$(X_\alpha)_{\alpha \in I}$ INDIPENDENTI

PER OGNI $n \geq 1$

" " $\alpha_1, \dots, \alpha_n \in I$

" " $A_1 \in \sigma(X_{\alpha_1}), \dots, A_n \in \sigma(X_{\alpha_n})$

$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdots P(A_n)$

OPPURE

PER OGNI $B_1 \dots B_n \in \mathcal{B}$

$P(X_{\alpha_1} \in B_1, \dots, X_{\alpha_n} \in B_n) = P(X_{\alpha_1} \in B_1) \cdot \dots \cdot P(X_{\alpha_n} \in B_n)$

$$E[g(X_1, X_2)] = ?$$

$g_1 = 1_{B_1}$ $\qquad$ $g_n = 1_{B_n}$

$$g_1(X_1) = 1_{\{X_1 \in B_1\}}$$

$$E\left[e^{i\, t^T X}\right] = \varphi_{X_1}(t_1) \cdot \ldots \cdot \varphi_{X_n}(t_n)$$

PER OGNI

$t = (t_1 \ldots t_n)$

$$t^T X = \begin{bmatrix} t_1 & \ldots & t_n \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = t_1 X_1 + \ldots + t_n X_n$$

$$\bar{F}_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y})(1+\alpha e^{-x-y})$$

$$\frac{\partial \bar{F}_{X,Y}}{\partial x \partial y}(x,y) = \frac{\partial}{\partial y}\Big(e^{-x}(1-e^{-y})(1+\alpha e^{-x-y}) + (1-e^{-x})(1-e^{-y})(-\alpha e^{-x-y}) \Big)$$

$$= \underbrace{e^{-x}e^{-y}}(1+\alpha e^{-x-y}) + e^{-x}(1-e^{-y})(-\alpha \underbrace{e^{-x-y}})$$

$$+ (1-e^{-x})e^{-y}(-\alpha \underbrace{e^{-x-y}}) + (1-e^{-x})(1-e^{-y})\alpha \underbrace{e^{-x-y}}$$

$$= e^{-x-y}\Big\{1+\alpha e^{-x-y} - \alpha e^{-x}(1-e^{-y}) - \alpha e^{-y}(1-e^{-x}) + \alpha(1-e^{-x})(1-e^{-y})\Big\}$$

$$= e^{-x-y}\Big\{1+\alpha\Big[e^{-x}\big(\overbrace{e^{-y}-(1-e^{-y})}^{2e^{-y}-1}\big) - (1-e^{-x})\big(\underbrace{e^{-y}-(1-e^{-y})}_{2e^{-y}-1}\big)\Big]\Big\}$$

$$= e^{-x-y}\Big\{1+\alpha(2e^{-x}-1)(2e^{-y}-1)\Big\} \qquad \begin{array}{l} x>0 \\ y>0 \end{array}$$

$$E[XY] = \int_0^{+\infty}\int_0^{+\infty} x \cdot y \cdot e^{-x-y}\{1+\alpha(2e^{-x}-1)(2e^{-y}-1)\}\,dx\,dy$$

$$= \ldots$$

U, V INDIPENDENTI $U(0,1)$

$X = \frac{U}{V}$ $\quad F_X$? $\quad X > 0$ q.c.

$$F_X(x) = P(X \le x) = P\left(\frac{U}{V} \le x\right) =$$

$x > 0$

$$= E\left[P\left(\frac{U}{v} \le x\right)\Big|_{v=V}\right] \overset{(*)}{=}$$

$$P(U \le vx) = \begin{cases} xv & xv < 1 \\ 1 & xv \ge 1 \end{cases} = xv\,1_{xv<1} + 1_{xv \ge 1}$$

$0 < v < 1$

$$\overset{(*)}{=} E\left[xV \mathbb{1}_{xV<1} + \mathbb{1}_{xV \geq 1} \right]$$

$xV < 1 \iff V < \frac{1}{x}$ $\qquad$ $\frac{1}{x} > 1 \iff x < 1$

<u>$x < 1$</u>

$$= E\left[xV \mathbb{1}_{\Omega} + \mathbb{1}_{\emptyset} \right] = x \cdot E[V] = \frac{x}{2}$$

$\underline{x \geq 1}$

$$= x\,E\left[V\,\mathbb{1}_{V<\frac{1}{x}}\right] + \underbrace{P\left(V \geq \frac{1}{x}\right)}$$

$$= x \int_0^{1/x} v\,dv + 1 - \frac{1}{x}$$

$$= x \left.\frac{v^2}{2}\right|_0^{1/x} + 1 - \frac{1}{x}$$

$$= x\,\frac{1}{2x^2} + 1 - \frac{1}{x} = \frac{1}{2x} + 1 - \frac{1}{x} = 1 - \frac{1}{2x}$$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \dfrac{x}{2} & 0 \leq x < 1 \\ 1 - \dfrac{1}{2x} & x \geq 1 \end{cases}$$

$$X \sim N(0,1) \qquad Y = X^2 = f(X)$$

$$E[X] = 0$$

$$E[XY] = E[X^3] = 0$$

$$\operatorname{cov}(X,Y) = 0$$

$$\underbrace{\mathrm{VAR}(\alpha X + Y)} \geq 0$$

$$\alpha^2\, \mathrm{VAR}(X) + \mathrm{VAR}(Y) + 2\alpha\, \mathrm{COV}(X,Y)$$

$$\Delta = 4\, \mathrm{COV}(X,Y)^2 - 4\, \mathrm{VAR}(X)\, \mathrm{VAR}(Y) \leq 0$$

(SE $\mathrm{VAR}(X) = 0$ LA DISUGUAGLIANZA È VERIFICATA)

DA CUI $\mathrm{COV}(X,Y)^2 \leq \mathrm{VAR}(X)\, \mathrm{VAR}(Y)$

QUINDI SE
VALE L'UGUAGLIANZA, ESISTE $\bar{\alpha} \in \mathbb{R}$
TALE CHE $\operatorname{VAR}(\bar{\alpha} X + Y) = 0$

CIOÈ $\bar{\alpha} X + Y = c$ CON $c \in \mathbb{R}$

QUINDI $Y = -\bar{\alpha} X + c$

$$\underset{m\times 1}{Y} = \overset{m\times n}{A}\,\underset{n\times 1}{X} + \underset{m\times 1}{e} \qquad \mathrm{cov}(Y_j, Y_k)$$

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}$$

$$Y_j = a_{j1} X_1 + \cdots + a_{jn} X_n + e_j$$

$$\operatorname{cov}(Y) = A \operatorname{cov}(X) A^T$$

$$z^T \operatorname{cov}(X) z = \overbrace{\underbrace{\begin{bmatrix} z^T \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} \operatorname{cov}(X) \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} z \end{bmatrix}}_{n \times 1}}^{1 \times 1} \geq 0$$

$$\mathrm{VAR}(Z^T X) = \mathrm{VAR}(z_1 X_1 + \dots + z_n X_n) \geq 0$$

$$\parallel$$

$$Z^T \mathrm{COV}(X) (Z^T)^T$$

$$\parallel$$

$$Z^T \mathrm{COV}(X)\, Z$$

QUANDO $\mathrm{RANGO}(\mathrm{COV}(X)) < n$

$\exists\ Z = (z_1 \dots z_n) \neq 0$ TALE CHE $\mathrm{VAR}(z_1 X_1 + \dots + z_n X_n) = 0$

QUINDI ESISTE $c \in \mathbb{R}$ CON

$$z_1 x_1 + \text{———} + z_n x_n = c \quad \text{Q.C.}$$

CIOÈ (SE $z_1 \neq 0$)

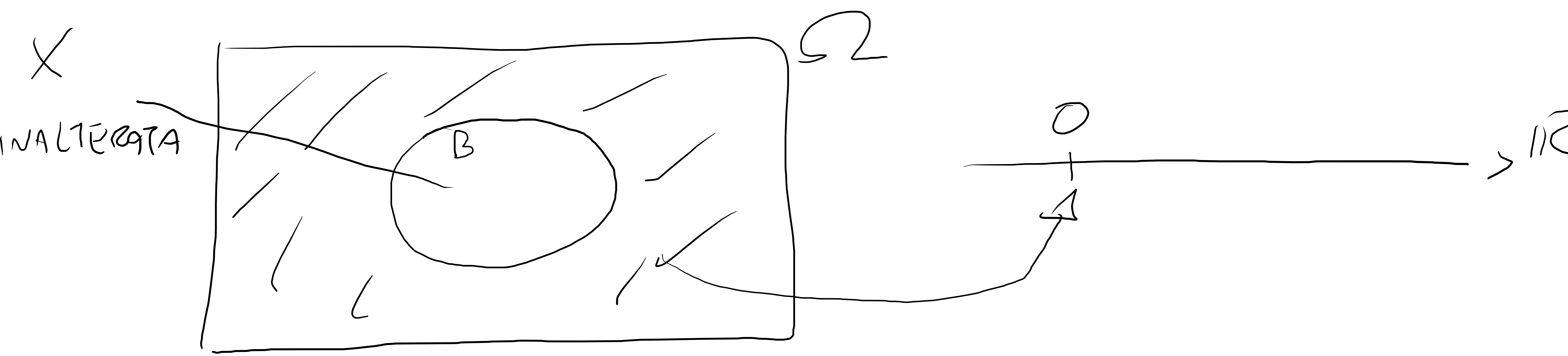
$$x_1 = c - \frac{z_2}{z_1} x_2 - - - - \frac{z_n}{z_1} x_n \quad \text{Q.C.}$$

$$Y = AX + b$$

X NORMALE MULTIVARIATA

$$Z^T Y = Z^T(AX + b) = \underbrace{(Z^T A)X}_{\text{NORMALE UNIVARIATA}} + \underbrace{Z^T b}_{\in \mathbb{R}}$$

$$(X 1_B)(\omega) = \begin{cases} 0 & \omega \notin B \\ X(\omega) & \omega \in B \end{cases}$$



$$P_B(A) = P(A|B)$$

P_B PROB. SU $(\Omega, \mathcal{F})$

$$E[X|B] = E^{P_B}[X]$$

X SEMPLICE $= \sum_{i=1}^{n} c_i \cdot 1_{A_i}$

$$E[X] = 0.20 \times 500 + 0.45 \times 750 + 0.35 \times 1500$$

LANCIO DI DUE DADI

$$E[X_1 | Z = 2] = 1$$

$$E[X_1 | Z = i] = \frac{i}{2}$$

$12 = 6$

$3 = 1.5$

$4 = 2$ $\quad (1,3), (2,2), (3,1)$

$5 = 2.5$ $\quad (1,4), (2,3), (3,2), (4,1)$

$$E[X \mid B] = E\left[\sum c_i 1_{A_i} \,\middle|\, B\right]$$

$$= \frac{E\left[\left(\sum c_i 1_{A_i}\right) 1_B\right]}{P(B)}$$

$$= \frac{E\left[\sum c_i 1_{A_i \cap B}\right]}{P(B)}$$

$$= \sum c_i \frac{P(A_i \cap B)}{P(B)} = P(A_i \mid B)$$

$$= P_B(A_i)$$

$$= \sum c_i \, P_B(A_i)$$

$$= E^{P_B}[X]$$