5 Dicembre

Formula dell'integrazione per parti

$$\int f' g \, dx = f g - \int f g'$$

$$\int_a^b f'(x) g(x) \, dx = f(x) g(x) \Big]_a^b - \int_a^b f(x) g'(x) \, dx$$

Esempi

$$\int x e^x \, dx = \int x (e^x)' \, dx = x e^x - \int (x)' e^x \, dx$$

$$= x e^x - \int e^x \, dx = x e^x - e^x + C$$

$$\int_0^1 x e^x \, dx = (x e^x - e^x) \Big]_0^1 = 0 - (-1) = 1$$

$$\int p(x) e^x \, dx = \int p(x) (e^x)' \, dx = p(x) e^x - \int p'(x) e^x \, dx$$

$$= p(x) e^x - \int p'(x) (e^x)' \, dx$$

$$= p(x) e^x - p'(x) e^x + \int p''(x) e^x \, dx$$

$$= \sum_{j=0}^{n-1} (-1)^j p^{(j)}(x) e^x + (-1)^n \int p^{(n)}(x) e^x \, dx$$

Se $p(x) = a_n x^n + \dots + a_0$ $\quad p^{(n)}(x) = a_n n!$

$$= \sum_{j=0}^{n-1} (-1)^j p^{(j)}(x) e^x + (-1)^n p^{(n)}(x) e^x + C$$

$$= \sum_{j=0}^{n} (-1)^j p^{(j)}(x) e^x + C$$

$$\int p(x)\sin(x)\,dx = \int p(x)\,(-\cos x)'\,dx =$$

$$= -p(x)\cos x + \int p'(x)\cos x\,dx =$$

$$= -p(x)\cos x + \int p'(x)\,\sin'(x)\,dx$$

$$= -p(x)\cos(x) + p'(x)\sin x - \int p''(x)\overset{(-\cos x)'}{\overset{\shortparallel}{\sin x}}$$

$$= -p(x)\cos x + p'(x)\sin x + \int p''(x)\cos'(x)\,dx$$

$$= -p(x)\cos x + p'(x)\sin x + p''(x)\cos x - \int p'''(x)\underset{\sin'(x)}{\underset{\shortparallel}{\cos x}}\,dx$$

$$= -p(x)\cos x + p'(x)\sin x + p''(x)\cos x - \int p'''(x)\sin'(x)\,dx$$

$$= -p(x)\cos x + p'(x)\sin x + p''(x)\cos x - p'''(x)\sin x - \int p^{(4)}(x)\sin x\,dx$$

$$\int \lg x \, dx = \int 1 \lg x \, dx = \int (x)' \lg x \, dx$$

$$= x \lg x - \int x \lg'(x) \, dx$$

$$= x \lg x - \int \cancel{x} \frac{1}{\cancel{x}} \, dx = x \lg x - x + C \, .$$

$$(\sqrt{x})' = \frac{1}{2} \frac{1}{\sqrt{x}}$$

$$(\arctan(x))' = \frac{1}{1+x^2}$$

$$\int \arctan(x) \, dx = \int (x)' \arctan(x) =$$

$$= x \arctan x - \frac{1}{2} \int 2x \frac{1}{1+x^2} \, dx$$

$$= x \arctan x - \frac{1}{2} \int \frac{(1+x^2)'}{1+x^2} \, dx =$$

$$= x \arctan x - \frac{1}{2} \lg(1+x^2) + C$$

Teor (cambio di variabile per l'integrale definito)

Siano I e J due intervalli e siano

$u : I \longrightarrow J$ e $f : J \rightarrow \mathbb{R}$ con

$u \in C^1(I)$ ed $f \in C^0(J)$. Allora, ~~denotato~~

$\int f(u)\,du$ le primitive di $f(u)$, si ha

$$\int f(u(x))\,u'(x)\,dx = \left(\int f(u)\,du\right)(u(x)) \quad (1)$$

Osservazione Impropriamente si usa

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du \quad (2)$$

Notare che può essere utile usare la notazione $\frac{du}{dx}$ al posto di u',

$$\int f(u(x))\,\frac{du(x)}{\cancel{dx}}\,\cancel{dx} = \int f(u)\,du$$

Dim Vogliamo dimostrare

$$\int f(u(x))\,u'(x)\,dx = \left(\int f(u)\,du\right)(u(x))$$

$$\frac{d}{dx}\int f(u(x))\,u'(x)\,dx = f(u(x))\,u'(x) \quad \checkmark$$

$$\frac{d}{dx}\left(\int f(u)\,du\right)(u(x)) = \left(\int f(u)\,du\right)'(u(x)) \quad u'(x) =$$

$$\frac{d}{dx}G(u(x)) = G'(u(x))\,u'(x)$$

$$= f(u(x))\,u'(x) \quad \checkmark$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$du = -\sin x \, dx$$

$$= -\int \frac{du}{u} = -\lg|u| + C$$

$$= -\lg|\cos x| + C$$

$$a \neq 0$$

$$\int \sqrt{ax+b}$$

$$\int \sqrt{2x+1}\, dx$$

$$u = 2x+1$$

$$\frac{du}{dx} = 2 \qquad du = 2\,dx$$

$$= \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{\cancel{2}} \frac{u^{\frac{3}{2}}}{\frac{3}{\cancel{2}}} + C = \frac{(2x+1)^{\frac{3}{2}}}{3} + C$$

$$R(x,y) = \frac{P(x,y)}{Q(x,y)}$$

$$R(\cos(x), \sin(x))$$

$$\int \cos^{2n+1}(x)\ \operatorname{sen}^m(x)\, dx =$$

$$= \int \cos^{2n}(x)\ \sin^m(x)\ \cos x\, dx$$

$$\cos^2(x) = 1 - \sin^2(x)$$

$$\left(\cos^2(x)\right)^n = \left(1 - \sin^2(x)\right)^n$$

$$= \int \left(1 - \sin^2(x)\right)^n \operatorname{sen}^m(x)\ \cos x\, dx$$

$$u = \sin x$$

$$du = \cos x\, dx$$

$$= \int \left(1 - u^2\right)^n u^m\, du$$

$$\int \cos^3(x)\ \sin^3(x)\, dx = \int \left(1 - u^2\right) u^3\, du =$$

$$= \int \left(u^3 - u^5\right) du = \frac{u^4}{4} - \frac{u^6}{6} + C$$

$$u = \sin x$$

$$\int \cos^2(x)\,\sin^2(x)\,dx = \qquad R(x, \sqrt{1-x^2})$$

$$= \int \cos(x)\,\sin^2(x)\,\cos x\,dx$$

$$= \int \sqrt{1-\sin^2(x)}\,\sin^2(x)\,\cos x\,dx \qquad u = \sin x$$

$$= \int \sqrt{1-u^2}\,u^2\,du$$

$\sin(2x)$

$$\int \cos^2(x)\,\sin^2(x)\,dx = \qquad \cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= 2\cos^2(x) - 1$$

$$= \int \frac{1+\cos(2x)}{2}\,\frac{1-\cos(2x)}{2}\,dx \qquad \cos^2(x) = \frac{\cos(2x)+1}{2}$$

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$= \frac{1}{4}\int \left(1 - \cos^2(2x)\right) dx \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$= \frac{x}{4} - \frac{1}{4}\int \cos^2(2x)\,dx = \qquad \cos^2(2x) = \frac{\cos(4x)+1}{2}$$

$$= \frac{x}{4} - \frac{1}{8}\int \left(\cos(4x) + 1\right) dx = \frac{x}{8} - \frac{1}{8}\int \cos(4x)\,dx$$

$$= \frac{x}{8} - \frac{1}{8}\,\frac{\sin(4x)}{4} + C$$

$$\int \frac{1}{ax^2+bx+c}\,dx \qquad \Delta = b^2-4ac<0$$

$$\int \frac{1}{x^2+1} = \arctan x + c \qquad \Delta = -4$$

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$

$$= a\left[x^2+2\,\frac{b}{2a}x+\frac{c}{a}\right] =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right] =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right] =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{|\Delta|}{4a^2}\right]$$

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\frac{|\Delta|}{4a^2}}\,dx$$

$$\sqrt{\frac{|\Delta|}{4a^2}} \qquad u = x+\frac{b}{2a}$$

$$\frac{\sqrt{|\Delta|}}{2a}u = x+\frac{b}{2a} \qquad \frac{\sqrt{|\Delta|}}{2a}du = dx$$

$$= \frac{1}{a}\int \frac{1}{\frac{|\Delta|}{4a^2}u^2+\frac{|\Delta|}{4a^2}}\,\frac{\sqrt{|\Delta|}}{2a}\,du$$

$$= \frac{1}{\cancel{a}}\,\frac{\sqrt{|\Delta|}}{\cancel{2}\cancel{a}}\,\frac{1}{\frac{|\Delta|}{{}_2\cancel{4}\cancel{a^2}}}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{\sqrt{|\Delta|}}\arctan(u)+C = \frac{1}{\sqrt{|\Delta|}}\arctan\left(\frac{x+\frac{b}{2a}}{\frac{\sqrt{|\Delta|}}{2a}}\right)+\cancel{C}$$

$$= \frac{1}{\sqrt{|\Delta|}}\arctan\left(\frac{2x+b}{\sqrt{|\Delta|}}\right)+c$$