Lezione 1 dicembre 2023

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DETERMINANTE DI UNA MATRICE

Cons. il caso particolare

$$V = \mathbb{K}^n$$

cons. $\mathcal{E} = \{e_1, \dots, e_n\}$ base canonica

$\exists!$ funzione determinante D, $D(e_1, \dots, e_n) = 1$

D verrà indicata con

$$\det : \underbrace{\mathbb{K}^n \times \mathbb{K}^n \times \cdots \times \mathbb{K}^n}_{n-\text{volte}} \longrightarrow \mathbb{K}$$

$$\det(e_1, \dots, e_n) = 1$$

Def. : determinante di una matrice quadrata $\underset{\dim = n^2}{A \in M_n(\mathbb{K})} \; \Big(\cong \underset{= n^2}{\mathbb{K}^n \times \cdots \times \mathbb{K}^n} \Big)$

$$\det A := \det(A_{(1)}, A_{(2)}, \dots, A_{(n)})$$

$$(= |A|)$$

non c'entra né con valori assoluti, moduli, né con ordine gruppo

Esempio: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $n = 2$

$$\det A = \det(A_{(1)}, A_{(2)})$$

$$= \underbrace{\sum_{\sigma \in S_2}{}' \operatorname{sgn}(\sigma) \cdot a_{1\sigma(1)} \cdot a_{2\sigma(2)}}$$

$A_{(1)} = (a_{11} \quad a_{12})$ le sue coordinate nelle e_1, e_2 sono a_{11} e a_{12}

$A_{(2)} = (a_{21} \quad a_{22})$ coordinate sono a_{21} a_{22}

$S_2 = \{$permutazioni di 2 numeri$\}$

$I_2 = \{1, 2\}$, $\# S_2 = 2! = 2$

$$S_2 = \Big\{ Id_{I_2} , \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{\tau} \Big\}$$

$$= a_{1 Id(1)} \, a_{2 Id(2)} + \underbrace{\operatorname{sgn}(\tau)}_{= (-1)} \cdot a_{1\tau(1)} \, a_{2\tau(2)}$$

$$= a_{11} a_{22} - 1 \cdot a_{12} \cdot a_{21}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

OSS: $\det(\mathbb{I}_n) = \det((\mathbb{I}_n)_{(1)}, \dots, (\mathbb{I}_n)_{(n)}) \overset{!}{=} \det(e_1, e_2, \dots, e_n) = 1$

$$\mathbb{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 1 \end{pmatrix}$$

OSS: $A = 0$ matrice nulla

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 0 \end{pmatrix}, \quad \det(0) = \det(0_{\mathbb{K}^n}, \dots, 0_{\mathbb{K}^n}) = 0$$

(visto ieri)

OSS: Se le righe di A sono n vettori LIN. DIP. $\Longleftrightarrow$ $\det(A) = 0$

$\Longleftrightarrow$ $\operatorname{rg} A < n$ rango massimo.

Esempio: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det A = \det(A_{(1)}, A_{(2)}, A_{(3)})$$

$$= \sum_{\sigma \in S_3} sgn(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \cdot a_{3\sigma(3)}$$ = somma di 6 termini

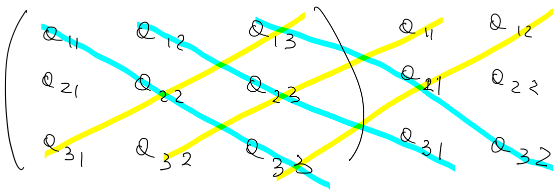
$\# S_3 = 6$

$$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} + a_{12} a_{23} a_{31}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$- a_{12} a_{21} a_{33} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

REGOLA DI SARRUS : 3 × 3



Prop.: Sia $A \in M_n(\mathbb{K})$ e sia ${}^tA$ la trasposta.

Allora
$$\det({}^tA) = \det A$$

Dim: $A = (a_{ij})_{i,j=1\dots n}$

${}^tA = (a'_{ij})$, $a'_{ij} = a_{ji}$

$$\det({}^tA) = \sum_{\sigma \in S_n} (sgn\,\sigma)\, a'_{1\sigma(1)} \cdot a'_{2\sigma(2)} \cdots a'_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} (sgn\,\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

$\forall \sigma$, σ è una biiezione, quindi $\exists\, \sigma^{-1}$

si dimostra $sgn(\sigma) = sgn(\sigma^{-1})$

$$= \sum_{\sigma^{-1} \in S_n} (sgn\,\sigma^{-1})\, a_{1\sigma^{-1}(1)}\, a_{2\sigma^{-1}(2)} \cdots a_{n\sigma^{-1}(n)} = \sum_{\mu \in S_n} (sgn\,\mu) \cdot a_{1\mu(1)} \cdots a_{n\mu(n)}$$

$$= \det A$$

eventualmente cambiando l'ordine delle molt.

infatti: $\exists\, k : a_{\sigma(k)k} = a_{1k} = a_{1\sigma^{-1}(1)}$

fissato σ $\quad \sigma(k) = 1 \iff k = \sigma^{-1}(1)$

$$\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$$

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$$

$$\sigma^{-1} : I_n \to I_n$$

$$\sigma^{-1}(\sigma(k)) = k$$

es. $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad (\operatorname{sgn}\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2}\, a_{\sigma(3)3}$

$$= a_{21}\, a_{32}\, a_{13}$$

$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $\quad \| \quad = a_{13}\, a_{21}\, a_{32} = a_{1\sigma^{-1}(1)}\, a_{2\sigma^{-1}(2)}\, a_{3\sigma^{-1}(3)}$

$\sigma^{-1}(\sigma(1)) = 1 \qquad \sigma^{-1}(\sigma(2)) = 2 \qquad \sigma^{-1}(\sigma(3)) = 3$

$\sigma^{-1}(2) = 1 \qquad \sigma^{-1}(3) = 2 \qquad \sigma^{-1}(1) = 3$

COMPORTAMENTO DI det per OE

Prop: (OE1) scambio di 2 righe = scambio tra 2 argomenti di det

$\Rightarrow$ (ALTERN.) il det cambia segno

$$-\det A = \det \tilde{A}$$

(OE2) moltiplico la riga i $A_{(i)}$ per $\lambda \in K$

chiamo $\tilde{A}$

$$\det \tilde{A} = \det(A_{(1)}, A_{(2)}, \dots, \lambda \cdot A_{(i)}, \dots, A_{(n)})$$

$$\big| \text{MULT.}$$

$$= \lambda \det A$$

(OE3) $A_{(i)} \leftrightarrow A_{(i)} + \lambda A_{(j)} \quad : \tilde{A}$

$$\det \tilde{A} = \det(A_{(1)}, \dots, A_{(i)} + \lambda A_{(j)}, \dots, A_{(j)}, \dots, A_{(n)})$$

$$\big| \text{MULT.}$$

$$= \overbrace{\det(A_{(1)}, \dots, A_{(i)}, \dots, A_{(j)}, \dots, A_{(n)})}^{= \det A}$$

$$+ \underbrace{\det(A_{(1)}, \dots, \lambda \cdot A_{(j)}, \dots, A_{(j)}, \dots, A_{(n)})}$$

$$= \lambda \det(A_{(1)}, \dots, \underbrace{A_{(j)}, \dots, A_{(j)}}_{=}, \dots, A_{(n)})$$

$$= 0$$

$$= \det A$$

DETERMINANTE DI UN PRODOTTO DI MATRICI

Teorema di BINET: $A, B \in M_n(K)$

Allora $\det(A \cdot B) = \det(A) \cdot \det(B)$

Dim.: Due casi:

① $\det B = 0 \Rightarrow \operatorname{rg} B < n$

$\Rightarrow B \cdot x = 0$ ammette soluz. non nulle

$\dim W = n - rg B > n - n = 0$

$\dim W \geq 1$

$\Rightarrow \exists\, s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ t.c. $B \cdot s = 0$

$\Rightarrow (A \cdot B) \cdot s = A \cdot (B \cdot s) = A \cdot 0 = 0$

$\Rightarrow$ anche $(A \cdot B) \cdot X = 0$ ha 1 soluz. non nulla

$\Rightarrow rg(A \cdot B) < n \Rightarrow \det(A \cdot B) = 0$

Quindi: se $\det B = 0 \Rightarrow \det(A \cdot B) = 0 = \det A \cdot \det B$

② $\det B \neq 0$; cons. la funzione

$$f_B : M_n(\mathbb{K}) \longrightarrow K$$

$$f_B(A) := \boxed{\frac{\det(A \cdot B)}{\det B}}$$

Voglio dimostrare $f_B(A) = \det(A)$

Per fare ciò, dimostro che f_B soddisfa:

① f_B è MULTILINEARE sulle righe di A

② f_B è ALTERNANTE

③ $f_B(\mathbb{I}_n) = 1$

Dim: ① & ② nelle note ("in progress")

③ $f_B(\mathbb{I}_n) = \dfrac{\det(\mathbb{I}_n \cdot B)}{\det B} = \dfrac{\det B}{\det B} = 1$

Per unicità della funzione determinante con

$$\det(\mathbb{I}_n) = 1$$

$$\Rightarrow f_B(A) = \det(A)$$

COR.: Sia $A \in M_n(\mathbb{K})$ invertibile. ($\Leftrightarrow rg A = n \Leftrightarrow \det A \neq 0$)

Allora $\det(A^{-1}) = (\det(A))^{-1} = \dfrac{1}{\det A}$

Dim.: $A \cdot A^{-1} = \mathbb{I}_n$

$\det(A \cdot A^{-1}) = \det \mathbb{I}_n$

Binet: $\det(A) \cdot \det(A^{-1}) = 1$

$\det(A^{-1}) = \dfrac{1}{\det A}$

COR.: Determinante di matrici simili: $B \sim A \Rightarrow \det(B) = \det(A)$

$B \sim A \Leftrightarrow B = P \cdot A \cdot P^{-1}, P \in GL(n, \mathbb{K})$

$$\det(B) = \det(\underbrace{P\cdot A\cdot P^{-1}})$$

$$\overset{\text{Binet}}{=} \det(P)\cdot\det(A)\cdot\det(P^{-1})$$

$$= \det(P)\cdot\det(A)\cdot\frac{1}{\det(P)} \in \mathbb{K}$$

$$\cancel{\det(P)}\cdot\frac{1}{\cancel{\det(P)}}\cdot\det(A) = \det(A)$$

Def.: Sia $f: V \to V$ endomorfismo (OL)

DETERMINANTE di f:

$$\det(f) := \det(M^{\mathcal{B}}_{\mathcal{B}}(f))$$

con $\mathcal{B}$ base di V qualunque

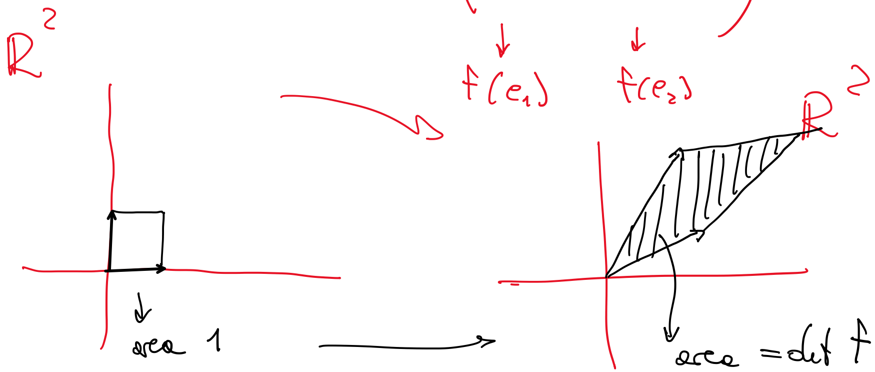
oss. se $\mathcal{C}$ altra base $M^{\mathcal{C}}_{\mathcal{C}}(f) = M^{\mathcal{B}}_{\mathcal{C}}(Id)\cdot M^{\mathcal{B}}_{\mathcal{B}}(f)\cdot\left(M^{\mathcal{B}}_{\mathcal{C}}(Id)\right)^{-1}$

e $\det M^{\mathcal{C}}_{\mathcal{C}}(f) = \det M^{\mathcal{B}}_{\mathcal{B}}(f)$

Interpretaz geometrica:

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{E} = \{e_1, e_2\}$

$$\det f = \det M^{\mathcal{B}}_{\mathcal{B}}(f) = \det(A^{(1)}, A^{(2)})$$

$$A = M^{\mathcal{B}}_{\mathcal{B}}(f) = \begin{pmatrix} \downarrow & \downarrow \\ f(e_1) & f(e_2) \end{pmatrix}$$