13 dec.

$d \geq 3$

$$\forall \qquad 2 < r < \frac{2d}{d-2}$$

$$\lim_{t \to +\infty} |u(t)|_{L^r(\mathbb{R}^d)} = 0$$

$$u_0 \in H^2 \qquad i\dot{u} = -\Delta u + |u|^{p-1}u$$

$$\mathcal{D}'((0,T), L^2) = \mathcal{L}(\mathcal{D}(0,T), L^2)$$

$$\frac{d}{dt}\frac{1}{2}\langle i\partial_r u, u\rangle = \frac{1}{2}\frac{d}{dt}\langle i(\partial_r + \frac{d-1}{2r})u, u\rangle$$

$$= -\langle (\partial_r + \frac{d-1}{2r})u, i\dot{u}\rangle =$$

$$= \cancel{-}\underbrace{\langle (\partial_r + \frac{d-1}{2r})u, \cancel{-}\Delta u\rangle}_{\leq 0}$$

$$-\langle (\partial_r + \frac{d-1}{2r})u, |u|^{p-1}u\rangle$$

$$\frac{d}{dt}\frac{1}{2}\langle i\partial_r u, u\rangle \leq -\langle (\partial_r + \frac{d-1}{2r})u, |u|^{p-1}u\rangle$$

We show now

$$\langle (\partial_r + \frac{d-1}{2r})u, |u|^{p-1}u\rangle = \frac{d-1}{2}\frac{p-1}{p+1}\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}$$

$$\frac{d-1}{2}\left\langle u, \frac{|u|^{p-1}}{r}u\right\rangle + \frac{1}{2}\mathrm{Re}\int |u|^{p-1} 2\partial_r u \;\bar{u}\, dx$$

$$= \frac{d-1}{2}\int \frac{|u|^{p+1}}{r} + \frac{1}{2}\int \left(|u|^2\right)^{\frac{p-1}{2}} \partial_r |u|^2\, dx$$

$$= \frac{d-1}{2}\int \frac{|u|^{p+1}}{r} + \frac{1}{2}\,\frac{2}{p+1}\int_{S^{d-1}}\int_0^{+\infty} \partial_r \left(|u|^2\right)^{\frac{p+1}{2}} r^{d-1}\, dr\, dS$$

$$= \frac{d-1}{2}\int \frac{|u|^{p+1}}{r} - \frac{d-1}{p+1} \underbrace{\int_{S^{d-1}}\int_0^{+\infty} |u|^{p+1} \frac{r^{d-1}}{r}\, dr\, d}_{\int \frac{|u|^{p+1}}{r} dx}$$

$$= (d-1)\underbrace{\left(\frac{1}{2} - \frac{1}{p+1}\right)}_{\frac{p-1}{2(p+1)}} \int \frac{|u|^{p+1}}{r}\, dx$$

We have shown that

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, |u|^{p-1}u \right\rangle = \frac{d-1}{2}\,\frac{p-1}{p+1}\int \frac{|u|^{p+1}}{r}\, dx$$

Lemma $\forall\, u_0 \in H^1$ then

$$\int_0^{+\infty} dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r}\, dx$$

$$\leq \frac{2}{d-1}\, \frac{p+1}{p-1}\, |u_0|_{L^2}\, |\nabla u|_{L^\infty(\mathbb{R}_+, L^2)}$$

$$E(u) = \frac{|\nabla u|_{L^2}^2}{2} + \frac{1}{p+1}\int |u|^{p+1}\, dx$$

$$\sqrt{2E(u_0)} \geq |\nabla u|_{L^2} \qquad \forall\, t$$

$$\leq \frac{2}{d-1}\, \frac{p+1}{p-1}\, |u_0|_{L^2}\, \sqrt{2E(u_0)}$$

Furthermore $u(t) \xrightarrow{t \to +\infty} 0$ in H^1

Pf If $u_0 \in H^2$, we have

$$-\frac{d}{dt}\frac{1}{\cancel{2}} \langle i\partial_r u, u\rangle \geq \frac{d-1}{\cancel{2}}\, \frac{p-1}{p+1} \int \frac{|u|^{p+1}}{r}\, dx$$

$$\int_0^T dt \int \frac{|u|^{p+1}}{r}\, dx \leq$$

$$\leq -\frac{p+1}{p-1}\,\frac{1}{d-1}\quad \langle i\partial_r u, u\rangle \Big]_0^T$$

$$\leq C_{pd}\,|u_0|\quad |\nabla u|_{L^\infty([0,T],L^2)}$$

For $T \to +\infty$ we obtain the formula in the statement

$$u_0 \in H^1 \qquad H^2 \ni u_{0n} \xrightarrow{n\to+\infty} u_0 \quad \text{in } H^1$$

Then $\forall$ $T>0$ we have

$$u_n \longrightarrow u \quad \text{in } C^0([0,T], H^1)$$

$$\nabla u_n \to \nabla u \quad \text{in } C^0([0,T], L^2)$$

$$\int_0^T dt \int_{\mathbb{R}^d} \frac{|u_n|^{p+1}}{r}\,dx \leq C\,|u_{0n}|_{L^2}\,|\nabla u_n|_{L^\infty([0,T],L^2)}$$

up to a subsequence we have

$$u_n(t,x) \longrightarrow u(t,x) \quad \text{for a.e. } (t,x) \in [0,T]\times\mathbb{R}^d$$

$$\int_0^T dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r}\,dx$$

$$\leq \liminf_{n\to+\infty} \int_0^T dt \int_{\mathbb{R}^d} \frac{|u_n|^{p+1}}{r}\,dx$$

$$\leq C \lim_{n\to+\infty} |u_{0n}|_{L^2} |\nabla u_n|_{L^\infty([0,T],L^2)}$$

$$= C \quad |u_0|_{L^2} \quad |\nabla u|_{L^\infty([0,T],L^2)}$$

$$\int_0^{+\infty} dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} dx < +\infty$$

$$u(t) \xrightarrow{t\to+\infty} 0 \qquad \text{in } H^1$$

This is equivalent to

$$< u(t), \psi > \xrightarrow{t\to+\infty} 0 \qquad \psi \in C_c^\infty(\mathbb{R}^d)$$

$$\left| < \frac{u}{r^{\frac{1}{p+1}}}, r^{\frac{1}{p+1}} \psi > \right| \leq$$

$$\leq \left| \frac{u}{r^{\frac{1}{p+1}}} \right|_{L^{p+1}(\mathbb{R}^d)} \cdot \underbrace{\left| r^{\frac{1}{p+1}} \psi \right|}_{} {}_{L^{\frac{p+1}{p}}(\mathbb{R}^d)}$$

$$|< u, \psi >| \leq C \left| \frac{u}{r^{\frac{1}{p+1}}} \right|_{L^{p+1}(\mathbb{R}^d)}$$

$$|< u, \psi >|^{p+1} \leq C^{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} dx \in L^1(\mathbb{R}_+)$$

$$|\langle u(t), \varphi\rangle|^{p+1} \in L^1(\mathbb{R}_+) \qquad u \in C^1([0,T], H^{-1})$$

$$i\dot{u} = -\Delta u + |u|^{p-1}u \quad \text{in} \quad C^0([0,T], H^{-1})$$

$$\mathcal{L}(\mathcal{D}(0,T), H^{-1})$$

$$-\int_0^T i u \dot{\varphi} = \underline{\int_0^T (-\Delta u + |u|^{p-1}u)\varphi\, dt} \quad \varphi(t)$$

$$2k \geq p+2 \qquad s < t$$

$$\left| |\langle u(t), \varphi\rangle|^{2k} - |\langle u(s), \varphi\rangle|^{2k} \right| \leq$$

$$\leq 2k \left| \int_s^t \left(\langle u(t'), \varphi\rangle^2 \right)^{k-1} \langle u(t'), \varphi\rangle \langle \dot{u}(t'), \varphi\rangle \, dt' \right|$$

$$\leq 2k \int_s^t |\langle u, \varphi\rangle|^{2k-1} \underbrace{|\langle -\Delta u + |u|^{p-1}u, \varphi\rangle|}_{\leq C(E(u_0), |u_0|_{L^2})} dt'$$

$$\leq C_{u_0} \int_s^t |\langle u, \varphi\rangle|^{p+1} dt' \xrightarrow{s \to +\infty} 0,$$

$$\cancel{\left\| |\langle u, \varphi\rangle|^{2k-(p+2)} \right\|_{L^\infty(\mathbb{R})}}$$

So $|\langle u(t), \varphi\rangle|^{2k} \xrightarrow{t \to +\infty} L$

$$\overset{\Uparrow}{L^1}(\mathbb{R}_+)$$

$$\Longrightarrow \quad L = 0$$

$$\Longrightarrow \quad \langle u(t), \varphi \rangle \xrightarrow{t \to +\infty} 0$$

$\partial_t u \in \mathcal{D}'((0,T], H^{-1})$

Lemma For any $u \in L^2([0,T], H^1) \cap$
$\underline{H^1([0,T], H^{-1})}$

then $u \in C^0([0,T], L^2)$

Furthermore $|u(t)|_{L^2}^2 \in AC\,[0,T]$

with $\dfrac{d}{dt} |u(t)|_{L^2}^2 = 2 \langle u(t), \dot{u}(t) \rangle$

Robinson, Rodrigo, Sadowski

$u(t) \xrightarrow{t \to \infty} 0$ in $L^{p+1}(\mathbb{R}^4)$

Lemma $$\int_{|x| \geq t \lg t} |u|^{p+1}\, dx \xrightarrow{t \to +\infty} 0$$

Pf $M > 0$

$$\Theta_M(x) = \begin{cases} \dfrac{|x|}{M} & |x| \leq M \\ 1 & |x| \geq M \end{cases}$$

$\Theta_M \in W^{1,\infty}(\mathbb{R}^d)$

$$|\nabla \Theta_M|_{L^\infty} \leq \frac{1}{M} .$$

$$|\partial_r \Theta_M|_{L^\infty} = \frac{1}{M}$$

$$u \in C^0(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$$
$[0,T]$

$$\sqrt{\Theta_M}\, u \in L^\infty([0,T], L^2) \cap C^1([0,T], H^{-1})$$
$$\cap \qquad\qquad H^1([0,T], H^1)$$
$$L^2([0,T], L^2)$$

$$\frac{1}{2}\frac{d}{dt}\langle \Theta_M u, u\rangle = \frac{d}{dt}\frac{1}{2}|\sqrt{\Theta_M}\, u|_{L^2}^2 = \langle \sqrt{\Theta_M} u, \sqrt{\Theta_M}\dot{u}\rangle$$

$$= \langle \Theta_M u, \dot{u}\rangle =$$

$$i\dot{u} = -\Delta u + |u|^{p-1} u$$

$$= \langle \Theta_M u, i\Delta u - i\cancel{|u|^{p-1}} u\rangle$$

$$= \langle \Theta_M u, i\nabla\cdot\nabla u\rangle =$$

$$= -\langle \nabla\Theta_M u + \cancel{\Theta_M \nabla u}, i\nabla u\rangle$$

$$\left| \frac{d}{dt} \frac{1}{2} \langle \Theta_M u, u \rangle \right| \le \underbrace{|\nabla \Theta_M|_{L^\infty}}_{\frac{4}{M}} |u|_{L^2} |\nabla u|_{L^2}$$

$$\le C_0 \frac{4}{M}$$

$$\langle \Theta_M u(t), u(t) \rangle = \langle \Theta_M u_0, u_0 \rangle +$$

$$+ \int_0^t \frac{d}{dt'} \langle \Theta_M u, u \rangle \, dt'$$

$$\le \langle \Theta_M u_0, u_0 \rangle + C_0 \frac{t}{M}$$

$\forall M > 0$

$M = t \lg t$

$$\int_{|x| \ge t \lg t} |u(t)|^2 dx \le$$

$$\le \int_{\mathbb{R}^d} \Theta_{t \lg t} |u(t)|^2 dx = \langle \Theta_{t \lg t} u(t), u(t) \rangle$$

$$\le \langle \Theta_{t \lg t} u_0, u_0 \rangle + \frac{C_0}{\lg t}$$

$$= \frac{C_0}{\lg t} + \int_{|x| \le t \lg t} \frac{|x|}{t \lg t} |u_0(x)|^2 dx$$

$\downarrow 0$ $\qquad \downarrow 0$

$$+ \int_{|x| \geq t \lg t} |u_0(x)|^2 \, dx \xrightarrow[t \to +\infty]{} 0$$

$$\int_{\mathbb{R}^d} \underbrace{\chi_{D_{\mathbb{R}^d}(0,\, t \lg t)} \frac{|x|}{t \lg t} |u_0(x)|^2}_{\leq |u_0(x)|^2 \in L^1(\mathbb{R}^d)} \, dx$$

$$\Longrightarrow \lim_{t \to +\infty} \int_{\mathbb{R}^d} \chi_{D_{\mathbb{R}^d}(0,\, t \lg t)} \frac{|x|}{t \lg t} |u_0(x)|^2 \, dx$$

$$= \int_{\mathbb{R}^d} \Big(\underbrace{\lim_{t \to +\infty} \chi_{D_{\mathbb{R}^d}(0,\, t \lg t)} \frac{|x|}{t \lg t} |u_0(x)|^2}_{0} \Big) dx = 0$$