21 Dicembre

lunedì 8 9-11 salta

$y'' + 2y' + 3y = \sin x$ $\qquad y_g = ?$

$L[y] = y'' + 2y' + 3y = 0 \qquad y = e^{rx}$

$L[e^{rx}] = (r^2 + 2r + 3)\, e^{rx} = 0$

$r^2 + 2r + 3 = 0$

$r_\pm = -1 \pm \sqrt{1-3} = -1 \pm \sqrt{-2} = -1 \pm i\sqrt{2}$

$y_h = A\, e^{-x} \sin(\sqrt{2}\, x) + B\, e^{-x} \cos(\sqrt{2}\, x)$

Cerchiamo ora una soluzione particolare y_p

$L[y_p] = \sin(x)$

La cerchiamo della forma $y_p = \alpha \cos(x) + \beta \sin(x)$

$L[y_p] = \sin(x) = \frac{1}{2i} e^{ix} - \frac{1}{2i} e^{-ix} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$y_p = y_{p+} + y_{p-}$

$\boxed{L[y_{p+}] = \frac{1}{2i} e^{ix}} \qquad y_{p+} = \lambda\, e^{ix}$

$\boxed{L[y_{p-}] = -\frac{1}{2i} e^{-ix}} \qquad y_{p-} = \mu\, e^{-ix}$

$L[y_{p+}] = L[\lambda e^{ix}] = \lambda L[e^{ix}] = \lambda\, p(i)\, e^{ix} \boxed{=} \frac{1}{2i} e^{ix}$

$\lambda = \frac{1}{2i\,p(i)}$

$L[y_{p+} + y_{p-}] = L[y_{p+}] + L[y_{p-}] = \frac{1}{2i} e^{ix} - \frac{1}{2i} e^{-ix}$
$= \sin(x)$

$y_p = y_{p+} + y_{p-}$

$y_p = \alpha \cos x + \beta \sin x$

$L[y] = y'' + 2y' + 3y$

$L[y_p] = \alpha L[\cos x] + \beta L[\sin x] =$

$= \alpha\,(\cos'' x + 2\cos' x + 3\cos x) + \beta\,(\sin''(x) + 2\sin' x + 3\sin x)$

$= \alpha\,(-\cos x - 2\sin x + 3\cos x) + \beta\,(-\sin x + 2\cos x + 3\sin x)$

$= \alpha\,(2\cos x - 2\sin x) + \beta\,(2\cos x + 2\sin x)$

$= \boxed{2(\alpha+\beta)\cos x + 2(-\alpha+\beta)\sin x = \sin x} + 0\cos x$

$\begin{cases} 2(\alpha+\beta) = 0 \\ 2(-\alpha+\beta) = 1 \end{cases} \qquad \begin{cases} \alpha = -\beta \\ 2(\beta+\beta) = 1 \end{cases}$

$L[y_p] = \sin x \qquad \beta = \frac{1}{4} \quad \alpha = -\frac{1}{4}$

$y_p = -\frac{1}{4}\cos x + \frac{1}{4}\sin x$

$L[y] = y'' + y = \sin(2x)$ $\qquad y^{(j)} \rightsquigarrow r^j$

$L[y] = 0 \qquad r^2 + 1 = 0 \qquad r = \pm i$

$$y_h = A\cos x + B\sin x$$

$L[y_p] \boxed{=} \sin(2x) \qquad y_p = \alpha\cos(2x) + \beta\sin(2x)$

$$L[\alpha\cos(2x) + \beta\sin(2x)] = \alpha L[\cos(2x)] + \beta L[\sin(2x)] =$$

$$= \alpha\left((\cos(2x))'' + \cos(2x)\right) + \beta\left((\sin 2x)'' + \sin(2x)\right) =$$

$$= \alpha(-4\cos(2x) + \cos(2x)) + \beta(-4\sin(2x) + \sin(2x))$$

$$= -3\alpha\cos(2x) - 3\beta\sin(2x) \boxed{=} \sin 2x$$

$\alpha = 0 \qquad -3\beta = 1 \qquad \beta = -\frac{1}{3}$

$$y_g = A\cos(x) + B\sin(x) - \frac{1}{3}\sin(2x)$$

$L[y] = y'' + y^{(0)} = \sin(x)$ $\qquad y^{(j)} \longrightarrow r^j$

$r^2 + 1 = 0 \qquad r = \pm i$

$y_h = A\cos(x) + B\sin(x)$

$L[y_p] = \sin(x)$

$\cancel{y_p = \alpha\cos(x) + \beta\sin(x)}$

$y_p = x\,(\alpha\cos x + \beta\sin x)$

$L[y] = y'' + y$

$L[y_p] = L[x(\alpha\cos x + \beta\sin x)] =$

$= x\,L[\alpha\cos x + \beta\sin x]$

$+ 2(x)'(\alpha\cos x + \beta\sin x)'$

$= x\,\underbrace{L[\alpha\cos x + \beta\sin x]}_{0} + 2(-\alpha\sin x + \beta\cos x)$

$= \sin x$

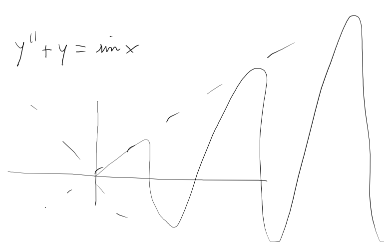
$-2\alpha\sin x + 2\beta\cos x = \sin x$

$\begin{cases} -2\alpha = 1 \\ 2\beta = 0 \end{cases} \qquad \begin{aligned} \alpha &= -\tfrac{1}{2} \\ \beta &= 0 \end{aligned}$

$y_p = -\frac{1}{2}\,x\cos x$

$y_g = A\cos x + B\sin x - \frac{1}{2}\,x\cos x$

$y'' + y = \sin x$



$$y'' + by' + y = \sin(x) \qquad b > 0$$

$$y'' = -by' - y + \sin(x)$$

$$y'' + by' + y = 0$$

$$r^2 + br + 1 = 0$$

$$r_\pm = -\frac{b}{2} \pm \frac{\sqrt{b^2-4}}{2} = -\frac{b}{2} \pm i\frac{\sqrt{4-b^2}}{2} =$$

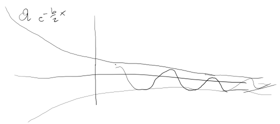
$$= -\frac{b}{2} \pm i\sqrt{1-\frac{b^2}{4}}$$

$$Y_h = A e^{-\frac{b}{2}x}\cos\left(\sqrt{1-\frac{b^2}{4}}\,x\right) + B e^{-\frac{b}{2}x}\sin\left(\sqrt{1-\frac{b^2}{4}}\,x\right)$$

$$Y_h = \sqrt{A^2+B^2}\, e^{-\frac{b}{2}x}\left(\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\cos(\varphi)}\cos\left(\sqrt{1-\frac{b^2}{4}}\,x\right) + \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\sin(\varphi)}\sin\left(\sqrt{1-\frac{b^2}{4}}\,x\right)\right)$$

$$Y_h = \sqrt{A^2+B^2}\, e^{-\frac{b}{2}x}\cos\left(\sqrt{1-\frac{b^2}{4}}\,x - \varphi\right)$$

$$Y_h = a\, e^{-\frac{b}{2}x}\cos\left(\sqrt{1-\frac{b^2}{4}}\,x + \varphi\right)$$



$$\underbrace{y''+by'+y}_{L[y]} = \sin x$$

$$Y_p = \alpha\cos x + \beta\sin x$$

$$L[Y_p] = \alpha L[\cos x] + \beta L[\sin x] =$$

$$= \alpha\left(\underbrace{\cos'' x + b\cos' x + \cos x}_{=0}\right) + \beta\left(\underbrace{\sin'' x + b\sin' x + \sin x}_{=0}\right)$$

$$= \alpha b\cos' x + \beta b\sin' x = -\alpha b\sin x + \beta b\cos x$$

$$= \sin x$$

$$-\alpha b = 1 \qquad \beta b = 0$$

$$\alpha = -\frac{1}{b} \qquad \beta = 0$$

$$Y_p = -\frac{1}{b}\cos x$$

$$Y = a\, e^{-\frac{b}{2}x}\cos\left(\sqrt{1-\frac{b^2}{4}}\,x + \varphi\right) - \frac{1}{b}\cos x$$

$$\underbrace{y'' - 3y' + y}_{L[y]} = 1\, e^{0x}$$

$$y'' - 3y' + y = 0 \qquad r^2 - 3r + 1 = 0$$

$$r_\pm = \frac{3}{2} \pm \frac{\sqrt{9-4}}{2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

$$y_h = A\, e^{\frac{3+\sqrt{5}}{2}x} + B\, e^{\frac{3-\sqrt{5}}{2}x}$$

$$y_p = ? \qquad y_p = \alpha$$

$$L[y_p] = L[\alpha] = \alpha\, L[1] =$$

$$= \alpha\left((1)'' - 3\,(1)' + 1\right) = \alpha = 1 \qquad \boxed{\alpha = 1}$$

$$y_g = A\, e^{\frac{3+\sqrt{5}}{2}x} + B\, e^{\frac{3-\sqrt{5}}{2}x} + 1$$

$$\lim_{x \to +\infty} \left(1 + x - \sqrt{x^2+1}\right)^x$$

$$\left(1 + x - \sqrt{x^2+1}\right)^x = e^{x \lg\left(1+x-\sqrt{x^2+1}\right)}$$

$$x \lg\left(1+x-\sqrt{x^2+1}\right) =$$

$$= x \lg\left[\left(1+x-\sqrt{x^2+1}\right) \frac{1+x+\sqrt{x^2+1}}{1+x+\sqrt{x^2+1}}\right]$$

$$= x \lg\left(\frac{(1+x)^2-(x^2+1)}{1+x+\sqrt{x^2+1}}\right) =$$

$$= x \lg\left(\frac{2x}{1+x+\sqrt{x^2+1}}\right) \xrightarrow{x \to +\infty} +\infty \ \underbrace{\lg(1)}_{0}$$

$$x \lg\left(1+x-\sqrt{x^2+1}\right) =$$

$$= x \lg\left(1+x-x\left(1+x^{-2}\right)^{\frac{1}{2}}\right) =$$

$$(1+y)^{\frac{1}{2}} = 1 + \frac{1}{2} y + o(y) \quad \text{per } y \text{ piccolo}$$

$$(1+x^{-2})^{\frac{1}{2}} = 1 + \frac{1}{2} x^{-2} + o(x^{-2})$$

$$= x \lg\left(1+x-x\left(1+\frac{1}{2}x^{-2}+o(x^{-2})\right)\right) =$$

$$= x \lg\left(1+\cancel{x}-\left(\cancel{x}+\frac{1}{2}x^{-1}+o(x^{-1})\right)\right)$$

$$= x \lg\left(1-\frac{1}{2}x^{-1}+o(x^{-1})\right) =$$

$$\lg(1+y) = y\,(1+o(1))$$

$$= x\left(-\frac{1}{2}x^{-1}+o(x^{-1})\right)\cancel{(1+o(1))}$$

$$= -\frac{1}{2} + o(1) \xrightarrow{x \to +\infty} \boxed{-\frac{1}{2}}$$