29 Nov

$\{u_n\}$

$$\begin{cases} i\partial_t u_n = -\cancel{P_n}\Delta u_n + \lambda Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right) \\ u_n(0) = Q_n u_0 \end{cases}$$

$[0,T]$

$u_n \longrightarrow u \quad \text{in } C^0([0,T], H^1)$

$$u(t) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u|^{p-1} u \, ds$$

$$u_n(t) = e^{it\Delta} Q_n u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} Q_n\left(|Q_n u_n|^{p-1} Q_n u_n\right)$$

$$u(t) - u_n(t) = (1-Q_n) e^{it\Delta} u_0$$

$$-i\int_0^t e^{i(t-s)\Delta} (1-Q_n) |u|^{p-1} u \, ds$$

$$-i\int_0^t e^{i(t-s)\Delta} Q_n\left(|u|^{p-1} u - |Q_n u|^{p-1} Q_n u\right) ds$$

$$-i\int_0^t e^{i(t-s)} Q_n\left(|Q_n u|^{p-1} Q_n u - |Q_n u_n|^{p-1} Q_n u_n\right) ds$$

$$|u-u_n|_{L^q([0,T],W^{1,p+1})} + |u-u_n|_{L^\infty([0,T],H^1)}$$

$$\leq c_0\,|(1-Q_n)u_0|_{H^1}$$

$$+c_0\,|(1-Q_n)|u|^{p-1}u|_{L^{q'}([0,T],W^{1,\frac{p+1}{p}})}$$

$$+c_0\,\big|\,|u|^{p-1}\underline{u} - |Q_n u|^{p-1}Q_n u\big|_{L^{q'}([0,T],W^{1,\frac{p+1}{p}})}$$

$$+c_0\,\big|\,|Q_n u|^{p-1}Q_n u - |Q_n u_n|^{p-1}Q_n u_n\big|_{L^{q'}W^{1,\frac{p+1}{p}}}$$

$$\leq c_0\,|(1-Q_n)u_0|_{H^1}$$

$$+c_0\,|(1-Q_n)|u|^{p-1}u|_{L^{q'}([0,T],W^{1,\frac{p+1}{p}})}$$

$$+c_0T^{\vartheta}\left(|u|^{p-1}_{L^\infty([0,T],H^1)} + |\cancel{Q_n}u|^{p-1}_{L^\infty([0,T],H^1)}\right)$$

$$|u-Q_n u|_{L^q([0,T],W^{1,p+1})}$$

$$+c_0T^{\vartheta}\left(|\cancel{Q_n}u|^{p-1}_{L^\infty([0,T],H^1)} + |\cancel{Q_n}u_n|^{p-1}_{L^\infty([0,T],H^1)}\right)$$

$$|\cancel{Q_n}(u-Q_n u_n|_{L^q([0,T],W^{1,p+1})}$$

$$|f^{p-1}g|_{L^{q'}([0,T],L^{\frac{p+1}{p}})}$$

$$\leq |f|^{p-1}_{L^{\beta}([0,T],L^{p+1})} |g|_{L^{q}([0,T],L^{p+1})}$$

$$\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'} \qquad q > 2$$

$\vartheta > 0$ $\qquad H^1$ $\beta < +\infty$

$$\leq T^{\vartheta} |f|^{p-1}_{L^{\infty}([0,T],\cancel{L^{p+1}})} |g|_{L^{q}([0,T],L^{p+1})}$$

$$p < d^{*}$$

$$H^1 \hookrightarrow L^{p+1}$$

$$|u - u_n|_{L^{q}([0,T],W^{1,p+1})} + \cancel{|u_n - u_n|_{L^{\infty}([0,T],H^1)}}$$

$$\leq c_e |(1-Q_n) u_0|_{H^1}$$

$$+ c_0 |(1-Q_n) |u|^{p-1} u|_{L^{q'}([0,T],W^{1,\frac{p+1}{p}})}$$

$$+ c_0 T^{\vartheta} (|u|^{p-1}_{L^{\infty}([0,T],H^1)} + |\cancel{Q}_n u|^{p-1}_{L^{\infty}([0,T],H^1)}$$

$$|u - Q_n u|_{L^{q}([0,T],W^{1,p+1})}$$

$$+ c_0 T^{\vartheta} (|\cancel{Q}_n u|^{p-1}_{L^{\infty}([0,T],H^1)} + |\cancel{Q}_n u_n|^{p-1}_{L^{\infty}([0,T],$$

$$|u - u_n|_{L^{q}([0,T],W^{1,p+1})}$$

$$|u-u_n|_{L^q([0,T],W^{1,p+1})} + \cancel{|u_n u_n|} - L^\infty([0,T],H^1)$$

$$\leq 2c_e \, |(1-Q_n)u_0|_{H^1} \longrightarrow 0$$

$$+2c_0 \, |(1-Q_n)|u|^{p-1}u|_{L^{q'}([0,T],W^{1,\frac{p+1}{p}})} \longrightarrow 0$$

$$+2c_0 \, T^{\vartheta} \left(|u|^{p-1}_{L^\infty([0,T],H^1)} + |\cancel{Q_n} u|^{p-1}_{L^\infty([0,T],H^1)}\right)$$

$$|u - Q_n u|_{L^q([0,T],W^{1,p+1})} \longrightarrow 0$$

$Q_n \longrightarrow 1$ strongly in ~~H~~

$Q_n \longrightarrow 1$ also in $L^{q'}([0,T],W^{1,\frac{p+1}{p}})$

$L^q([0,T],W^{1,p+1})$

$C^0([0,T],W^{1,p+1})$ is dense in $\uparrow$

$Q_n \longrightarrow 1 \qquad Q_n u \longrightarrow u$

$Q_n u(t) \longrightarrow u(t)$

$u_n \longrightarrow u$ in $C^0([0,T],H^1)$

Prop Let $u \in C^0([-S,T), H^1)$ be a maximal solution with $u_0 \in H^2(\mathbb{R}^d)$. Then

$$u \in C^0((-S,T), H^2)$$

$$1 < p < d^*$$

$$|u|^{p-1} u$$

Global existence

Lemma Let $u \in C^0([0,T^*), H^1)$ be maximal. Then if $T^* < +\infty$

$$\Rightarrow \quad |\nabla u(t)|_{L^2} \xrightarrow{t \to T^*} +\infty$$

Corollary If $\lambda = 1$ the solutions are globally defined

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{\cancel{\lambda}}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} dx$$

$$\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 \qquad \text{if } \lambda = 1$$

Corollary If $\lambda = -1$ and

$$1 < p < 1 + \frac{4}{d}$$

$$2E(u) = \|\nabla u\|_{L^2_x}^2 - \frac{2}{p+1} \|u\|_{L^{p+1}_x}^{p+1} \geq$$

$$\|u\|_{L^{p+1}_x} \leq \|\nabla u\|_{L^2}^{\alpha} \|u\|_{L^2}^{1-\alpha} \qquad \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}$$

$$0 < \alpha < 1 \qquad \alpha = \frac{d}{2} - \frac{d}{p+1}$$

$$2E(u) \geq \|\nabla u\|_{L^2_x}^2 - \frac{2}{p+1} \|\nabla u\|_{L^2}^{\alpha(p+1)} \|u_0\|_{L^2}^{(1-\alpha)(p+1)}$$

$$\alpha(p+1) < 2$$

$$= \|\nabla u\|_{L^2_x}^2 \left(1 - \underbrace{\frac{2}{p+1} \|\nabla u\|_{L^2}^{\overbrace{\alpha(p+1)-2}^{<0}} \|u_0\|^{(1-\alpha)(p+1)}}_{O(1)}\right)$$

$$\alpha(p+1) = (p+1)\, d\left(\frac{1}{2} - \frac{1}{p+1}\right) =$$

$$= \frac{d}{2}(p+1) - d < 2$$

$$\Updownarrow$$

$$p+1-2 < \frac{4}{d}$$

$$p-1 < \frac{4}{d}$$

$$p < 1 + \frac{4}{d}$$

Local Existence L^2 critical case

Thm $\forall\, u_0 \in L^2(\mathbb{R}^d)$ $\exists$ unique maximal solution with $p = 1 + \frac{4}{d}$ and

$$u \in C^0([0,T^*), L^2(\mathbb{R}^d)) \cap L^{p+1}_{loc}([0,T^*), L^{p+1}(\mathbb{R}^d))$$

with $\frac{2}{p+1} + \frac{d}{p+1} = \frac{d}{2}$

Mass is preserved

There is continuity in terms of initial data

If $T^* < \infty$ then

$$\lim_{t \to T^*} \|u\|_{L^a([0,t], L^b)} = +\infty$$

$\forall (a,b)$ admissible with $b \geq p+1$.

1A $p = 5$ in 1 D.

$1 + \frac{4}{d} = 3$ $p = 3$ 2 D

Prop $\exists\, \delta > 0$ st if for $T > 0$

$$\|e^{it\Delta} u_0\|_{L^{p+1}([0,T] \times \mathbb{R}^d)} < \delta$$

then $\exists$ and is unique

$$u \in C^0([0,T], L^2(\mathbb{R}^d)) \cap L^{p+1}([0,T] \times \mathbb{R}^d)$$

Pf $E(T,\delta) = \{ v \in L^{p+1}([0,T] \times \mathbb{R}^d) :$

$\|v\|_{L^{p+1}([0,T] \times \mathbb{R}^d)} \leq 2\delta \}$

$$\phi(u) = e^{it\Delta} u_0 - i\lambda \int_0^t e^{i(t-s)} |u|^{p-1} u \, ds$$

$$|\phi(u)|_{L^{p+1}([0,T]\times\mathbb{R}^d)} \leq \delta$$

$$+ c_0 | \quad u \quad |^{p}_{L^{p+1}([0,T]\times\mathbb{R}^d)}$$

$$\leq \delta + c_0 2^p \delta^p < (1 + \underbrace{c_0 2^{p-1} \delta^{p-1}}_{<1}) \delta$$

$$|\phi(u) - \phi(v)|_{L^{p+1}([0,T]\times\mathbb{R}^d)}$$

$$\leq c_0 \left| |u|^{p-1} u - |v|^{p-1} v \right|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d)}$$

$$\lesssim c_0 \left| (|u|^{p-1} + |v|^{p-1}) |u-v| \right|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d)}$$

$$\leq c_0 \left(|u|^{p-1}_{L^{p+1}([0,T]\times\mathbb{R}^d)} + |v|^{p-1}_{L^{p+1}([0,T]\times\mathbb{R}^d)} \right) |u-v|_{L^{p+1}}$$

$$\leq \underbrace{2 c_0 2^{p-1} \delta^{p-1}}_{< \frac{1}{2}} |u-v|_{L^{p+1}([0,T]\times\mathbb{R}^d)}$$

Proof of Theore

$$\|e^{it\Delta}u_0\|_{L^{p+1}(\underbrace{[0,T]\times\mathbb{R}^d}_{S_T})} \xrightarrow{T\to 0^+} 0$$

For T small S_T

$$\|e^{it\Delta}u_0\|_{L^{p+1}(S_T)} < \delta$$

Therefore I can apply the Prop to S_T

Let u ~~be~~ $[0,T^*)$ be maximal

We want to show that:

$$\|u\|_{L^a([0,T^*),L^b)} = +\infty$$

$b \geq p+1$

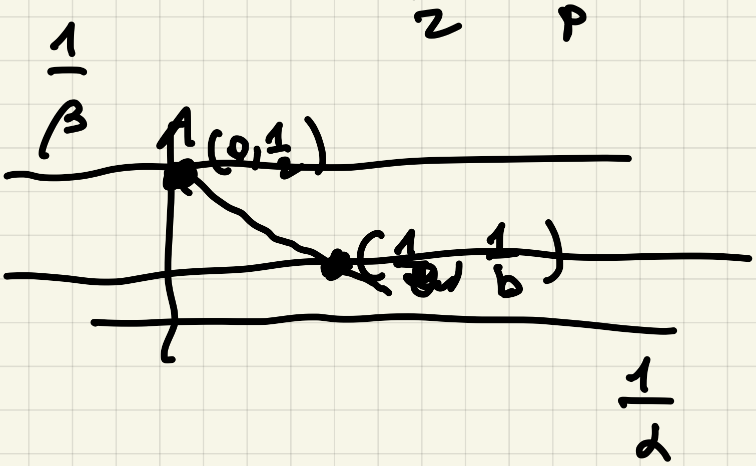
(a,b) admissible ~~$b=p+1$~~ if $b>p+1$

$$\|u\|_{L^a([0,T^*),L^b)} < +\infty$$

$(p+1,p+1)$ is intermediate between $(\infty,2)$ and (a,b)

$$|u|_{L^{p+1}(S_{T^*})} \le |u|^{\mu}_{L^\infty([0,T^*),L^2)}\, |u|^{1-\mu}_{L^a([0,T^*),L^b)}$$

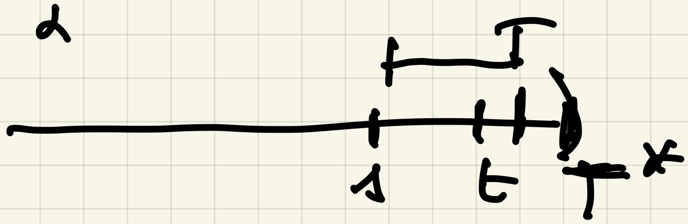
$$\mu = \frac{\frac{1}{p+1} - \frac{1}{b}}{\frac{1}{2} - \frac{1}{p}}$$



$$\left(\frac{1}{p+1}, \frac{1}{p+1}\right)$$

$$\frac{1}{2} > \frac{1}{p+1} > \frac{1}{b}$$

$$(\alpha, \beta)$$



$$u(t) = e^{i(t-s)\Delta} u(s) - i\lambda \int_s^t e^{i(t-t')\Delta} |u|^{p-1} u \, dt'$$

$$e^{i(t-s)\Delta} u(s) = u(t) + i\lambda \int_s^t e^{i(t-t')\Delta} |u|^{p-1} u \, dt'$$

$$\left|e^{i(t-s)\Delta} u(s)\right|_{L^{p+1}([s,T]\times\mathbb{R}^d)} \le$$

$$\le |u|_{L^{p+1}([s,T]\times\mathbb{R}^d)} +$$

$$+ C_0 \, |u|^p_{L^{p+1}([s,T]\times\mathbb{R}^d)} \xrightarrow{s<T} \overset{s\to T^*}{0}$$

so $\exists$ $\varepsilon^* > 0$ and an $s < T^*$

s.t. $\| e^{i(t-s)} u(s) \|_{L^{p+1}([s, T^* + \varepsilon^*] \times \mathbb{R}^d)} < \delta$

$\Rightarrow$ u can be defined also in $[s, T^* + \varepsilon^*]$