Dec 6

Thm $d \geq 3$

$$\lim_{t \to +\infty} |u(t)|_{L^r(\mathbb{R}^d)} = 0$$

$$2 < r < \frac{2d}{d-2}$$

$$|u|_{L^q(\mathbb{R}, W^{1,p+1})}$$

Lemma $p \in [1, +\infty)$, $0 \leq q \leq p$

$q < d$. Then

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q} \, dx \leq \frac{p}{d-q} |u|^{p-q}_{L^p(\mathbb{R}^d)} |\nabla u|^q_{L^p(\mathbb{R}^d)}$$

Pf $u \in W^{1,p}(\mathbb{R}^d)$

$u \in C_c^\infty(\mathbb{R}^d)$

$u_n \longrightarrow u$ in $W^{1,p}(\mathbb{R}^d)$

up to a subsequence $u_n(x) \longrightarrow u(x)$

a. e.

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q}\,dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}^d} \frac{|u_n(x)|^p}{|x|^q}\,dx$$

$$\le \lim_{n\to+\infty} \frac{p}{d-q} |u_n|_{L^p}^{p-q} |\nabla u_n|_{L^p}^{q}$$

$$= \frac{p}{d-q} |u|_{L^p}^{p-q} |\nabla u|_{L^p}^{q}$$

$$z(x) = |x|^{-q} x$$

$$\nabla\cdot z(x) = \nabla(|x|^{-q})\cdot x + |x|^{-q} d$$

$$= -q|x|^{-q-1} \frac{x}{|x|}\cdot x + d|x|^{-q}$$

$$= (d-q)|x|^{-q}$$

$$\int_{|x|>r} |u|^p \nabla\cdot z = \int_{|x|>r} \nabla\cdot(|u|^p z) - p\int_{|x|>r} |u|^{p-1} \nabla|u|\cdot z$$

$$(d-q)\int_{|x|>r} \frac{|u|^p}{|x|^q}\,dx = \int_{|x|=r} -\frac{\cancel{x}}{|x|}\cdot\frac{x}{|x|^q}|u|^p\,dS$$

$$+ p\int_{|x|>r} |u|^{p-1}\nabla|u|\cdot\frac{x}{|x|^q}\,d\cancel{x}$$

$$\le p\int_{|x|>r} \frac{|u|^{p-1}|\nabla u|}{|x|^{q-1}}\,dx$$

$$\nabla |u| = \frac{u}{|u|} \nabla u$$

$$(d-q)\int_{|x|>r} \frac{|u|^p}{|x|^q}\, dx$$

$$\le p \int_{|x|>r} \frac{|u|^{p-1} |\nabla u|}{|x|^{q-1}}\, dx$$

$$= p \int_{|x|>r} \frac{|u|^{\frac{p(q-1)}{q}}}{|x|^{q-1}} |u|^{\frac{p-q}{q}} |\nabla u| \, dx$$

$$\frac{q-1}{q} + \frac{p-q}{pq} + \frac{1}{p} = 1$$

$$\le p \left(\int_{|x|>r} \left(\frac{|u|^{p\frac{q-1}{q}}}{|x|^{q-1}} \right)^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}}$$

$$\left| |u|^{\frac{p-q}{q}} \right|_{L^{\frac{pq}{p-q}}} \quad |\nabla u|_{L^p}$$

$$= p \left(\int_{|x|>r} \frac{|u|^p}{|x|^q} \right)^{\frac{q-1}{q}} |u|_{L^p}^{\frac{p-q}{q}} |\nabla u|_{L^p}$$

$$(d-q)\int_{|x|>r} \frac{|u|^p}{|x|^q}\, dx$$

$$\le p \left(\int_{|x|>r} \frac{|u|^p}{|x|^q} \right)^{\frac{q-1}{q}} |u|_{L^p}^{\frac{p-q}{q}} |\nabla u|_{L^p}$$

$$\left(\int_{|x|>r} \frac{|u|^p}{|x|^q}\,dx\right)^{1-\frac{1}{q'}=\frac{1}{q}}$$

$$\leq \frac{p}{d-q} \qquad |u|_{L^p}^{\frac{p-q}{q}} \quad |\nabla u|_{L^p}$$

$$\int_{|x|>r} \frac{|u|^p}{|x|^q}\,dx \leq \left(\frac{p}{d-q}\right)^q |u|_{L^p}^{p-q} \quad |\nabla u|_{L^p}$$

$r \to 0^+$ we get statement.

Lemma $d \geq 3$ $\exists\, C_d$ st

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^3}\,dx \leq C_d\, |u|^2_{H^2(\mathbb{R}^d)}$$

Pf Apply previous lemma for

$q=3 \qquad p=2$

$$(d-q)\int_{|x|>r} \frac{|u(x)|^p}{|x|^q} \leq p\int_{|x|>r} \frac{|u|^{p-1}|\nabla u|}{|x|^{q-1}}\,dx$$

$$(d-3)\int_{|x|>r} \frac{|u(x)|^2}{|x|^3} \leq 2\int_{|x|>r} \frac{|u|\,|\nabla u|}{|x|^2}\,dx$$

$$\leq 2\left(\int_{|x|>r} \frac{|u|^2}{|x|^2}\,dx\right)^{\frac{1}{2}}\left(\int_{|x|>r} \frac{|\nabla u|^2}{|x|^2}\,dx\right)^{\frac{1}{2}}$$

Apply case $p = q = 2$ of previous lemma to both factor

$$\leq C \quad |\nabla u|_{L^2} \quad |\nabla^2 u|_{L^2} \qquad \square$$

$$\frac{d}{dt} \langle \partial_r u, i u \rangle \geq (d-1) \frac{p-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} dx$$

$$u_0 \in H^2(\mathbb{R}^d) \implies u \in C^0(\mathbb{R}, H^2)$$

$i u_t = -\Delta u + |u|^{p-1} u$ is true in $\mathcal{D}'(\mathbb{R}, L^2(\mathbb{R}^d))$

$$\mathcal{L}(\mathcal{D}(\mathbb{R}), L^2(\mathbb{R}^d))$$

$$\left.\begin{aligned} -\Delta u &\in C^0(\mathbb{R}, L^2(\mathbb{R}^d)) \\ |u|^{p-1} u &\in C^0(\mathbb{R}, L^2(\mathbb{R}^d)) \end{aligned}\right)$$

$$|u|_{L^{2p}(\mathbb{R}^d)} \qquad H^2(\mathbb{R}^d) \hookrightarrow L^{2p}(\mathbb{R}^d)$$

$$\frac{1}{2p} \geq \frac{1}{2} - \frac{2}{d} > 0$$

$$\frac{1}{\cancel{2}p} \geq \frac{d-4}{\cancel{2}d} \iff p < \frac{d}{d-4} \quad ?$$

$$p < d^* = \frac{d+2}{d-2} < \frac{d}{d-4}$$

$$(d+2)(d-4) < (d-2)d$$

$$d^2 - 2d - 8 < d^2 - 2d \quad \checkmark$$

$u \in C^1(\mathbb{R}, L^2)$ $\qquad$ $u \in C^1(\mathbb{R}, L^2)$

$\qquad\qquad\qquad\qquad\qquad\quad\;\;$ $u \in C^0(\mathbb{R}, H^2)$

$$\frac{d}{dt}\frac{1}{2}\left\langle i\left(\partial_r + \frac{d-1}{2r}\right)u, u\right\rangle =$$

$$= \frac{1}{2}\left\langle i\left(\partial_r + \frac{d-1}{r}\right)\dot{u}, u\right\rangle + \frac{1}{2}\left\langle i\left(\partial_r + \frac{d-1}{r}\right)u, \dot{u}\right\rangle$$

$$= \left\langle i\left(\partial_r + \frac{d-1}{r}\right)u, \dot{u}\right\rangle$$

$$= -\left\langle \left(\partial_r + \frac{d-1}{r}\right)u, i\dot{u}\right\rangle = -\left\langle \left(\partial_r + \frac{d-1}{r}\right)u, -\Delta u + |u|^{p-1}u\right\rangle$$

$$i u_t = -\Delta u + |u|^{p-1}u$$

Suppose that we can say

$$\left\langle \left(\partial_r + \frac{d-1}{r}\right)u, \Delta u\right\rangle \leq 0$$

then we conclude

$$\frac{d}{dt}\frac{1}{2}\langle u_r, iu\rangle \geq +\langle u_r, |u|^{p-1}u\rangle + (d-1)\iint \frac{|u|^{p+1}}{r}\,dx$$

Lemma. $u \in H^2(\mathbb{R}^d)$

$$\langle (\partial_r + \frac{d-1}{2r})u, \Delta u \rangle \leq 0$$

Pf.

$$\mathrm{Re}\{\Delta u\,(\bar{u}_r + \frac{d-1}{2r}\bar{u})\} =$$

$$= \nabla\cdot \mathrm{Re}\{\nabla u(\bar{u}_r + \frac{d-1}{2r}\bar{u})\} - \nabla\cdot\{\frac{x}{2r}|\nabla u|^2\}$$

$$+ \nabla\cdot\{\frac{d-1}{4}\frac{x}{r^3}|u|^2\} - \frac{1}{r}(|\nabla u|^2 - |u_r|^2)$$

$$- \frac{(d-1)(d-3)}{4r^3}|u|^2$$

$\mathbb{R}^d \setminus D(0,a)$

$$\langle (\partial_r + \frac{d-1}{2r})u, \Delta u \rangle \leq -\int_{\mathbb{R}^d} \frac{|\nabla u|^2 - |u_r|^2}{r}\,dx$$

$$- \frac{(d-1)(d-3)}{4} \lim_{a\to 0^+} \int_{r\geq a} \frac{|u|^2}{r^3}\,dx$$

$$- \liminf_{a\to 0^+} \int_{r=a} \left[\partial_r u(\bar{u}_r + \frac{d-1}{2r}\bar{u}) - \cancel{\frac{|\nabla u|^2}{2}} + \frac{d-1}{4}\frac{|u|^2}{a^2}\right]$$

$$\lim_{a\to 0^+} \int_{r=a} |\nabla u|^2\,dS = 0$$

$$\lim_{a\to 0^+} \int_{r=a} |u_r|^2\,dS = 0$$

$d > 3$

$$\frac{1}{a^2}\int_{r=a} |u|^2\,dS \xrightarrow{a\to 0^+} 0$$

For $d=3$

$$\lim_{a\to 0^+}\int_{r=a}\frac{|u|^2}{a^2}\,dS = C_3\,|u(0)|^2$$

$$\underbrace{\left|\int_{r=a}\partial_r u\,\frac{\bar u}{r}\right|^2}_{\downarrow\, 0} \le \underbrace{\int_{r=a}|\partial_r u|^2}_{\downarrow\, 0}\ \underbrace{\int_{r=a}\frac{|u|^2}{r^2}}_{\downarrow\, |u(0)|^2}$$

$$u_r = \frac{x}{|x|}\cdot\nabla u$$

$$\mathrm{Re}\left\{\Delta u\left(\bar u_r+\frac{d-1}{2r}\bar u\right)\right\} =$$

$$= \nabla\cdot\mathrm{Re}\left\{\nabla u\left(\bar u_r+\frac{d-1}{2r}\bar u\right)\right\} - \nabla\cdot\left\{\frac{x}{2r}|\nabla u|^2\right\}$$

$$+\ \nabla\cdot\left(\frac{d-1}{4}\,\frac{x}{r^3}\,|u|^2\right) - \frac{1}{r}\left(|\nabla u|^2-|u_r|^2\right)$$

$$-\ \frac{(d-1)(d-3)}{4r^3}|u|^2$$

$$\nabla\cdot\mathrm{Re}\left\{\nabla u\left(\bar u_r+\frac{d-1}{2r}\bar u\right)\right\} =$$

$$= \mathrm{Re}\left\{\Delta u\left(\bar u_r+\frac{d-1}{2r}\bar u\right)\right\} +$$

$$+\ \mathrm{Re}\left\{\partial_j u\ \partial_j\left(\frac{x_k}{r}\partial_k\bar u\right)\right\}$$

$$+ \operatorname{Re}\left\{ \partial_j u \; \partial_j \left(\frac{d-1}{2r} \bar{u} \right) \right\}$$

$$= \operatorname{Re}\left\{ \Delta u \left(\bar{u}_r + \frac{d-1}{2r} \bar{u} \right) \right\} +$$

$$+ \frac{x_k}{2r} \partial_k |\nabla u|^2 + \frac{1}{r} |\nabla u|^2$$