7 Dicembre

Teor I, J intervalli, $u: I \to J$ $u \in C^1(I)$

$f: J \to \mathbb{R}$. $f \in C^0(J)$ Allora

$$\int f(u(x))\, u'(x)\, dx = \left(\int f(u)\, du\right)(u(x))$$

Teor Sia $[a,b]$ un intervallo e $u: [a,b] \to J$

$u \in C^1([a,b])$, e sia $f \in C^0(J)$. Allora

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

Dim

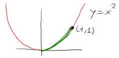
1° teor fond calcolo

$$\int_a^b f(u(x))\, u'(x)\, dx = \left.\int f(u(x))\, u'(x)\, dx\, \right]_a^b$$

cambio di variabile in integrali indef.

$$= \left.\left(\int f(u)\, du\right)(u(x))\, \right]_a^b = \int_{u(a)}^{u(b)} f(u)\, du$$

Esempio $\int_0^1 \sqrt{1+4x^2}\, dx$



In generale data $f \in C^1([a,b])$, l'integrale

$\int_a^b \sqrt{1+(f'(x))^2}\, dx$ rappresenta la lunghezza del grafico

$f(x)=x^2 \quad f'(x)=2x \quad (f'(x))^2=4x^2$

$$\int_0^1 \sqrt{1+4x^2}\, dx$$

$y = 2x$

$\frac{dy}{dx} = 2 \qquad dy = 2\,dx$

$$= \frac{1}{2}\int_0^2 \sqrt{1+y^2}\, dy =$$

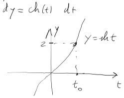
$ch^2(t) - sh^2(t) = 1$

$ch^2(t) = 1 + sh^2(t)$

$$= \frac{1}{2}\int_0^{t_0} \sqrt{1+sh^2(t)}\; ch(t)\, dt$$

$y = sh(t)$

$dy = ch(t)\, dt$



$2 = sh(t) \Rightarrow t_0 > 0$ per un qualche valore t_0 da determinare

$$= \frac{1}{2}\int_0^{t_0} ch^2(t)\, dt$$

Ho $ch^2(t) = \frac{1+ch(2t)}{2}$

$$\frac{1+ch(2t)}{2} = \frac{1+\frac{e^{2t}+e^{-2t}}{2}}{2} = \frac{\frac{e^{2t}+e^{-2t}+2}{2}}{2}$$

$$= \frac{e^{2t}+e^{-2t}+2}{4} = \frac{(e^t+e^{-t})^2}{4} = \left(\frac{e^t+e^{-t}}{2}\right)^2$$

$\cos(t) = \frac{e^{it}+e^{-it}}{2}$

$$= \frac{1}{2}\int_0^{t_0} \frac{1+ch(2t)}{2}\, dt = \frac{1}{4}\int_0^{t_0} (1+ch(2t))\, dt$$

$$= \frac{1}{4}\int_0^{t_0} dt + \frac{1}{4}\int_0^{t_0} ch(2t)\, dt$$

$$= \frac{t_0}{4} + \frac{1}{4}\left.\frac{sh(2t)}{2}\right]_0^{t_0}$$

$$= \frac{t_0}{4} + \frac{1}{8}\, sh(2t_0)$$

$sh(t_0) = 2$

$$= \frac{t_0}{4} + \frac{1}{\cancel{8}\,4}\,\cancel{2}\, sh(t_0)\, ch(t_0)$$

$$= \frac{t_0}{4} + \frac{1}{\cancel{4}\,2}\,\cancel{2}\sqrt{1+sh^2(t_0)}$$

$$= \frac{t_0}{4} + \frac{1}{2}\sqrt{5}$$

$sh(t) = y$

$\frac{e^t - e^{-t}}{2} = y \qquad e^t - e^{-t} = 2y$

$e^{2t} - 1 = 2e^t y \qquad (e^t)^2 - 2ye^t - 1 = 0$

$(e^t)_\pm = y \pm \sqrt{y^2+1}$

$y + \sqrt{y^2+1} > 0$

$y - \sqrt{y^2+1} < 0$

$\Rightarrow e^t = y + \sqrt{y^2+1} \Rightarrow t = \lg(y + \sqrt{y^2+1})$

$t_0 = \lg(2+\sqrt{5})$

$$\int_1^2 \lg^2(x)\,dx = \qquad\qquad y = \lg x \,, \quad x = e^y$$

$$dy = \frac{1}{x}\,dx$$

$$= \int_0^{\lg 2} y^2 \left(e^y\right)' dy = \dots \qquad\qquad dx = x\,dy = e^y\,dy$$

$$= y^2 e^y \Big]_0^{\lg 2} - 2\int_0^{\lg 2} y \left(e^y\right)' dy$$

$$= \lg^2(2)\; 2 - 2\, y e^y \Big]_0^{\lg 2} + 2 \int_0^{\lg 2} e^y\, dy$$

$$= 2\lg^2(2) - 4 \lg 2 + 2\; e^y \Big]_0^{\lg 2} = 2\lg^2(2) - 4\lg 2 + 4 - 2$$

$$\int_0^1 \sqrt{1-x^2}\,dx$$

$$x = \sin t$$

$$dx = \cos t \; dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(t)}\,\cos t \; dt$$

$$\sqrt{1-\sin^2 t} = \cos t$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t \; dt =$$

$$\cos^2 t = \frac{1+\cos(2t)}{2}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} dt + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2t)\,dt$$

$$= \frac{\pi}{4} + \left.\frac{\sin(2t)}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{4}\big(\underbrace{\sin(\pi)-\sin(0)}_{0}\big)$$

Funzioni razionali $R(x) = \dfrac{P(x)}{Q(x)}$

$$\boxed{\deg P(x) < \deg Q(x)}$$

Se $\deg P(x) \geq \deg Q(x)$ sappiamo che esistono due polinomi $D(x)$ ed $r(x)$ t.c.

$$P(x) = D(x)\, Q(x) + r(x)$$

con $\deg r(x) < \deg Q(x)$

$$\frac{P(x)}{Q(x)} = \frac{D(x)\, Q(x) + r(x)}{Q(x)} = D(x) + \frac{r(x)}{Q(x)}$$

$R(x) = \frac{P(x)}{Q(x)}$ $\qquad$ $\deg P(x) < \deg Q(x)$

Espansioni di Hermite

$$\frac{1}{x^3-3x^2+2x} = \frac{1}{x(x^2-3x+2)} = \frac{1}{x(x-1)(x-2)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$\int \frac{1}{x^3-3x^2+2x}\,dx = A\int \frac{1}{x}\,dx + B\int \frac{1}{x-1} + C\int \frac{1}{x-2}$$

$$= A \lg|x| + B \lg|x-1| + C \lg|x-2| + c$$

$$\frac{1}{x^3-3x^2+2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$= \frac{A(x-1)(x-2) + Bx(x-2) + Cx(x-1)}{x^3-3x^2+2x}$$

$$1 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1) =$$

$$= A(x^2-3x+2) + B(x^2-2x) + C(x^2-x)$$

$$1 = (A+B+C)x^2 + (-3A-2B-C)x + 2A$$

$$\begin{cases} A+B+C = 0 \\ 3A+2B+C = 0 \\ 2A = 1 \end{cases}$$

$$\frac{1}{x^3-3x^2+2x} = \frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$R(x) = \frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \qquad \cdot x$$

$$R(x)\,x = \frac{1}{(x-1)(x-2)} = A + x\left(\frac{B}{x-1} + \frac{C}{x-2}\right)$$

$$R(x)\,x\Big|_{x=0} = \frac{1}{2} = A \qquad \boxed{A = \frac{1}{2}}$$

$$R(x)(x-1)\Big|_{x=1} = \frac{1}{x(x-2)}\Big|_{x=1} = B + \cancel{(x-1)\left(\frac{A}{x} + \frac{C}{x-2}\right)}\Big|_{x=1}$$

$$B = -1$$

$$R(x)(x-2)\Big|_{x=2} = \frac{1}{x(x-1)}\Big|_{x=2} = C + \cancel{(x-2)\left(\frac{A}{x} + \frac{B}{x-1}\right)}\Big|_{x=2}$$

$$C = \frac{1}{2}$$

$$\frac{1}{x^3-3x^2+2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \qquad \cdot x$$

$$\frac{x}{x^3-3x^2+2x} = \frac{Ax}{x} + \frac{Bx}{x-1} + C\frac{x}{x-2}$$

per $x \to +\infty$

$$0 = A + B + C$$

$$R(x) = \frac{x+5}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$A = R(x)\,x \Big|_{x=0} = \frac{x+5}{(x-1)(x-2)} \Big|_{x=0} = +\frac{5}{2}$$

$$B = R(x)(x-1) \Big|_{x=1} = \frac{x+5}{x(x-2)} \Big|_{x=1} = \frac{6}{-1} = -6$$

$$C = R(x)(x-2) \Big|_{x=2} = \frac{x+5}{x(x-1)} \Big|_{x=2} = \frac{7}{2}$$

$$A + B + C = \frac{5}{2} - 6 + \frac{7}{2} = 0$$

$$R(x) = \frac{x^2+5}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$A = R(x)\,x\Big|_{x=0} = \frac{x^2+5}{(x-1)(x-2)}\Big|_{x=0} = \frac{5}{2}$$

$$B = R(x)(x-1)\Big|_{x=1} = \frac{x^2+5}{x(x-2)}\Big|_{x=1} = -6$$

$$C = R(x)(x-2)\Big|_{x=2} = \frac{x^2+5}{x(x-1)}\Big|_{x=2} = \frac{9}{2}$$

$$\frac{14}{2} - 6 = 7 - 6 = 1$$

$$\frac{x^3+5}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$