15 dec

Lemma $\displaystyle\int_{|x|\geq t\log t} |u|^{p+1}dx \xrightarrow{t\to+\infty} 0$

(we want to move $\displaystyle\int_{\mathbb{R}^d} |u|^{p+1}dx \xrightarrow{t\to+\infty} 0$)

Lemma $\forall\, \varepsilon>0 \quad t>1\,,\ \tau>0$

$\exists\ t_0 > \max\{t, 2\tau\}$ s.t.

$$\int_{t_0-2\tau}^{t_0} ds \int_{|x|\leq s\log s} |u|^{p+1}dx \leq \varepsilon.$$

Pf

$$+\infty > \int_{0}^{+\infty} ds \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r}dx$$

$$\geq \int_2^{+\infty} \frac{ds}{s\log s} \int_{r\leq s\log s} |u|^{p+1}dx$$

$$\geq \sum_{k=0}^{\infty} \int_{t+2k\tau}^{t+2(k+1)\tau} \frac{ds}{s\lg s} \int_{r\leq s\lg s} |u|^{p+1} dx$$

$$\geq \sum_{k=0}^{\infty} \int_{t+2k\tau}^{t+2(k+1)\tau} \frac{ds \int_{r\leq s\lg s} |u|^{p+1} dx}{(t+2(k+1)\tau)\lg(t+2(k+1)\tau)}$$

$$+\infty > \sum_{k=0}^{\infty} \frac{1}{(t+2(k+1)\tau)\lg(t+2(k+1)\tau)}$$

$$\cdot \int_{t+2k\tau}^{t+2(k+1)\tau} ds \int_{r\leq s\lg s} |u|^{p+1} dx$$

$$\liminf_{k\to+\infty} \int_{t+2k\tau}^{t+2(k+1)\tau} ds \int_{r\leq s\lg s} |u|^{p+1} dx = 0$$

so $\forall \; \varepsilon > 0 \quad \exists \; k_0$ arbitrarily large s.t.

$$\int_{t+2k_0\tau}^{t+2(k_0+1)\tau} ds \int_{r\leq s\lg s} |u|^{p+1} dx < \varepsilon$$

$$t_0 = t + 2(k_0+1)\tau$$

$$t + 2k_0\tau = t_0 - 2\tau$$

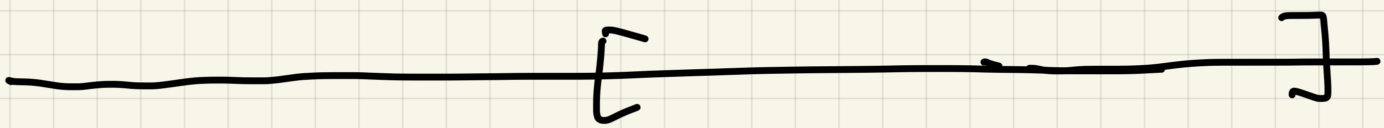
$$p > 1 + \frac{4}{d} \qquad \|e^{it\Delta}\|_{L^1 \to L^\infty} \le \frac{C}{t^{\frac{d}{2}}}$$

$$1 + \frac{2}{d} < p \le 1 + \frac{4}{d}$$

Lemma $\forall \ \varepsilon, a, b \in \mathbb{R}_+ \ \exists$

$t_0 > \max\{a, b\}$ s.t.

$$\sup_{s \in [t_0 - b, t_0]} \|u(s)\|_{L^{p+1}(\mathbb{R}^d)} \le \varepsilon$$



Pf

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}|u|^{p-1}u\,ds$$

$$= e^{it\Delta}u_0 - i\int_0^{t-\tau} e^{i(t-s)\Delta}|u|^{p-1}u\,ds$$

$$- i\int_{t-\tau}^{t} e^{i(t-s)\Delta}|u|^{p-1}u\,ds$$

$$= e^{it\Delta}u_0 + w(t,\tau) + z(t,\tau)$$

Claim $\| e^{it\Delta} u_0 \|_{L^{p+1}} \xrightarrow{t \to +\infty} 0$

Pf $\| e^{it\Delta} u_0 \|_{L^{p+1}} \leq C \, t^{-d\left(\frac{1}{2} - \frac{1}{p+1}\right)} \| u_0 \|_{L^{\frac{p+1}{p}}}$

if $u_0 \in L^{\frac{p+1}{p}} \cap H^1$

is dense in H^1

Claim $\| w(t,\tau) \|_{L^{p+1}} \leq C \, \tau^{-\frac{d(p-1) - 2\max\{1, p-1\}}{2(p+1)}}$

Remark Exponent is negative.

If $p - 1 \leq 1$

$d(p-1) - 2 > 0$ ✓

$d(p-1) > 2$ $\quad p - 1 > \frac{2}{d}$

$p > 1 + \frac{2}{d}$ $\quad$ (we chose $p > 1 + \frac{4}{d}$)

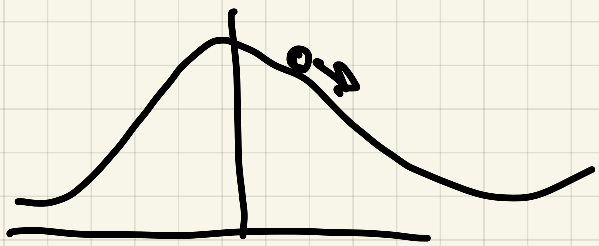
If $p - 1 > 1$

$d(p-1) - 2(p-1) = (d-2)(p-1) > 0$

$\iff d \geq 3$

$$i\partial_t u = (-\Delta + |u|^{p-1})\, u$$

Claim $\quad \|w(t,\tau)\|_{L^{p+1}} \le C\, \tau^{-\frac{d(p-1)-2\max\{1,\,p-1\}}{2(p+1)}}$



Pf $\quad q = \begin{cases} \infty & p \geq 2 \\ \frac{2}{2-p} & p < 2 \end{cases} \qquad q > 2$

$$\|w\|_{L^q} = \left\| \int_0^{t-\tau} e^{i(t-s)\Delta}\, |u|^{p-1} u \, ds \right\|_{L^q}$$

$$\leq \int_0^{t-\tau} \left\| e^{i(t-s)\Delta}\, u^{p-1} u \right\|_{L^q} ds$$

$$\leq \int_0^{t-\tau} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{q}\right)} \, \|u\|^p_{L^{q'p}} \, ds$$

Claim $\quad d\left(\frac{1}{2} - \frac{1}{q}\right) > 1$

If $\quad q = \infty \quad (p \geq 2) \qquad \frac{d}{2} \geq \frac{3}{2} > 1$

If $q < \infty$ $\quad p < 2$

$$d\left(\frac{1}{2} - \frac{1}{q}\right) = d\left(\frac{1}{2} - \frac{2-p}{2}\right) =$$

$$= \frac{d}{2}(p-1) \overset{?}{>} 1$$

$$p - 1 > \frac{2}{d} \qquad p > 1 + \frac{2}{d}$$

$$|w|_{L^q} = \left| \int_0^{t-\tau} e^{i(t-s)\Delta} |u|^{p-1} u \, ds \right|_{L^q}$$

$$\leq \int_0^{t-\tau} \left| e^{i(t-s)\Delta} u^{p-1} u \right|_{L^q} ds$$

$$\leq \int_0^{t-\tau} (t-s)^{-d\left(\frac{1}{2} - \frac{1}{q}\right)} |u|^p_{L^{q'p}} ds$$

$$\leq \int_0^{t-\tau} (t-s)^{-d\left(\frac{1}{2} - \frac{1}{p}\right)} ds \sup_s |u(s)|^p_{L^{q'p}}$$

$$\leq C \, \tau^{-d\left(\frac{1}{2} - \frac{1}{q}\right) + 1} \sup_s |u(s)|^p_{L^{q'p}}$$

$$2 \leq p q' \leq p + 1 \qquad *$$

If $p \geq 2$ $\quad q = \infty \implies q' = 1$

$$2 \leq p q' = p < p + 1$$

If $\quad p < 2$

$$\frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{2-p}{2} = \frac{2-2+p}{2} = \frac{p}{2}$$

$q' p = 2$. $\forall$ is crucial

$2 \leq p q' \leq p + 1$ implies

$$H^1(\mathbb{R}^d) \hookrightarrow L^{pq'}(\mathbb{R}^d)$$

$$\Rightarrow \quad \sup_s |u(s)|^p_{L^{pq'}} \leq C(\|u_0\|_{L^2}, E(u_0))$$

$$|w(t,\tau)|_{L^q} \leq C \quad \tau^{-d\left(\frac{1}{2} - \frac{1}{q}\right) + 1}$$

$$|w(t,\tau)|_{L^2} \leq 2 \, |u_0|_{L^2}$$

$$w(t,\tau) = -i \int_0^{t-\tau} e^{i(t-s)\Delta} |u|^{p-1} u \, ds$$

$$= e^{i\tau\Delta} \left(-i \int_0^{t-\tau} e^{i(t-\tau-s)\Delta} |u|^{p-1} u \, ds \right)$$

$$= e^{i\tau\Delta}\left(u(t-\tau) - e^{i(t-\tau)\Delta} u_0 \right)$$

$$w(t,\tau) = e^{i\tau\Delta} u(t-\tau) - e^{it\Delta} u_0$$

$$|w(t,\tau)|_{L^2} \le |\cancel{e^{i\tau\Delta}} u(t-\tau)|_{L^2} + |\cancel{e^{it\Delta}} u_0|_{L^2}$$

$$= 2|u_0|_{L^2}$$

$$|w(t,\tau)|_{L^q} \le C\, \tau^{-d\left(\frac{1}{2}-\frac{1}{q}\right)+1}$$

$$|w(t,\tau)|_{L^2} \le 2|u_0|_{L^2}$$

We claim that $2 < p+1 \le q$

If $q = \infty$ $\qquad p \ge 2$ this is true

If $q < \infty$ $\qquad q = \dfrac{2}{2-p}$

$\underline{q > p+1} \iff \dfrac{2}{2-p} > p+1$

$$\iff \cancel{2} > (p+1)(2-p) = -p^2 + p + \cancel{2}$$

$p(1-p) < 0$ ✓

$$\frac{1}{p+1} = \frac{1-\alpha}{2} + \frac{\alpha}{q} \qquad \alpha \in (0,1)$$

$$|w(t,\tau)|_{L^{p+1}} \leq |w|_{L^2}^{1-\alpha} \, |w|_{L^q}^{\alpha}$$

$$\alpha = \frac{\frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{q}}$$

$$|w(t,\tau)|_{L^{p+1}} \leq C \quad \tau^{\left(-d\left(\frac{1}{2}-\frac{1}{q}\right)+1\right)\frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}}$$

$$\left(-d\left(\frac{1}{2}-\frac{1}{q}\right)+1\right)\frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}$$

$$= -d\left(\frac{1}{2}-\frac{1}{p+1}\right) + \frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}$$

If $\quad q = \infty \qquad p \geq 2$

$$= -d\left(\frac{1}{2}-\frac{1}{p+1}\right) + 2\left(\frac{1}{2}-\frac{1}{p+1}\right) =$$

$$= -(d-2) \quad \frac{p-1}{2(p+1)}$$

$$= - \quad \frac{d(p-1) - 2(p-1)}{2(p+1)}$$

$$= - \frac{d(p-1) - 2\max\{1, p-1\}}{2(p+1)}$$

$q = \frac{2}{2-p}$ $\qquad p < 2$

$$-d\left(\frac{1}{2} - \frac{1}{p+1}\right) + \frac{\frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{q}} =$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(-d + \frac{2}{p-1}\right)$$

$$= \frac{\cancel{p-1}}{2(p+1)} \quad \frac{-d(p-1)+2}{\cancel{p-1}}$$

$$= - \frac{d(p-1) - 2\max\{1, p-1\}}{2(p+1)}$$

$$|w(t,\tau)|_{L^{p+1}} \leq C\,\tau^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}}$$

$$z(t,\tau) = -i \int_{t-\tau}^{t} e^{i(t-\tau)\Delta} |u|^{p-1} u \, ds$$

$$|z(t,\tau)|_{L^{p+1}} \leq \int_{t-\tau}^{\tau} \left| e^{i(t-\tau)\Delta} |u|^{p} \right|_{L^{p+1}} ds$$

$$\leq C \int_{t-\tau}^{t} (t-\tau)^{-\underbrace{d\left(\frac{1}{2}-\frac{1}{p+1}\right)}} \|u\|_{L^{p+1}}^{p} ds$$

$$p < d^* \implies d\left(\frac{1}{2}-\frac{1}{p+1}\right) < 1$$

$$d^* = \frac{d+2}{d-2}$$

$$H^1 \hookrightarrow L^{d^*+1}$$

$$p+1 < d^*+1 = \frac{d+2}{d-2} + 1 = \frac{2d}{d-2}$$

$$p+1 < \frac{2d}{d-2}$$

$$\frac{1}{p+1} > \frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$$

$$\frac{1}{d} > \frac{1}{2} - \frac{1}{p+1}$$

$$1 > d\left(\frac{1}{2} - \frac{1}{p+1}\right)$$

$$q \in \left(1, \frac{2(p+1)}{d(p-1)}\right)$$

$$q\, d\left(\frac{1}{2} - \frac{1}{p+1}\right) < 1$$

$$q\, d\, \frac{p-1}{2(p+1)} < 1$$

$$q < \frac{2(p+1)}{d(p-1)} \qquad \frac{1}{q} + \frac{1}{q'} = 1$$

$$|z(t,\tau)|_{L^{p+1}} \leq C \int_{t-\tau}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} |u(s)|_{L^{p+1}}^{p} ds$$

$$\leq C \left(\int_{t-\tau}^{t} (t-s)^{-q d\left(\frac{1}{2}-\frac{1}{p+1}\right)} ds \right)^{\frac{1}{q}}$$

$$\left(\int_{t-\tau}^{t} |u|_{L^{p+1}}^{pq'} ds \right)^{\frac{1}{q'}}$$

$$= C \quad \tau^{\alpha} \left(\int_{t-\tau}^{t} |u|_{L^{p+1}}^{pq'} ds \right)^{\frac{1}{q'}}$$

$\alpha > 0$

$$|z(t,\tau)|_{L^{p+1}} \leq C \quad \tau^{\alpha} \left(\int_{t-\tau}^{t} |u|_{L^{p+1}}^{pq'} ds \right)^{\frac{1}{q'}}$$