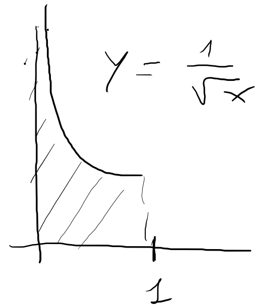
12 dicembre

Integrali generalizzati (Impropri)

$\frac{1}{\sqrt{x}}$ è integrabile in senso generalizzato

in $(0,1]$



$$\int_1^{+\infty} \frac{1}{x^2}\,dx$$

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx$$

Def Sia $[a, b)$ e sia $f \in L_{loc}[a, b)$

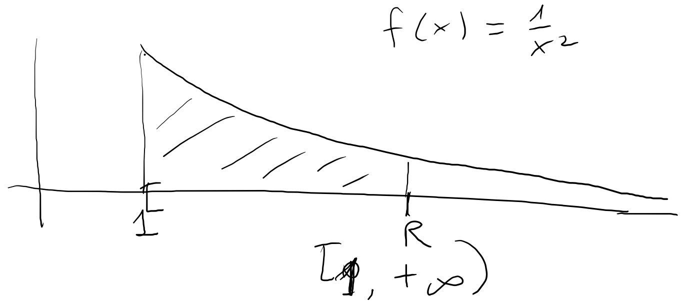
($f \in L[\alpha, \beta]$ $\forall$ intervallo chiuso e limitato $[\alpha, \beta] \subseteq [a, b)$)

Diremo che f è integrabile in senso generalizzato (o improprio) in $[a, b)$ se

$$\lim_{R \to b^-} \int_a^R f(x)\,dx \quad \text{esiste ed è finito}$$

e si denota il limite con

$$\int_a^b f(x)\,dx$$



Diremo che f è assolutamente integrabile in $[a, b)$ se $|f|$ è integrabile in senso generalizzato in $[a, b)$

Esempi x^{-p} $p>0$. Allora $x^{-p} \in L[1,+\infty)$

se e solo se $p>1$.

$$\lim_{R\to+\infty} \int_1^R x^{-p}\,dx$$

$p \neq 1$

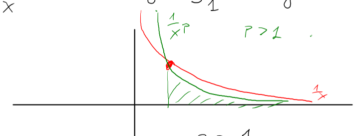
$$\int_1^R x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right]_1^R =$$

$$= \frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1} \xrightarrow{R\to+\infty} \begin{cases} +\infty & \text{se } p<1 \\ \dfrac{1}{p-1} & p>1 \end{cases}$$

$$\boxed{\text{Se } p>1 \quad \int_1^{+\infty} x^{-p}\,dx = \frac{1}{p-1}}$$

$p=1$

$$\int_1^R \frac{1}{x}\,dx = \left.\lg x\right]_1^R = \lg R \xrightarrow{R\to+\infty} +\infty$$



$p>1$

$$\int_2^{+\infty} x^{-p}\,dx = \lim_{R\to+\infty} \int_2^R x^{-p}\,dx$$

$$= \lim_{R\to+\infty} \left.\frac{x^{-p+1}}{-p+1}\right]_2^R \qquad p>1$$

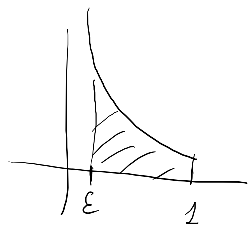
$$= \lim_{R\to+\infty} \left(\frac{R^{-p+1}}{-p+1} - \frac{2^{-p+1}}{-p+1} \right) =$$

$$= \frac{2^{-p+1}}{p-1}$$

Esempio $(0,1]$ $x^{-p} \in L(0,1]$ se e solo se $p < 1$.

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-p}\, dx$$

$p \neq 1$



$$\int_\varepsilon^1 x^{-p}\, dx =$$

$$= \frac{x^{-p+1}}{-p+1}\Big]_\varepsilon^1 = \frac{1}{-p+1} - \frac{\varepsilon^{-p+1}}{-p+1} \xrightarrow{\varepsilon \to 0^+} \begin{cases} \frac{1}{1-p} & \text{se } p<1 \\ +\infty & p>1 \end{cases}$$

Se $p < 1$ $\displaystyle\int_0^1 x^{-p}\, dx = \frac{1}{1-p}$

$p = 1$

$$\int_\varepsilon^1 \frac{1}{x}\, dx = \lg x \Big]_\varepsilon^1 = -\lg \varepsilon \xrightarrow{\varepsilon \to 0^+} +\infty$$

Teor (Aut-aut) Se $f \in L_{loc}[a,b)$ e se $f \geq 0$ ovunque, allora

$$\lim_{R \to b^-} \int_a^R f(x)\,dx \quad \text{esiste in } [0,+\infty]$$

Dim Poniamo $F(R) \doteq \int_a^R f(x)\,dx$

Se $R_1 > R$

$$F(R_1) = \int_a^{R_1} f(x)\,dx =$$

$$F(R_1) = \underbrace{\int_a^R f(x)\,dx}_{F(R)} + \underbrace{\int_R^{R_1} f(x)\,dx}_{\geq 0}$$

$$F(R_1) \geq F(R)$$

cioè $F(R)$ è crescente

$$\lim_{R \to b^-} F(R) = \sup\{F(R) : a \leq R < b\}$$

Teor Siano $f, g \in L_{loc}[a,b)$ con $0 \le f(x) \le g(x)$

Allora, se $g \in L[a,b)$ si ha $f \in L[a,b)$

(o, equivalentemente, se $f \notin L[a,b)$ allora $g \notin L[a,b)$)

Dim $g \in L[a,b) \Rightarrow \lim_{R \to b^-} \int_a^R g(x)\,dx \in [0,+\infty)$

~~A1~~ Siccome $f(x) \le g(x) \quad \forall x \Rightarrow$

$$\int_a^R f(x)\,dx \le \int_a^R g(x)\,dx \qquad \forall R \quad a<R<b$$

Siccome $f(x) \ge 0$, $\lim_{R\to b^-} \int_a^R f(x)\,dx$ esiste

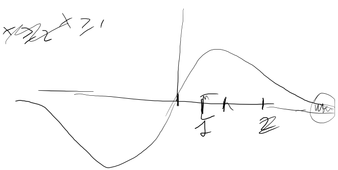
ed inoltre,

$$\lim_{R\to b^-} \int_a^R f(x)\,dx \le \lim_{R\to b^-} \int_a^R g(x)\,dx = \underset{\uparrow\,\mathbb{R}}{\int_a^b g(x)\,dx}$$

$$\Rightarrow \lim_{R\to b^-} \int_a^R f(x)\,\underset{\uparrow\,\mathbb{R}}{dx} < +\infty$$

Esempio $\frac{x^3}{1+x^4}$ è ~~integr~~ sommabile in $[1,+\infty)$?

Risposta: no! ~~$x \ge 2$~~ $x \ge 1$



$$\frac{x^3}{1+x^4} = \frac{1}{x^4}\,\frac{x^3}{1+x^{-4}} = \frac{1}{x}\,\frac{1}{1+x^{-4}} \ge \frac{1}{x}\,\frac{1}{2}$$

$$\frac{x^3}{1+x^4} \ge \frac{1}{x}\,\frac{1}{2} \qquad \forall\, x \ge 1$$

$$\frac{1}{x} \notin L[1,+\infty) \Rightarrow \cancel{x^3}\ \frac{1}{2}\,\frac{1}{x} \notin L[1,+\infty)$$

$$\Rightarrow \frac{x^3}{1+x^4} \notin L[1,+\infty)$$

Teor Siano $f, g \in L_{loc}([a,b))$, $g > 0$

$f \geq 0$. Vale quanto segue

1) Se $\lim_{x \to b^-} \frac{f(x)}{g(x)} = L$ esiste con $0 < L < +\infty$

allora $f \in L[a,b) \iff g \in L[a,b)$

2) Se $L = 0$

$g \in L[a,b) \implies f \in L[a,b)$

3) Se $L = +\infty$

$f \in L[a,b) \implies g \in L[a,b)$

$[1, +\infty$

Ex

$$\lim_{x \to +\infty} \frac{\frac{x^3}{1+x^4}}{\frac{1}{x}} = 1 \quad \text{e allora}$$

$$\frac{1}{x} \notin L[1,+\infty) \implies \frac{x^3}{1+x^4} \notin L[1,+\infty).$$

$$\int_1^{+\infty} \lg\left(1 + \frac{1}{x}\right) dx$$

$y = \frac{1}{x}$

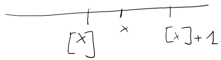
$$\lim_{x \to +\infty} \frac{\lg\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{y \to 0} \frac{\lg(1+y)}{y} = 1$$

$$\frac{1}{x} \notin L[1,+\infty) \implies \lg\left(1 + \frac{1}{x}\right) \notin L[1,+\infty)$$

Verificare che $\displaystyle\int_1^{+\infty} \frac{1}{[x]^p}\,dx$ esiste

finito se e solo se $p > 1$

$$\underset{\substack{\Uparrow \\ \mathbb{Z}}}{[x]} \le x < [x] + 1$$



Si confronta con $\dfrac{1}{x^p}$

perché per $p = 1$

$$\lim_{x \to +\infty} \frac{\frac{1}{[x]}}{\frac{1}{x}} = \lim_{x \to +\infty} \frac{x}{[x]} =$$

$$= \lim_{x \to +\infty} \frac{[x] + (x - [x])}{[x]} =$$

$$= \lim_{x \to +\infty} \left(1 + \frac{x - [x]}{[x]} \right)$$

$$0 \le x - [x] < 1$$

$$\lim_{x \to +\infty} \frac{x - [x]}{[x]} = 0$$

$$\lim_{x \to +\infty} [x] = +\infty$$

$$[x] > x - 1$$

$$0 \le \frac{x - [x]}{[x]} < \frac{1}{[x]} \to 0$$

$$\Rightarrow \lim_{x \to +\infty} \frac{\frac{1}{[x]^p}}{\frac{1}{x^p}} = 1$$

$$\frac{1}{x^p} \in L[1, +\infty)$$
$$\Updownarrow$$
$$\frac{1}{[x]^p} \in L[1, +\infty)$$