11 dicembre

Teor Sia $R(z) = \frac{P(z)}{Q(z)}$ con $P(z)$ e $Q(z)$ due polinomi con $\deg P(z) < \deg Q(z)$ e sia $1 \le n = \deg Q$ e ~~sono~~ si abbia $Q(z) = a_n z^n + \dots + a_0$ e la fattorizzazione del Teor. fond. dell'Algebra

$$Q(z) = a_n (z-z_1)^{m_1} \cdots (z-z_k)^{m_k}$$

dove $z_1, \dots, z_k$ sono le radici di $Q(z)$ con le relative molteplicità $m_1, \dots, m_k$ dove $m_1 + \dots + m_k = n$

Allora esistono per ogni $j = 1, \dots, k$ delle costanti A_{jl}, $l = 1, \dots, m_j$ tali che

$$R(z) = \sum_{j=1}^{k} \sum_{l=1}^{m_j} \frac{A_{jl}}{(z-z_j)^l}$$

Le A_{jl} sono univocamente determinate dalla seguente formula

$$A_{jl} = \frac{1}{(m_j - l)!} \left(\frac{d}{dz}\right)^{m_j - l} R(z)(z-z_j)^{m_j} \Big|_{z=z_j}$$

Es 1) $R(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$ $\qquad z_j = j$, $j = 1,2,3$

$$= \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3} \qquad m_j = 1$$

$$A_j = R(x)(x-j)\Big|_{x=j}$$

2) $$R(x) = \frac{x^2+x+1}{(x-1)^2(x-2)} = \frac{A_{11}}{(x-1)} + \frac{A_{12}}{(x-1)^2} + \frac{A_2}{x-2}$$

$$A_2 = R(x)(x-2)\Big|_{x=2} = \frac{x^2+x+1}{(x-1)^2}\Big|_{x=2} = 7$$

$$A_{12} = \frac{1}{(2-l)!}\left(\frac{d}{dx}\right)^{2-l} R(x)(x-1)^2\Big|_{x=1} = R(x)(x-1)^2\Big|_{x=1}$$
$l=2$

$$= \frac{x^2+x+1}{x-2}\Big|_{x=1} = -3$$

$l=1$

$$A_{11} = \frac{d}{dx} R(x)(x-1)^2\Big|_{x=1} = \frac{d}{dx}\frac{x^2+x+1}{x-2}\Big|_{x=1}$$

$$= \frac{(2x+1)(x-2) - (x^2+x+1)}{(x-2)^2}\Big|_{x=1}$$

$$= -3 - 3 = -6$$

$$\frac{x^2+x+1}{(x-1)^2(x-2)} = \frac{-6}{x-1} - \frac{3}{(x-1)^2} + \frac{7}{x-2}$$

Notare che moltiplico per x e faccio il limite $x \to +\infty$

$$\lim_{x\to+\infty} \frac{x^3+x^2+x}{(x-1)^2(x-2)} = \lim_{x\to+\infty}\left[-\frac{6x}{x-1} - \frac{3x}{(x-1)^2} + \frac{7x}{x-2}\right]$$

$$\begin{array}{c} \| \\ 1 \end{array} = -6 + 7$$

$$R(z) = \frac{1}{(z^2+1)^2} =$$

$z^2+1 = 0$ ha radici $z = \pm i$

$z^2+1 = (z-i)(z+i)$

$$= \frac{1}{((z-i)(z+i))^2} = \frac{1}{(z-i)^2(z+i)^2}$$

$\pm i$ sono ~~molte~~ radici del denomin di molteplicità 2.

$$R(z) = \frac{A}{z-i} + \frac{B}{z+i} + \frac{C}{(z-i)^2} + \frac{D}{(z+i)^2}$$

$$C = R(z)(z-i)^2\Big|_{z=i} = \frac{1}{(z+i)^2}\Big|_{z=i} = \frac{1}{(2i)^2} =$$

$$= \frac{1}{4(-1)} = -\frac{1}{4}$$

$$D = R(z)(z+i)^2\Big|_{z=-i} = \frac{1}{(z-i)^2\cancel{(z+i)^2}}\cancel{(z+i)^2}\Big|_{z=-i}$$

$$= \frac{1}{(-2i)^2} = -\frac{1}{4}$$

$$A = \frac{1}{(m-l)!}\left(\frac{d}{dz}\right)^{m-l} R(z)(z-i)^2\Big|_{z=i} \qquad m = 2,\ l = 1$$

$$= \frac{d}{dz}\frac{1}{(z+i)^2}\Big|_{z=i} = -2\frac{1}{(z+i)^3}\Big|_{z=i} =$$

$$= -\frac{2}{(2i)^3} = \frac{1}{4}\frac{1}{i} = \frac{-i}{4}$$

$$B = \frac{d}{dz}\frac{1}{(z-i)^2}\Big|_{z=-i} = -\frac{2}{(z-i)^3}\Big|_{z=-i}$$

$$= -\frac{2}{(-2i)^3} = \frac{i}{4}$$

$$\frac{1}{(z^2+1)^2} = \frac{-i/4}{z-i} + \frac{i/4}{z+i} - \frac{1}{4}\frac{1}{(z-i)^2} - \frac{1}{4}\frac{1}{(z+i)^2}$$

$$= \frac{-i/4}{z-i} + \frac{i/4}{z+i} - \frac{1}{4}\frac{d}{dz}\frac{-1}{z-i} - \frac{1}{4}\frac{d}{dz}\frac{-1}{z+i}$$

$$= \frac{1}{4}\left[\frac{-i}{z-i} + \frac{i}{z+i} + \frac{d}{dz}\left(\frac{1}{z-i} + \frac{1}{z+i}\right)\right]$$

$$= \frac{1}{4}\left[\frac{-i(\cancel{z}+i) + i(\cancel{z}-i)}{(z-i)(z+i)} + \frac{d}{dz}\frac{z+\cancel{i}+z-\cancel{i}}{(z-i)(z+i)}\right]$$

$$= \frac{1}{4}\left[\frac{2}{z^2+1} + \frac{d}{dz}\frac{2z}{z^2+1}\right]$$

$$\frac{1}{(z^2+1)^2} = \frac{\frac{1}{2}}{z^2+1} + \frac{d}{dz}\frac{\frac{1}{2}z}{z^2+1}$$

$$\frac{1}{(x^2+1)^2} = \frac{\frac{1}{2}}{x^2+1} + \frac{d}{dx}\frac{\frac{1}{2}x}{x^2+1}$$

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{d}{dx}\frac{x}{x^2+1}$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{x^2+1} + C$$

Nel libro si considera il caso $R(x) = \dfrac{P(x)}{Q(x)}$

dove $P(x)$ e $Q(x)$ hanno coefficienti reali

In Particolare

$Q(x) = a_n x^n + \dots + a_0$, dove $a_n, \dots, a_0 \in \mathbb{R}$.

Poi c'è da osservare che se un numero complesso $z_0 \in \mathbb{C}$

è una radice di $Q(x)$, anche $\bar{z}_0$ lo è

ed inoltre z_0 e $\bar{z}_0$ hanno la stessa molteplicità

$Q(x)$ supponiamo che $x_1 \dots x_\alpha$ (molteplicità $m_1 \dots m_\alpha$) siano le radici reali

e $\underbrace{z_1, \bar{z}_1}_{n_1}, \dots, \underbrace{z_\beta, \bar{z}_\beta}_{n_\beta}$ siano le radici complesse $z_j = u_j + i v_j$

$$Q(x) = a_n (x-x_1)^{m_1} \dots (x-x_\alpha)^{m_\alpha} (x-z_1)^{n_1} (x-\bar{z}_1)^{n_1} \dots (x-z_\beta)^{n_\beta} (x-\bar{z}_\beta)^{n_\beta}$$

$$= a_n (x-x_1)^{m_1} \dots (x-x_\alpha)^{m_\alpha} \left((x-z_1)(x-\bar{z}_1) \right)^{n_1} \dots \left((x-z_\beta)(x-\bar{z}_\beta) \right)^{n_\beta}$$

$$= a_n (x-x_1)^{m_1} \dots (x-x_\alpha)^{m_\alpha} \left(x^2 - (z_1+\bar{z}_1)x + |z_1|^2 \right)^{n_1} \dots \left(x^2 - (z_\beta + \bar{z}_\beta)x + |z_\beta|^2 \right)^{n_\beta}$$

$$= a_n (x-x_1)^{m_1} \dots (x-x_\alpha)^{m_\alpha} \left(x^2 - 2u_1 x + u_1^2 + v_1^2 \right)^{n_1} \dots \left(x^2 - 2u_\beta x + u_\beta^2 + v_\beta^2 \right)^{n_\beta}$$

In altre parole, esiste una fattorizzazione

$$Q(x) = a_n (x-x_1)^{m_1} \dots (x-x_\alpha)^{m_\alpha} (x^2 + c_1 x + d_1)^{n_1} \dots (x^2 + c_\beta x + d_\beta)^{n_\beta}$$

per opportune costanti $c_1, d_1, \dots, c_\beta, d_\beta$

con $c_1^2 - 4d_1 < 0, \dots, c_\beta^2 - 4d_\beta < 0$

Allora

$$R(x) = \frac{A_{11}}{x-x_1} + \dots + \frac{A_{1m_1}}{(x-x_1)^{m_1}} + \dots + \frac{A_{\alpha 1}}{x-x_\alpha} + \dots + \frac{A_{\alpha m_\alpha}}{(x-x_\alpha)^{m_\alpha}}$$

$$+ \frac{B_{11}x + C_{11}}{x^2 + c_1 x + d_1} + \dots + \frac{B_{1n_1}x + C_{1n_1}}{(x^2 + c_1 x + d_1)^{n_1}}$$

$$+ \dots + \frac{B_{\beta 1}x + C_{\beta 1}}{x^2 + c_\beta x + d_\beta} + \dots + \frac{B_{\beta n_\beta}x + C_{\beta n_\beta}}{(x^2 + c_\beta x + d_\beta)^{n_\beta}}$$

$$R(x) = \frac{A_{11}}{x-x_1} + \dots + \frac{A_{\alpha 1}}{x-x_\alpha} + \frac{B_{11}x + C_{11}}{x^2 + c_1 x + d_1} + \dots + \frac{B_{\beta 1}x + C_{\beta 1}}{x^2 + c_\beta x + d_\beta}$$

$$+ \frac{d}{dx} R_1(x)$$ dove $R_1(x)$ è una nuova funzione razionale

Notare: nella prima espansione di Hermite, con riferimento alle radici z_1 e $\bar{z}_1$, entrambe di molteplicità n_1

avevo

$$\frac{A_{11}}{z-z_1} + \dots + \frac{A_{1n_1}}{(z-z_1)^{n_1}} +$$

$$+ \frac{B_{11}}{z-\bar{z}_1} + \dots + \frac{B_{1n_1}}{(z-\bar{z}_1)^{n_1}}$$

$$= \frac{A_{11}(z-\bar{z}_1) + B_{11}(z-z_1)}{(z-z_1)(z-\bar{z}_1)} + \dots + \frac{A_{1n_1}(z-\bar{z}_1)^{n_1} + B_{1n_1}(z-z_1)^{n_1}}{(z-z_1)^{n_1}(z-\bar{z}_1)^{n_1}}$$

$$\int \frac{1}{(x^2+1)\ (x-1)}\,dx =$$

$$= \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1} \right) dx$$

$$A = \left.\frac{1}{x^2+1}\right|_{x=1} = \frac{1}{2}$$

$$\frac{1}{(x^2+1)(x-1)} = \frac{\frac{1}{2}}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\frac{x}{(x^2+1)(x-1)} = \frac{\frac{1}{2}x}{x-1} + \frac{Bx^2+Cx}{x^2+1}$$

e poi faccio il limite per $x \to +\infty$ ottengo

$$0 = \frac{1}{2} + B \qquad B = -\frac{1}{2}$$

$$\frac{1}{(x^2+1)(x-1)} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}x + C}{x^2+1} =$$

$$= \frac{\frac{1}{2}x^2 + \frac{1}{2} - \frac{1}{2}x(x-1) + C\,(x-1)}{(x^2+1)\ (x-1)} = \frac{\ldots + \frac{1}{2} - C}{(x^2+1)(x-1)}$$

$$\frac{1}{2} - C = 1 \qquad C = -\frac{1}{2}$$

$$= \frac{1}{2}\,\frac{1}{x-1} - \frac{1}{2}\,\frac{x+1}{x^2+1}$$

$$\int \frac{1}{(x^2+1)(x-1)}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx$$

$$= \frac{1}{2}\lg|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\lg|x-1| - \frac{1}{4}\lg(x^2+1) - \frac{1}{2}\arctan x + C$$