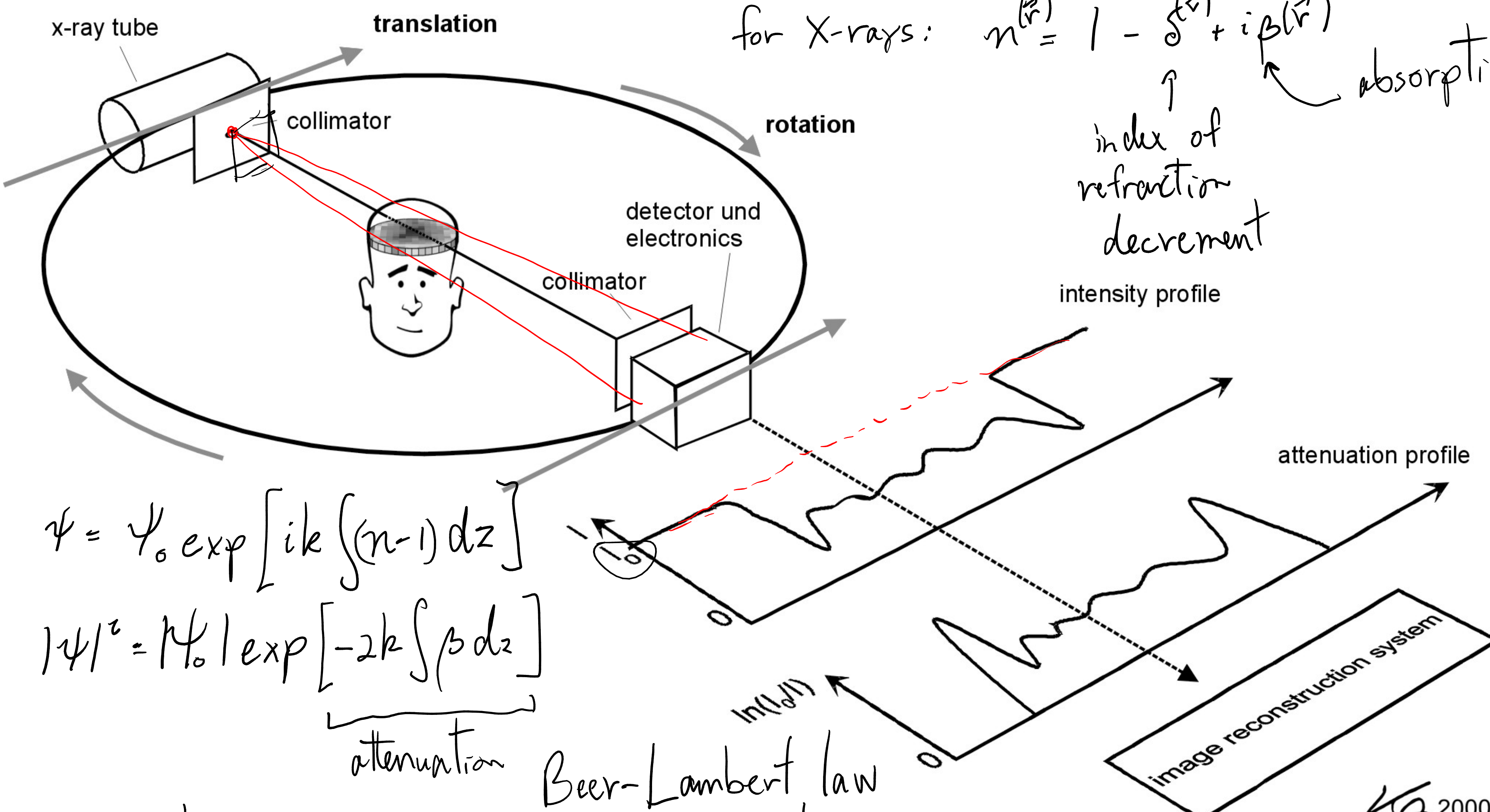
Principles of X-ray CT

sample: index of refraction $\neq 1$

for X-rays: $n(\vec{r}) = 1 - \delta(\vec{r}) + i\beta(\vec{r})$

index of refraction decrement

absorption



$$\psi = \psi_0 \exp\left[ik\int(n-1)\,dz\right]$$

$$|\psi|^2 = |\psi_0| \exp\Big[\underbrace{-2k\int\beta\,dz}_{\text{attenuation}}\Big]$$

$$\mu = 2k\beta$$

$$\ln\left(\frac{I_0}{I}\right) = \int \mu\,dz$$

Beer-Lambert law

$$I = I_0 e^{-\mu t}$$

thickness

attenuation coefficient

$$= I_0 \exp\left(-\int \mu\,dz\right)$$

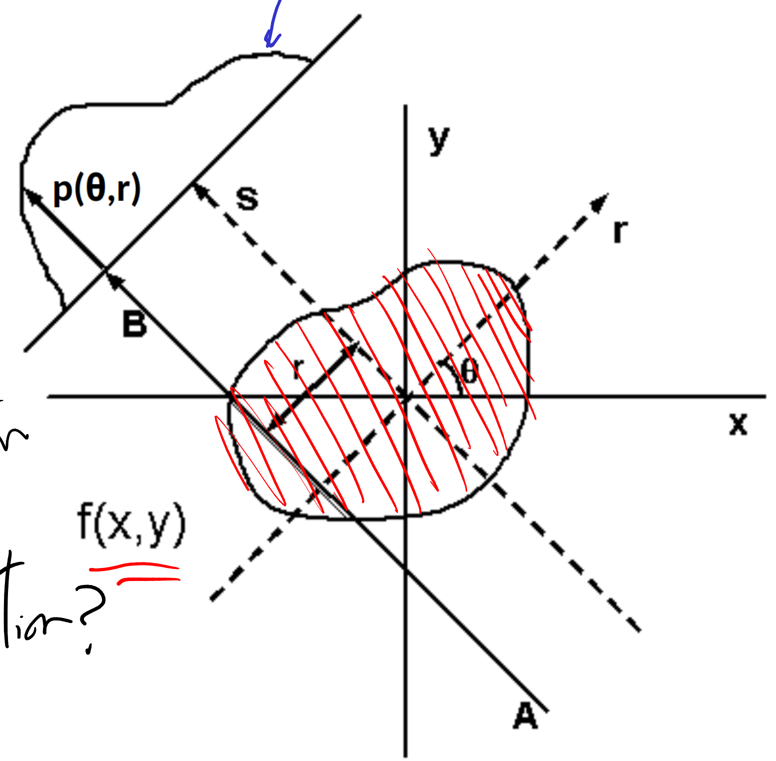
Source: W. Kalender, Publicis, 3rd ed. 2011

Radon transform

Rotated coordinate system

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

projection at θ



Radon transform

$$p(\theta, r) = \int f(x = r\cos\theta - s\sin\theta,\ y = s\cos\theta + r\sin\theta)\, ds$$

Radon transform is a linear transformation

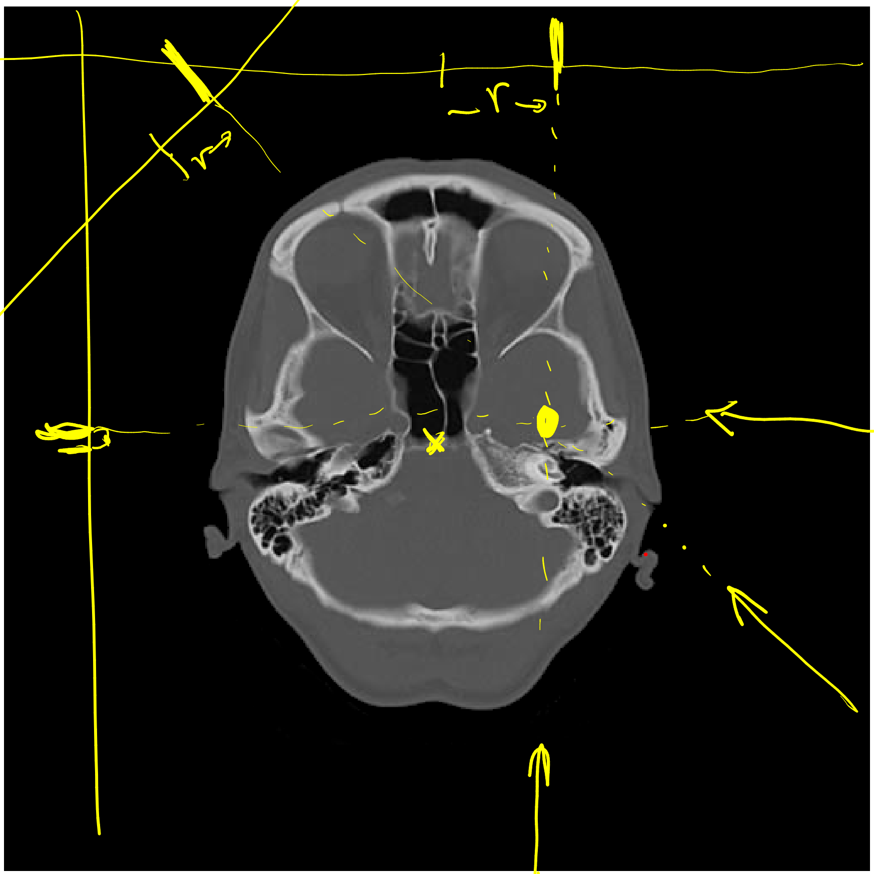
measured: $p(\theta, r)$; can we invert this relation?

<u>inverse Radon transform</u>

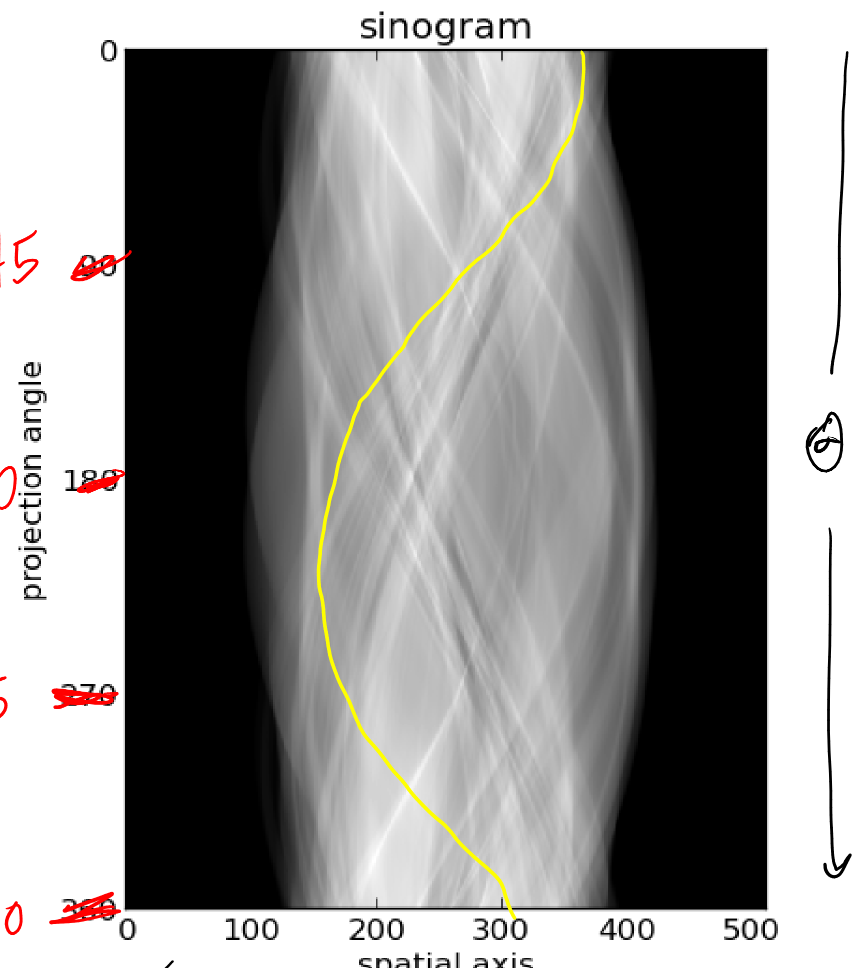
Sinogram

Representation of projection measured by
a single detector line as a function of angle

$f(x,y)$



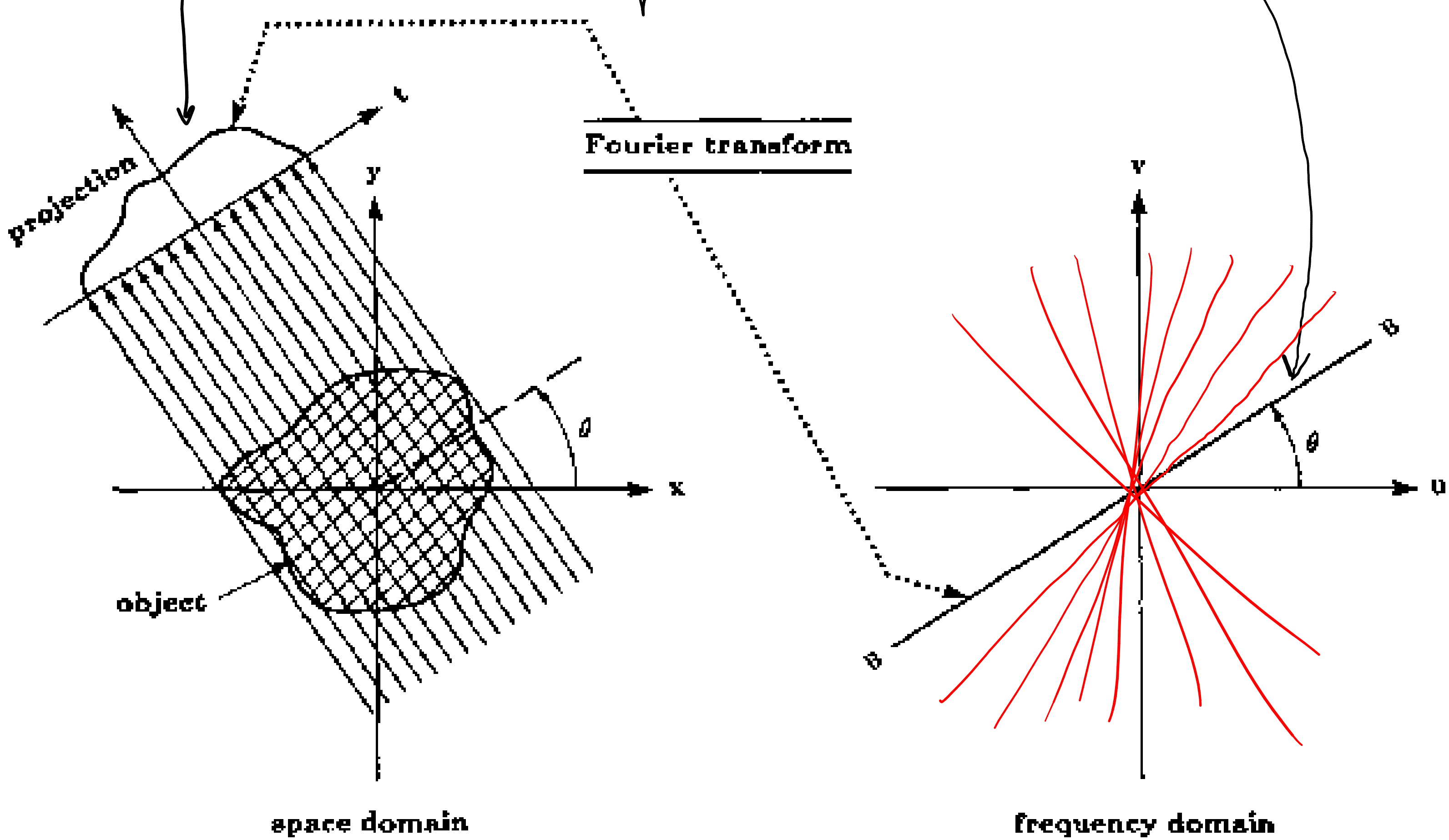
$p(\theta,r)$



computed Tomography

The Fourier slice theorem

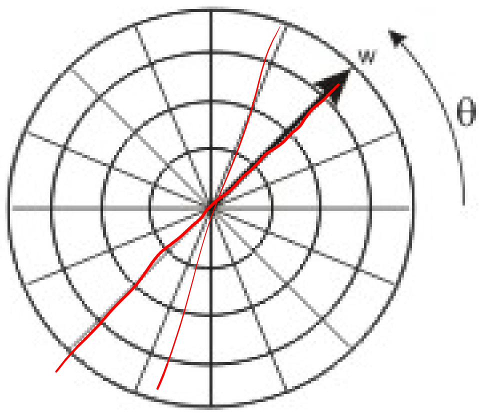
$p(\theta, r) \longrightarrow \mathcal{F}_r\{p(\theta, r)\} = P(\theta, w)$

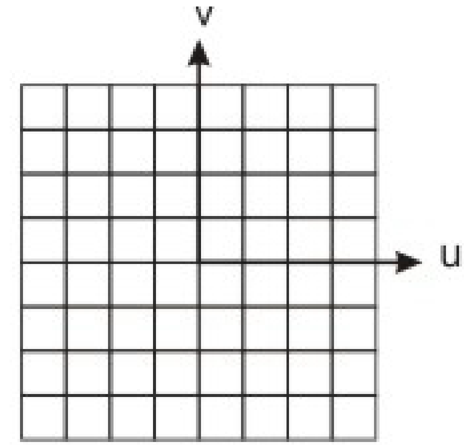


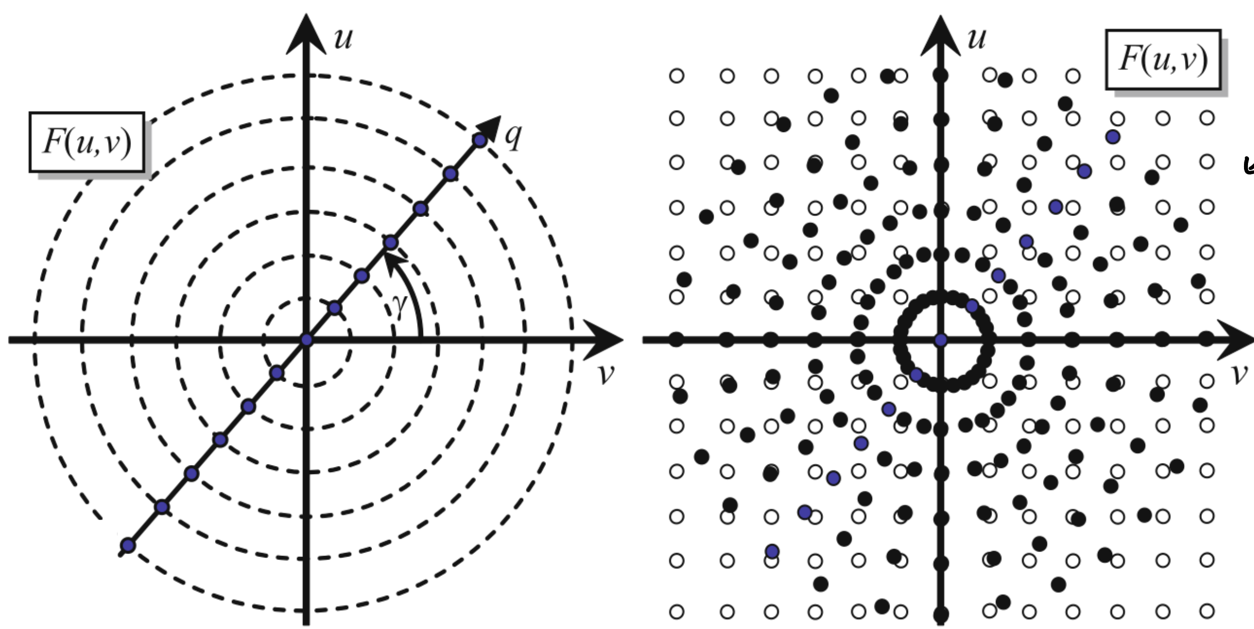
$f(x,y) \longrightarrow \mathcal{F}^{2D} \longrightarrow F(u,v)$

Frequency space sampling

Change of sampling grid from polar
to rectangular requires interpolation







a

b

black circles:
from measurements

white circles:
required values
to compute
inverse F.T.

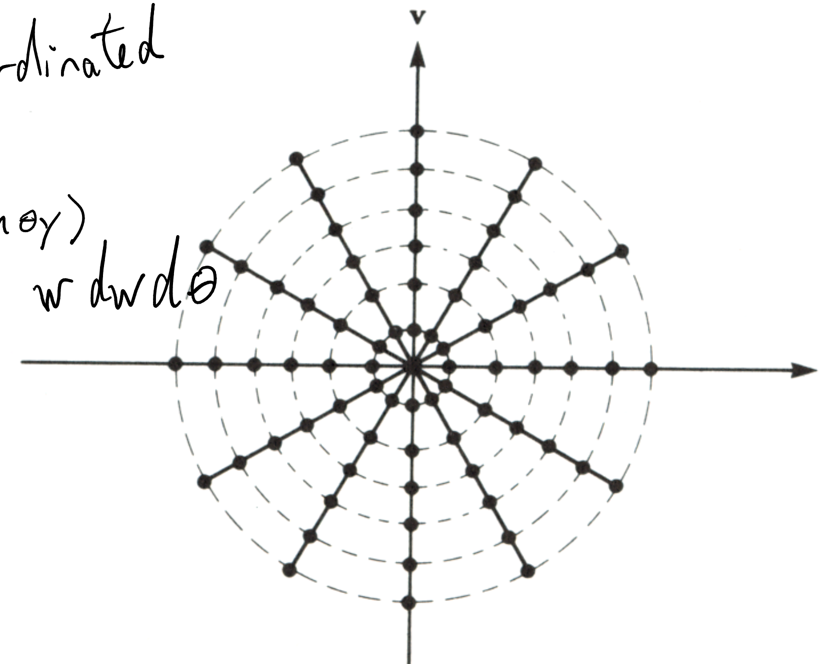
Filtered back-projection

$$f(x,y) = \mathcal{F}^{-1}\{F(u,v)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i(ux+vy)}\, du\, dv$$

polar coordinated

$$= \int_{0}^{2\pi}\int_{0}^{\infty} F(w\cos\theta, w\sin\theta)\, e^{2\pi i(w\cos\theta\, x + w\sin\theta\, y)}\, w\, dw\, d\theta$$

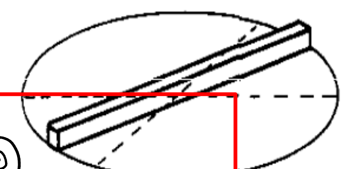


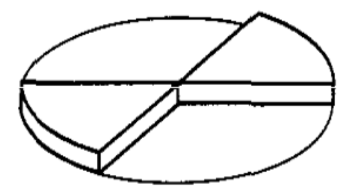
frequency domain

1D. F.T. of $p(\theta, r)$:

$$\mathcal{F}_r\{p(\theta,r)\} = F(u,v)\Big|_{\text{on the line at angle } \theta}$$

$$= F(w\cos\theta, w\sin\theta)$$

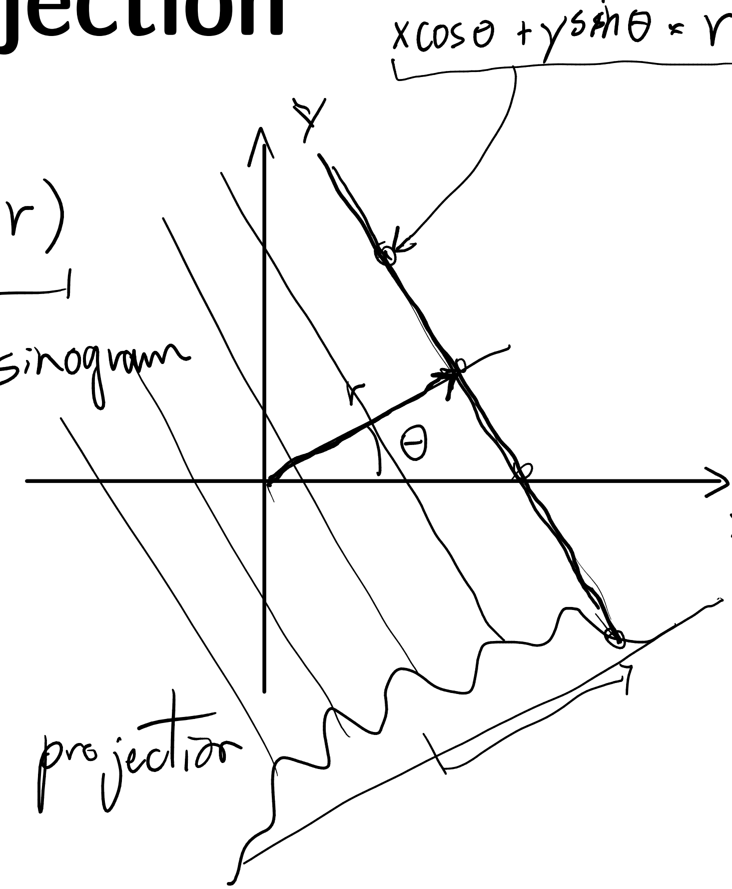




$$f(x,y) = \int_{0}^{\pi} \boxed{\int_{-\infty}^{\infty} \mathcal{F}_{r\to w}\{p(\theta,r)\}\, e^{2\pi i w(x\cos\theta + y\sin\theta)}\, |w|\, dw}\, d\theta$$

Filtered back-projection

$$\int_{-\infty}^{\infty} \mathcal{F}_{r\to w}\{p(\theta,r)\}\, e^{2\pi i w r}\, |w|\, dw = \underbrace{p'(\theta,r)}_{\text{filtered sinogram}}$$



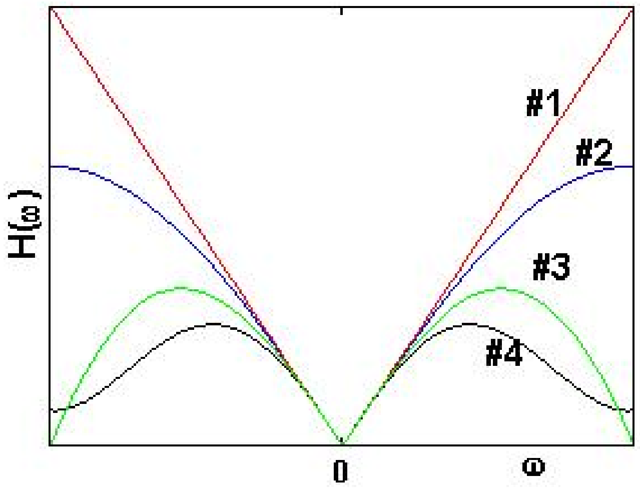
$$f(x,y) = \int_0^{\pi} \underbrace{p'(\theta, r = x\cos\theta + y\sin\theta)}_{\text{back-projected filtered projection}}\, d\theta$$

Recipe: 1) apply Ramp filter (also called "Ram-Lak") on sinogram

2) Back-project

Filtered back-projection

- Filter can be tuned to achieve image enhancement
- Trade-off between noise and sharpness

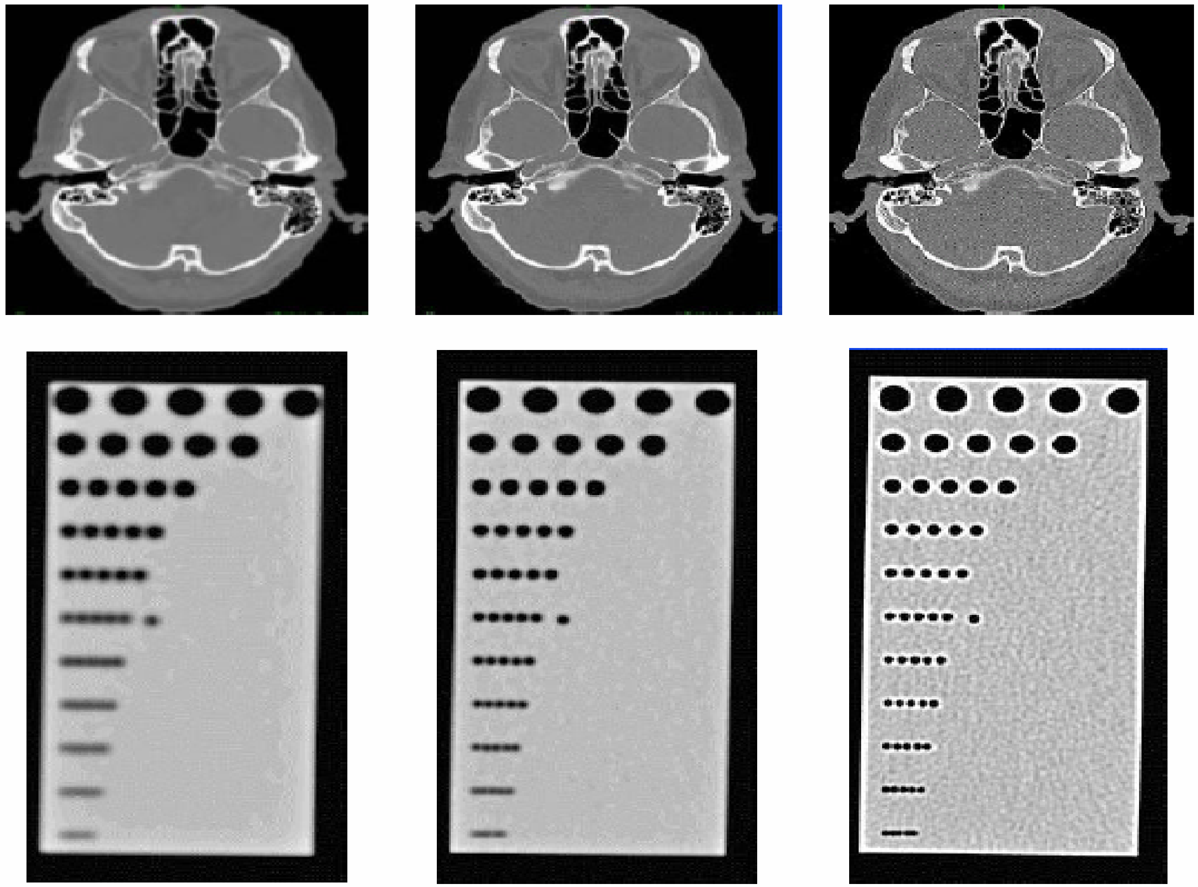


#1 ram-lak (ramp)

#2 Shepp-Logan

#3 cosine

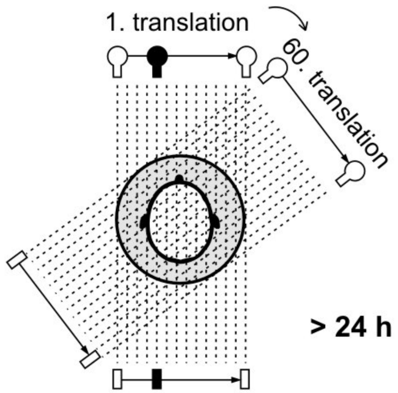
#4 Hamming



smoothing standard edge enhanced

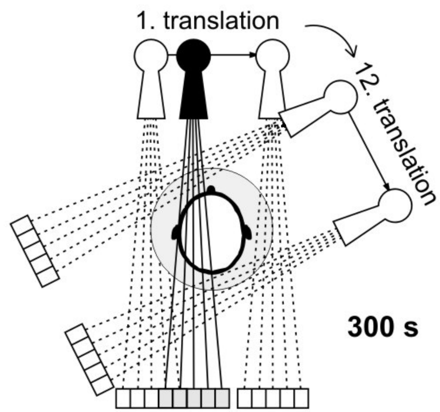
Geometries

pencil beam (1970)

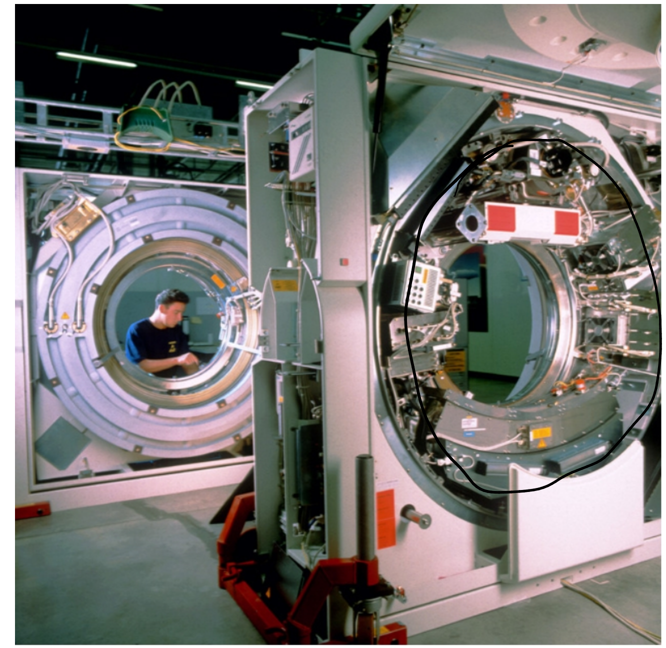


1^{st} generation: translation / rotation

partial fan beam (1972)

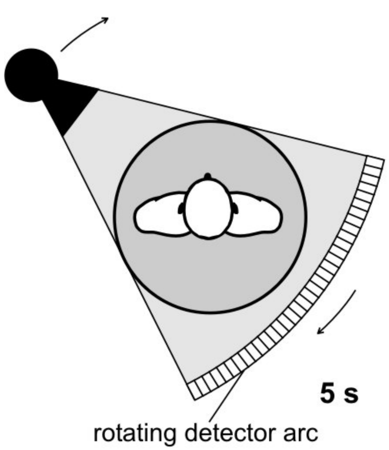


2^{nd} generation: translation / rotation



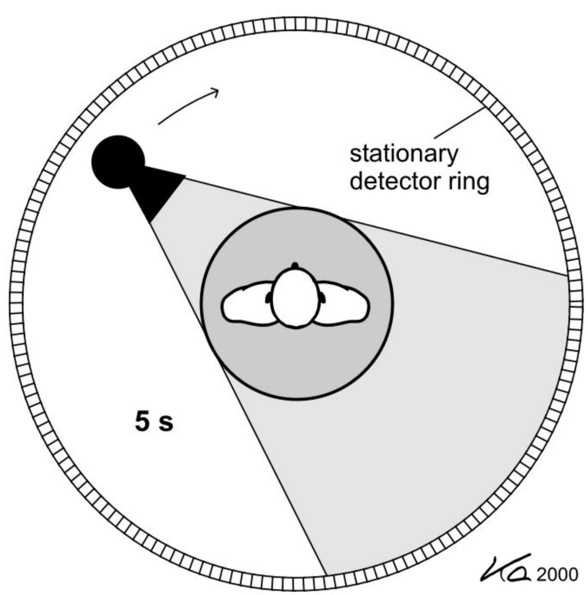
gantry

fan beam (1976)

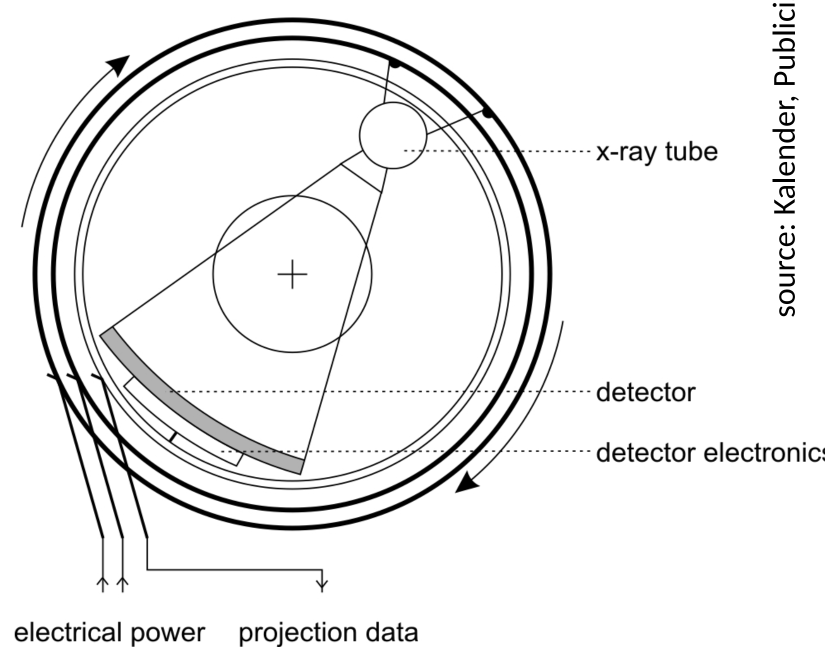


3^{rd} generation: continuous rotation

fan beam (1978)



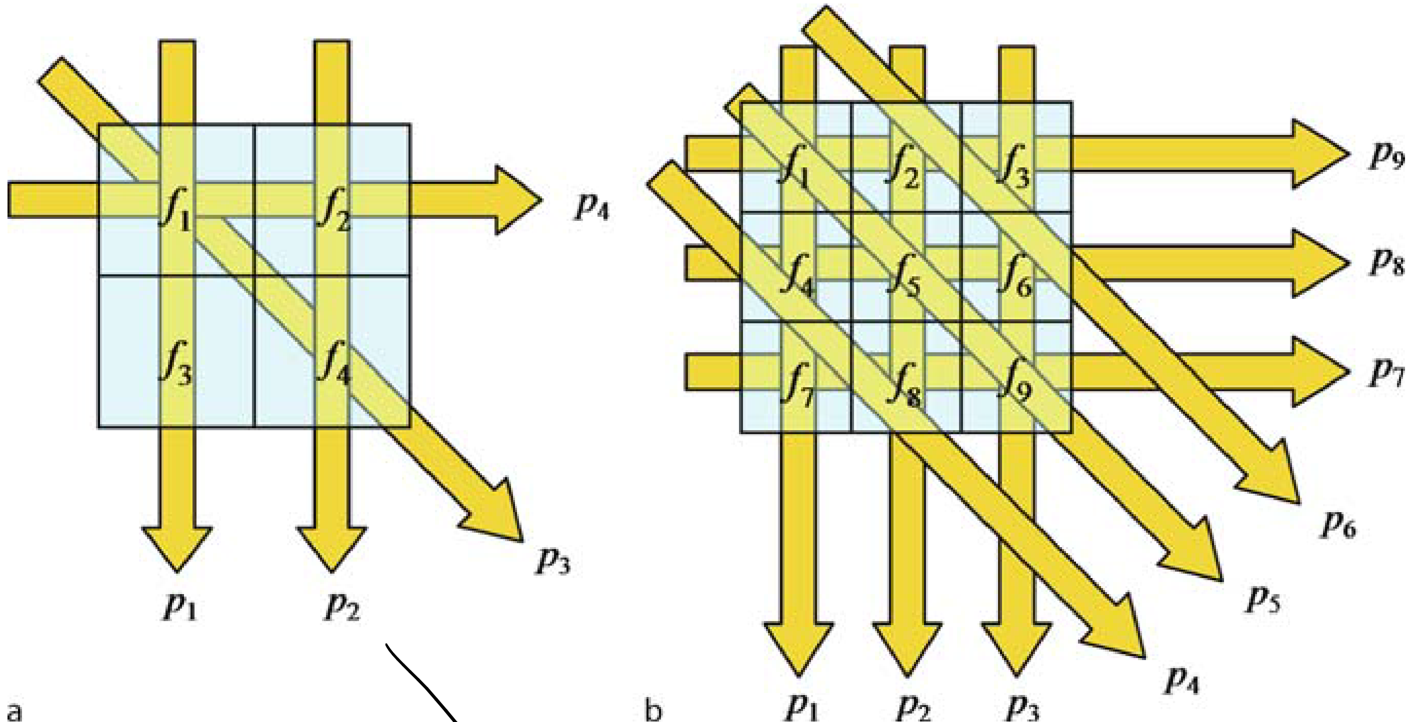
4^{th} generation: continuous rotation



source: Kalender, Publicis, 3rd ed. 2011

Algebraic formulation

Tomography can be formulated as a set of linear equations



$p_1 = f_1 + f_4 + f_7$

$p_2 = f_2 + f_5 + f_8$ ...

$p_3 = f_3 + f_6 + f_9$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

"$Ax = b$"

$x = A^{-1} b$

source: Buzug, Springer, 1st ed. 2008

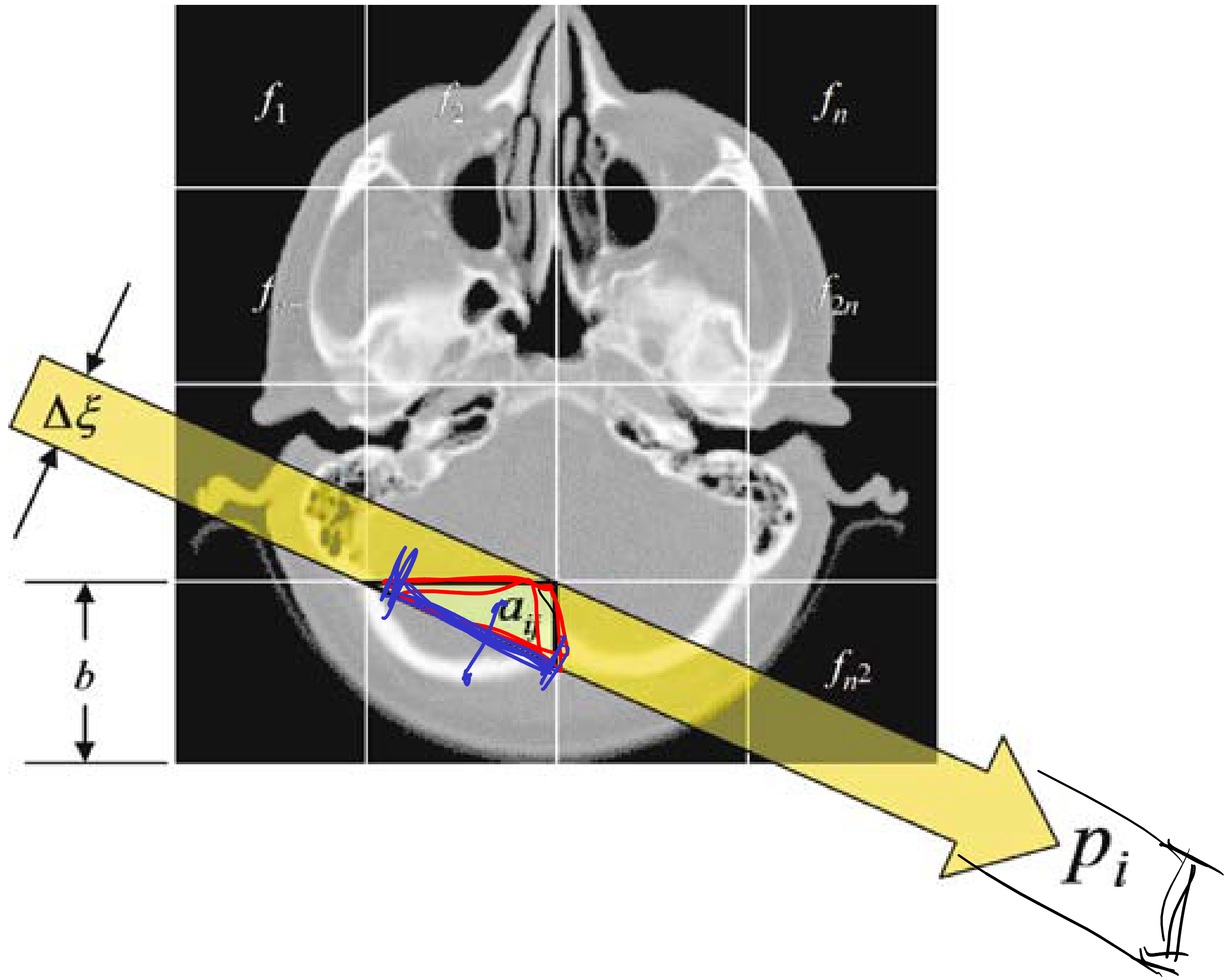
Weighting coefficients

Weighting measures:

- Logic

  0 or 1

- Area
- Path length
- Distance to pixel center



Differences in calculation effort, smoothness, noise sensitivity, ...

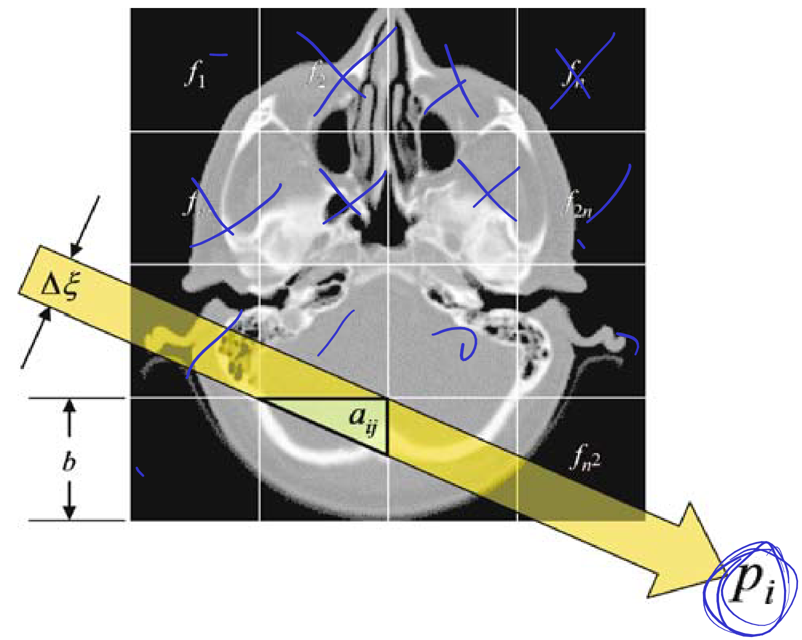
source: Buzug, Springer, 1st ed. 2008

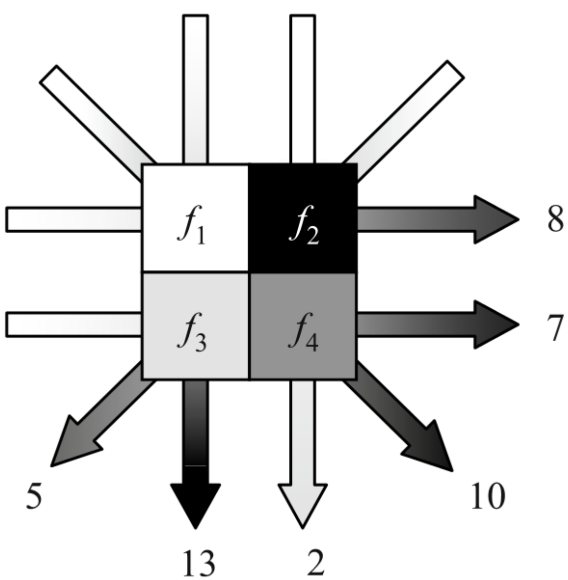
System Matrix

system matrix in general is

→ made of entries between 0 and 1

→ sparse





$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}$$

source: Buzug, Springer, 1st ed. 2008

Matrix (pseudo)-inversion

Tomographic reconstruction = linear system inversion

only square matrices can be inverted. rectangular matrices are pseudo-inverted

system matrix tomo slice sinogram

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} S \end{bmatrix}$$

$$M^{-1} = \underbrace{(M^T M)^{-1} M^T}_{\text{Moore-Penrose pseudo-inverse}}$$

$$T = M^{-1} S$$

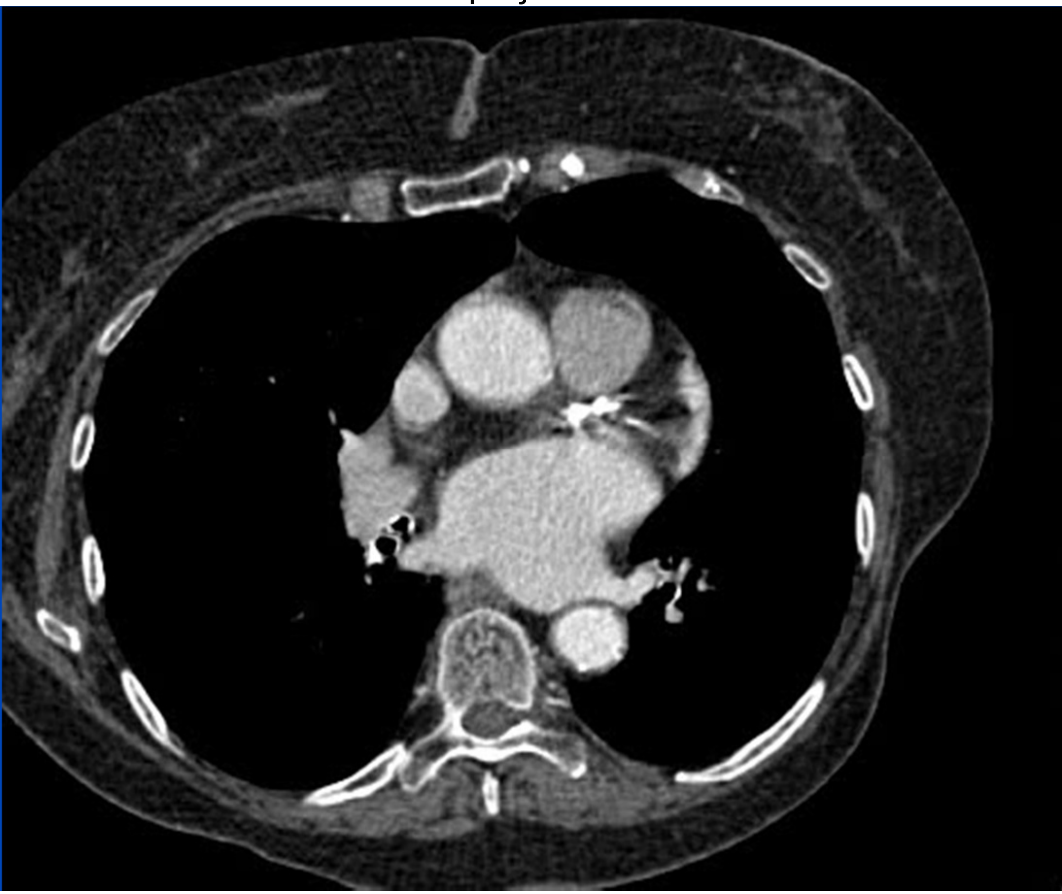
Iterative methods:

- ART   Algebraic reconstruction technique
- SART   Simultaneous algebraic reconstruction technique
- SIRT   Simultaneous iterative reconstruction technique
- MART   Multiplicative algebraic reconstruction technique
- MLEM   Maximum likelihood expectation maximization
- OSEM   Ordered subset expectation maximization
- ... and many, many more

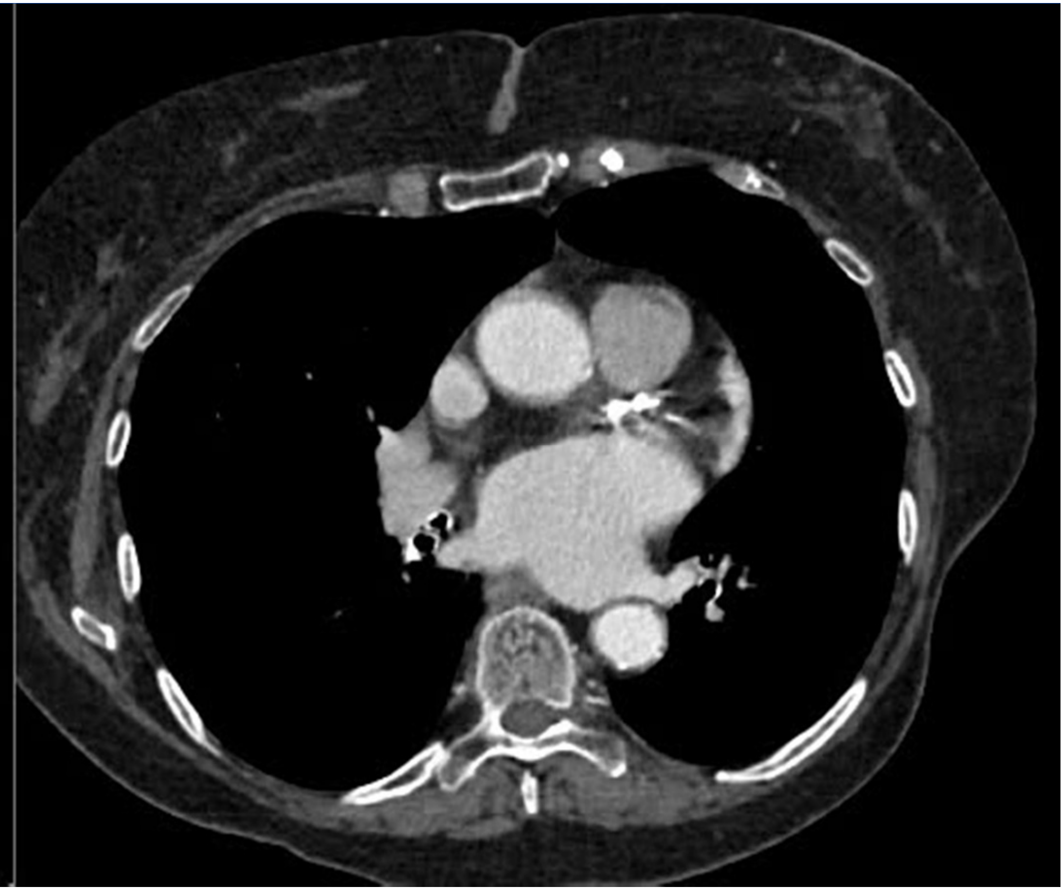
M^T corresponds to Back-projection operation

FBP vs algebraic methods

Filtered backprojection 100% dose



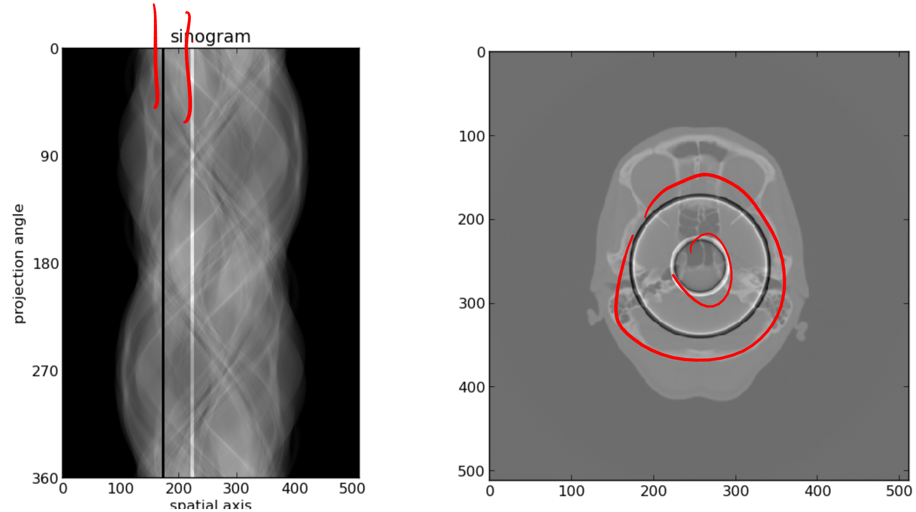
iterative 40% dose



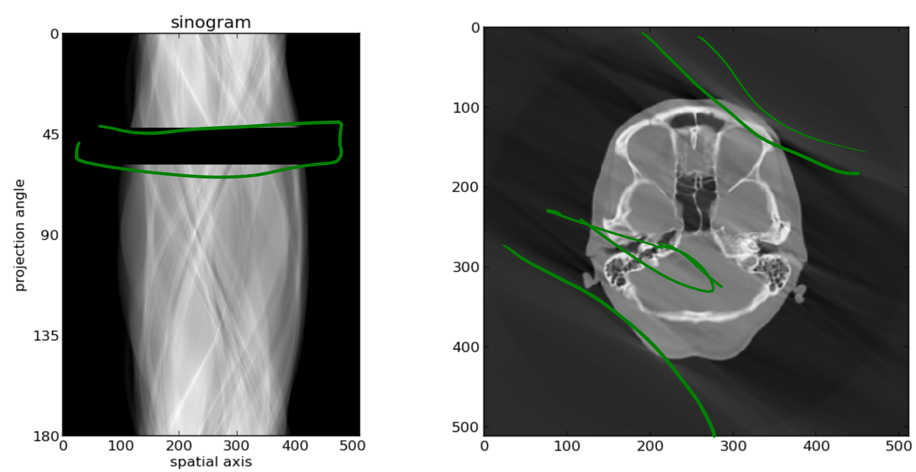
source: Kachelries, http://www.dkfz.de/en/medphysrad/workinggroups/ct/ct_conference_contributions/BasicsOfCTImageReconstruction_Part2.pdf

Artifacts

Detector imperfections → ring artifacts ← very common and painful somewhat solved with low-pass filters

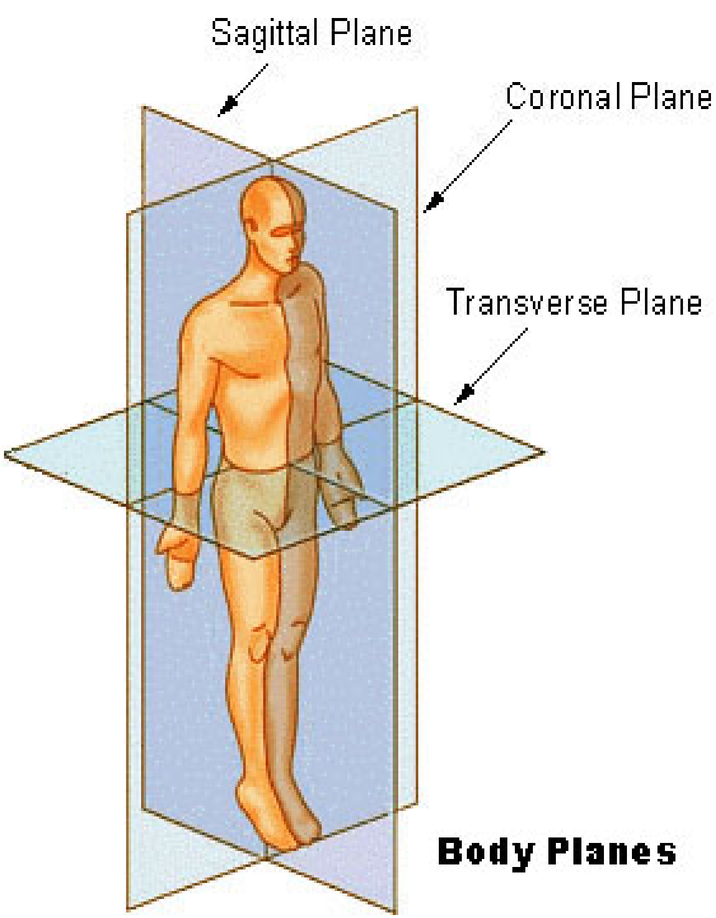


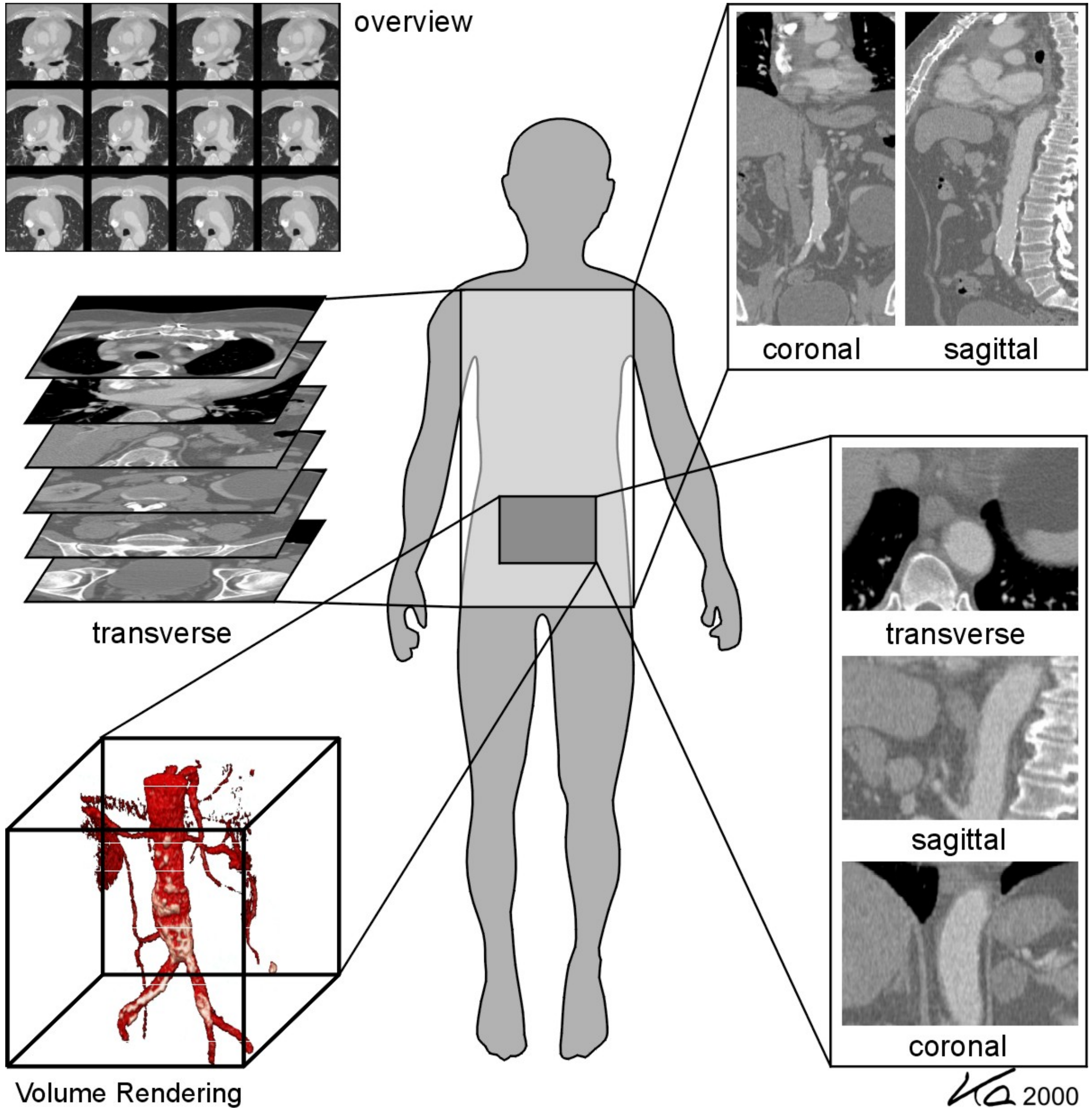
Missing projections → "streak" artifacts



Also: sample motion, beam hardening, ...

Tomographic Display

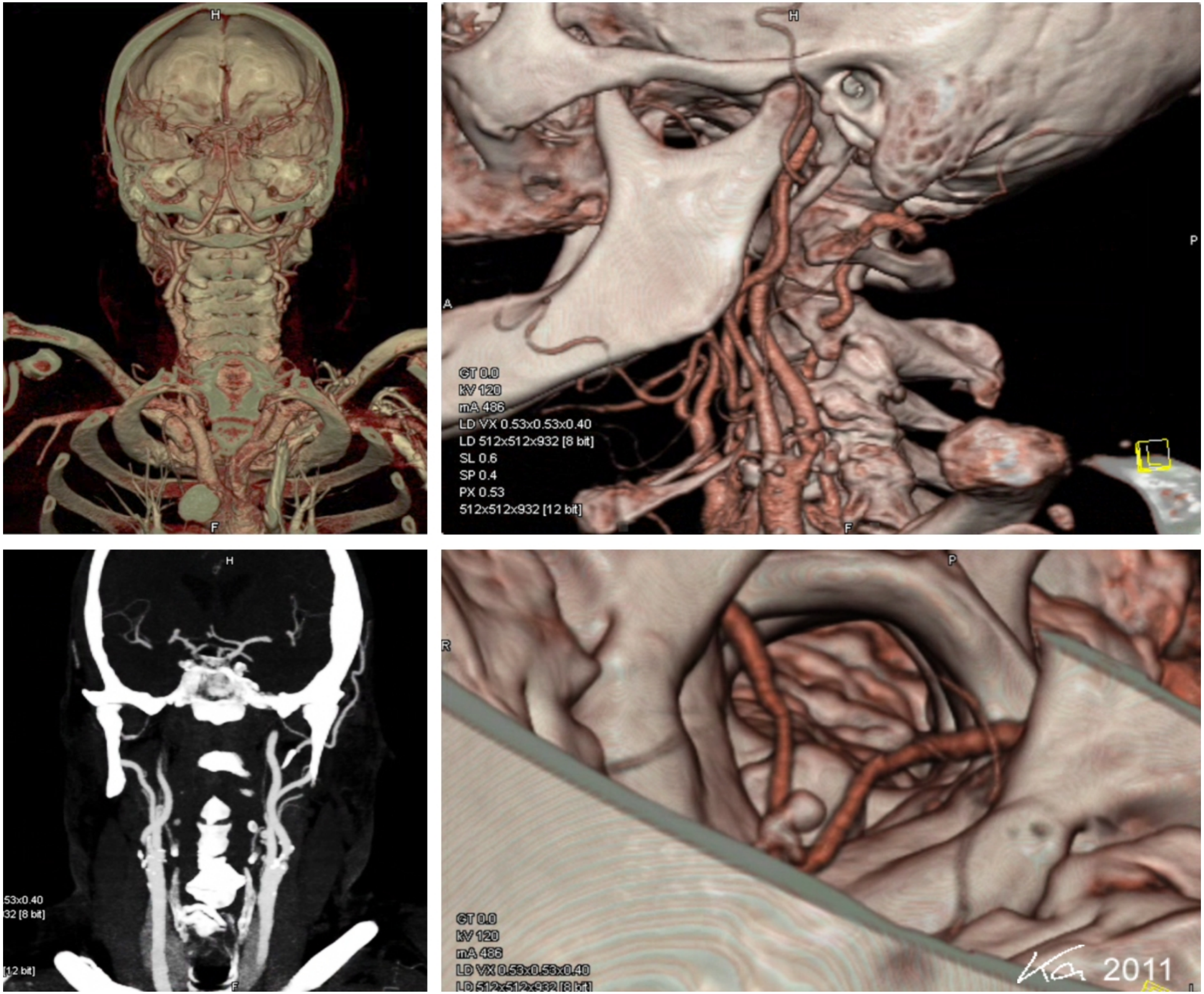




source: http://wikipedia.org

source: W. Kalender, Publicis, 3rd ed. 2011

Volume rendering display



source: Kalender, Publicis, 3rd ed. 2011

Summary

- Computed tomography: reconstruction from projections
- Analytic approach:
  - Projections and tomographic slices are related by the Fourier slice theorem
  - Standard algorithm uses filtered back-projection
- Algebraic approach:
  - Tomography as a system of linear equations
  - Iterative methods are used for large matrix inversions
  - More powerful but computationally more costly
- Imperfect data leads to artifacts