Dec 1

$$i\dot{u} = -\Delta u + |u|^{p-1}u \simeq (-\Delta + |u|^{p-1})u$$

$$1 + \frac{4}{d} < p < d^*$$

$$E(u) = \frac{|\nabla u|^2_{L^2}}{2} + \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}\,dx$$

Teor (Scattering) Consider a solution $u \in C^0(\mathbb{R}, H^1)$.
Then $u \in L^a(\mathbb{R}, W^{1,b}(\mathbb{R}^d))$ for any admissible pair. $L^q(\mathbb{R}, W^{1,p+1}(\mathbb{R}^d))$
and $\exists$ $u_\pm \in H^1(\mathbb{R}^d)$ s.t.

$$\lim_{t \to \pm\infty} \left(u(t) - e^{it\Delta}u_\pm\right) = 0 \qquad (1)$$

$H \qquad H_0$

$$i\dot{u} = \underbrace{(-\Delta + V)}_{H} u$$

$$\lim_{t\to\pm\infty}\left(e^{it(-\Delta+V)}u_0 - e^{it(-\Delta)}u_\pm\right)$$

Remark For $1 < p \leq 1 + \frac{2}{d}$ (1)

is false.

Walter Strauss AMS

$1+\frac{2}{d} < p \leq 1 + \frac{4}{d}$ Theorem unknown

$$e^{it\Delta} f(x) = \frac{C}{t^{\frac{d}{2}}} \int e^{i\frac{|x-y|^2}{4t}} f(y)\,dy$$

$\overline{u(-t,x)}$

$u \in L^q(\mathbb{R}_+, W^{1,p+1})$

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s)\,ds$$

$$e^{-it\Delta} u(t) = u_0 - i \int_0^t e^{-is\Delta} |u(s)|^{p-1} u(s)\,ds$$

$t_1 < t_2$

$e^{-it_2\Delta} u(t_2) - e^{-it_1\Delta} u(t_1) =$

$$= - \int_{t_1}^{t_2} e^{-is\Delta} |u(s)|^{p-1} u(s)\, ds$$

$$\| e^{-it_2\Delta} u(t_2) - e^{-it_1\Delta} u(t_1) \|_{H^1} \le$$

$$\lesssim \| |u|^{p-1} u \|_{L^{q'}((t_1,t_2),\, W^{1,\frac{p+1}{p}})}$$

$$\lesssim \|u\|^{p-1}_{L^{\alpha}((t_1,t_2),\, L^{p+1})} \, \|u\|_{L^{q}((t_1,t_2),\, W^{1,p+1})}$$

$$\frac{p-1}{\alpha} + \frac{1}{q} = \frac{1}{q'} \qquad W^{1,\beta} \hookrightarrow L^{p+1}$$

here $\alpha > q$, ✓ if not $\alpha \le q$

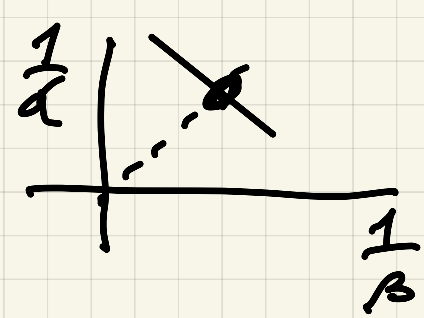
$$\frac{1}{\alpha} \ge \frac{1}{q}$$

$$\frac{p}{q} \le \frac{1}{q'} = 1 - \frac{1}{q}$$

$$\frac{p+1}{q} \le 1$$

$$p+1 \le q$$

$$\frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}$$



$$q \geq p + 1 > 2 + \frac{4}{d}$$

It can be checked $\left(2+\frac{4}{d},\ 2+\frac{4}{d}\right)$ is an admissible pair

$$\frac{2}{2+\frac{4}{d}} + \frac{d}{2+\frac{4}{d}} = \frac{d}{2} \quad ? \ \checkmark$$

$$\frac{1}{2}(d+2)\frac{1}{1+\frac{2}{d}} = \frac{1}{2}\cancel{(d+2)}\frac{d}{\cancel{d+2}}$$

$$\Rightarrow \quad q < 2 + \frac{4}{d} \qquad \text{contradiction}$$

$$\frac{p-1}{d} + \frac{1}{p} = \frac{1}{q'} \ \checkmark$$

here $d > q'$,

$q < \alpha < +\infty$. We pick β s.t.

(α, β) is admissible

(it is intermediate between $(q, p+1)$ and

$(\infty, 2)$)

$$\alpha > q \implies \beta < p+1$$

We claim that

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{\alpha} \quad \text{for a} \quad \tau \in (0,1)$$

$$t_1 < t_2$$

$$\| e^{-it_2\Delta} u(t_2) - e^{-it_1\Delta} u(t_1) \|_{H^1} \leq$$

$$\leq \left(\| u \|^{p-1}_{L^\alpha((t_1,t_2), W^{1,\beta})} \right) \| u \|_{L^q((t_1,t_2), W^{1,p+1})}$$

since $u \in L^\alpha(\mathbb{R}, W^{1,\beta}) \cap L^q(\mathbb{R}, W^{1,p+1})$

$$\xrightarrow{t_1 \to +\infty} 0$$

$$\implies \exists \lim_{t \to +\infty} e^{-it\Delta} u(t) = u_+ \in H^1$$

$$\lim_{t \to +\infty} \| e^{it\Delta} (e^{-it\Delta} u(t) - u_+) \|_{H^1} = 0$$

$e^{it\Delta}$ is an isometry in H

$$\| \langle \xi \rangle e^{-it|\xi|^2} (\ldots) \|_{L^2}$$

$$\lim_{t \to +\infty} \| u(t) - e^{it\Delta} u_+ \|_{H^1} = 0$$

(2) $$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} \quad \text{for a} \quad \tau \in (0,1)$$

$$2 < \beta < p+1 < +\infty$$

$d = 1, 2$ $\qquad W^{1,\beta}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$

$\Rightarrow \quad \tau \in (0,1)$ s.t. (2) is true.

$$W^{1,2} \xhookrightarrow{\text{almost}} L^\infty$$

$$W^{1,2} \xhookrightarrow{\text{almost}} L^\infty \qquad H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$$

$$d \geq 3 \qquad p < d^* = \frac{d+2}{d-2}$$

$$p+1 < \frac{d+2}{d-2} + 1 = \frac{2d}{d-2}$$

$$\frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d} \quad . \quad H^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$$

$$\begin{tikzcd}
W^{1,\beta}(\mathbb{R}^d) \arrow[r, hook] \arrow[dr, bend right] \arrow[rd, bend left] & L^{\frac{2d}{d-2}}(\mathbb{R}^d) \\
 & L^{p+1}(\mathbb{R}^d) \\
 & L^{\beta}(\mathbb{R}^d)
\end{tikzcd}$$

$$\beta < p+1 < \frac{2d}{d-2}$$

$$\Rightarrow \qquad \tau < 1 \qquad \frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d}$$

$$W^{\tau,\beta}(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$$

$$\tau \leq 1$$

Thm $\quad d \geq 3 \qquad u \in C^0(\mathbb{R}, H^1)$

Then

$$\lim_{t \to +\infty} \| u(t) \|_{L^r(\mathbb{R}^d)} = 0$$

$$\forall \quad 2 < r < \frac{2d}{d-2} .$$

<u>Lemma</u> $f(x) = a - x + b x^{\alpha}$

$a, b > 0 \quad \alpha > 1, \quad x \geq 0$

Assume that $0 < x_0 < x_1$ are such that $f(x_0) = f(x_1) = 0$

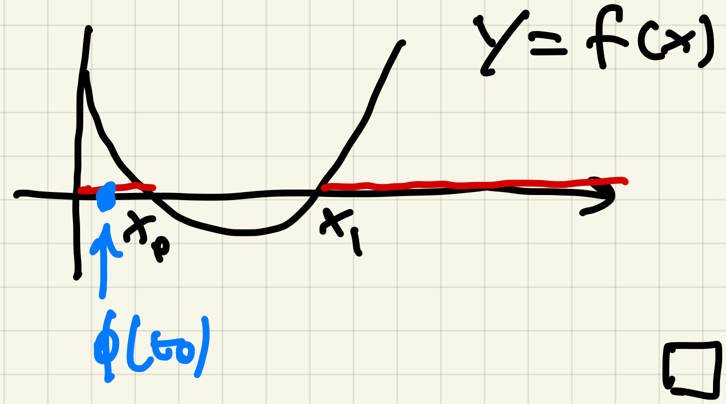
And let $\phi \in C^0(I, [0, +\infty))$ s.t.

$\phi(t) \leq a + b\, \phi^{\alpha}(t) \quad (\Longleftrightarrow f(\phi(t)) \geq 0)$

so that $\exists\ t_0 \in I$ t.s.

$\phi(t_0) \leq x_0$. Then $\phi(t) \leq x_0 \ \forall t \in I$

<u>Pf</u>



$\square$

$$u(t) = e^{i(t-s)\Delta} u(s) - i \int_s^t e^{i(t-s)} |u(s)|^{p-1} u(s)\, ds$$

$$\| u \|_{L^q((s,t), W^{1,p+1})} \leq c_0 \| u(s) \|_{H^1} \#$$

$$c\, \left\| |u|^{p-1}u \right\|_{L^{q'}((s,t),\, W^{1,\frac{p+1}{p}})}$$

$$\le \|u(s)\|_{H^1} + \left\| \|u\|^{p-1}_{L^{p+1}} \|u\|_{W^{1,p+1}} \right\|_{L^{q'}((s,t))}$$

$$= \|u(s)\|_{H^1} + \left(\int_s^t \|u(s)\|^{(p-1)q'}_{L^{p+1}} \|u(s)\|^{q'}_{W^{1,p+1}}\, ds \right)^{\frac{1}{q'}}$$

$$\lesssim \|u(s)\|_{H^1} + \left(\int_s^t \|u(s)\|^{(p-1)q'-(q-q')}_{L^{p+1}} \|u(s)\|^{q}_{W^{1,p+1}}\, ds \right)^{\frac{1}{q'}}$$

$$\le \|u(s)\|_{H^1} + \|u\|^{\frac{(p-1)q'-(q-q')}{q'}}_{L^\infty((s,t),L^{p+1})} \left(\int_s^t \|u(s)\|^{q}_{W^{1,p+1}}\, ds \right)^{\frac{1}{q'}}$$

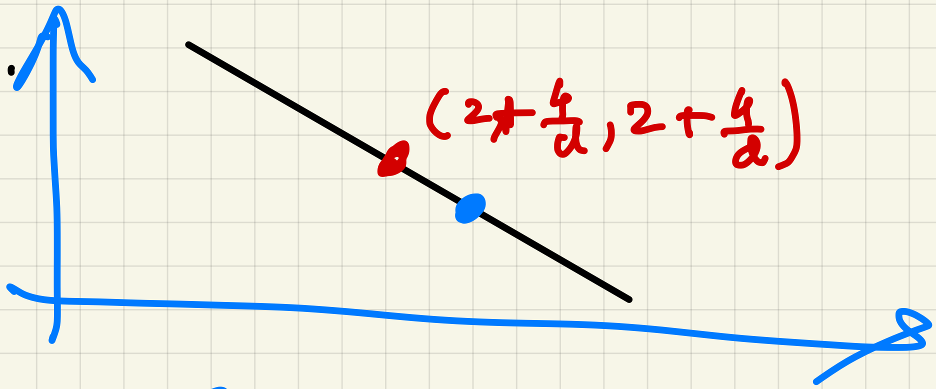
$$= \|u(s)\|_{H^1} + \|u\|^{\frac{pq'-q}{q'}}_{L^\infty((s,t),L^{p+1})} \|u\|^{\frac{q}{q'}}_{L^q((s,t),W^{1,p+1})}$$

$$p\,\frac{q}{q-1} - q$$

$$= \|u(s)\|_H + \|u\|^{p-\frac{q}{q'}}_{L^\infty((s,t),L^{p+1})} \|u\|^{\frac{q}{q'}}_{L^q((s,t),W^{1,p+1})}$$

$$p - \frac{q}{q'} = p - q\left(1 - \frac{1}{q}\right) = p + 1 - q > 0$$

$$\iff p > 1 + \frac{4}{d}$$



$\varepsilon > 0$ $\quad C_0$

$$\|u\|_{L^q([s,t), W^{1,p+1})} \le C_0 + \varepsilon \|u\|^{\frac{q}{q'}}_{L^q([s,t), W^{1,p+1})}$$

$s_0 < s < t$ $\qquad f(x) = \underline{C_0 - x + \varepsilon x^{\frac{q}{q'}}}$

$$\|u\|_{L^q([s,t), W^{1,p+1})} \le X_0 \qquad \forall\ t > s$$

$$\Longrightarrow \|u\|_{L^q([0,\infty), W^{1,p+1})} < +\infty$$