19 dicembre

Esercizio, continuazione $\quad g(x) = \sin(x + x^3)$

polinomio di McLaurin $\quad q_5(x)$

$$q_5(x) \equiv \sum_{j=0}^{5} \frac{g^{(j)}(0)}{j!} x^j \qquad \text{per definizione}$$

$$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} + o(y^5)$$

$$\sin(x+x^3) = (x+x^3) - \frac{(x+x^3)^3}{3!} + \frac{(x+x^3)^5}{5!} + \underbrace{o\left((x+x^3)^5\right)}_{o(x^5)}$$

$$o\left((x+x^3)^5\right) = o\left(x^5 \cancel{(1+x^2)^5}\right) = o(x^5)$$

$$\lim_{x\to 0} \frac{o\left(x^5(1+x^2)^5\right)}{x^5(1+x^2)^5} \quad (1+x^2)^5 \quad = 0 \cdot 1 = 0$$

$$\sin(x+x^3) = x + x^3 - \frac{(x+x^3)^3}{3!} + \frac{(x+x^3)^5}{5!} + o(x^5)$$

$$= x + x^3 - \frac{x^3 + 3x^5}{3!} + \frac{x^5}{5!} + o(x^5)$$

$$\int_0^x \sin(t+t^3)\,dt \qquad e^{-x^2}$$

$$e^{-x^3}$$

$L[y] = e^x$

$L[y] = y'' + by' + cy \qquad b, c \in \mathbb{R}$

Premesse: Formule di Euler. Per $z \in \mathbb{C}$

$$e^z = e^{x+iy} = e^x e^{iy} = \qquad z = x+iy$$

$$\boxed{e^{x+iy} = e^x (\cos y + i \sin y)}$$

Qui si è definito
Notare che
$$\begin{cases} e^{iy} = \cos(y) + i\sin(y) & \forall y \in \mathbb{R} \\ e^{-iy} = \cos(y) - i\sin(y) \end{cases}$$

Da questo, risolvendo rispetto a $\cos(y)$ e $\sin y$, si ricava

$$\begin{cases} \cos(y) = \dfrac{e^{iy} + e^{-iy}}{2} \\ \sin(y) = \dfrac{e^{iy} - e^{-iy}}{2i} \end{cases}$$

Un altro modo alternativo di definire l'esponenziale è di usare la serie

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

(1) $L[y] = e^x$ $\qquad b, c \in \mathbb{R}$

$$L[y] = y'' + by' + cy$$

(i) (2) $L[y] = 0$ $\qquad y_h = ?$

(ii) Si cerca una soluzione particolare $y_p$ di (1)

Con queste, la soluzione generale di (1) sarà

$$y_g = y_p + y_h$$

$L[y] = y'' + y' + y$

$$\begin{cases} L[y] = e^x \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

Per prima cosa cerchiamo $y_g$

$\underbrace{y''+y'+y}_{L[y]} = 0$   $y_h = ?$

Cerchiamo soluzioni dell' ~~omogenea~~ della forma

$y = e^{rx}$

$L[e^{rx}] = (\underbrace{r^2 + r + 1}_{p(r)})\, e^{rx} = 0$

$r^2 + r + 1 = 0$

$r_\pm = -\frac{1}{2} \pm \frac{\sqrt{1-4}}{2} = -\frac{1 \pm i\sqrt{3}}{2}$

$y_h = c_1\, e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} + c_2\, e^{\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x} =$

$= c_1\, e^{-\frac{x}{2} + i\frac{\sqrt{3}}{2}x} + c_2\, e^{-\frac{x}{2} - i\frac{\sqrt{3}}{2}x}$

$= e^{-\frac{x}{2}}\left(c_1 e^{i\frac{\sqrt{3}}{2}x} + c_2 e^{-i\frac{\sqrt{3}}{2}x}\right)$

Se scelgo $c_1 = c_2 = \frac{1}{2}$ ottengo la funzione

$e^{-\frac{x}{2}} \; \frac{e^{i\frac{\sqrt{3}}{2}x} + e^{-i\frac{\sqrt{3}}{2}x}}{2} = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$

se scelgo $c_1 = \frac{1}{2i}$ e $c_2 = -\frac{1}{2i}$ ottengo la funzione

$e^{-\frac{x}{2}} \; \frac{e^{i\frac{\sqrt{3}}{2}x} - e^{-i\frac{\sqrt{3}}{2}x}}{2i} = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$

$y_h = A\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) + B\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$

Ora cerchiamo $y_p$   $L[y_p] = e^x$

$y_p = \alpha e^x$

$L[y_p] = L[\alpha e^x] = \alpha\, L[e^x] =$

$= \alpha\, p(1)\, e^x = e^x$

$\alpha\, p(1)\, e^x = \alpha \cdot 3\, e^x = e^x$

$\alpha \cdot 3 = 1$   $\alpha = \frac{1}{3}$

$y_p = \frac{1}{3} e^x$

$y = \frac{1}{3}e^x + A\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) + B\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$

Trovare $y$ tale che $y(0) = 1$, $y'(0) = 2$

$y(0) = \frac{1}{3} + A = 1$   $\boxed{A = \frac{2}{3}}$

$y'(x) = \frac{1}{3}e^x - \frac{A}{2} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{\sqrt{3}}{2} A\, e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$

$- \frac{B}{2} e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2} B\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$

$y'(0) = \underbrace{\frac{1}{3} - \frac{A}{2}}_{0} + \frac{\sqrt{3}}{2} B = 2$   $\boxed{B = \frac{4}{\sqrt{3}}}$

$L[y] = e^x$

$p(r) = (r-1)(r-2) =$
$= r^2 - 3r + 2$

$L[y] = y'' - 3y' + 2y$

$L[y] = 0 \qquad y_h\,?$

$L[e^{rx}] = p(r)\, e^{rx} = 0 \qquad p(r) = 0 \qquad r = 1, 2$

$y_h = A e^x + B e^{2x}$

Cerchiamo $y_p$ t.c. $L[y_p] = e^x$

$\cancel{y_p = c e^x}$

$L[c e^x] = e^x$

$c \underbrace{L[e^x]}_{0} = e^x$

$y_p = C x e^x$

$L[C x e^x] = e^x$

$C\left( (x e^x)'' - 3(x e^x)' + 2 x e^x \right) = C x \cancel{\underbrace{\left( (e^x)'' - 3 (e^x)' + 2 e^x \right)}_{L[e^x] = 0}}$

$+ C\, 2\, (x)'\, e^x - C\, 3\, (x)' e^x$

$= C(2-3)\, e^x = -C e^x = e^x \qquad \boxed{C = -1}$

$L[x e^{rx}] = x\, p(r)\, e^{rx} + \underbrace{(2r-3)}_{p'(r)}\, e^{rx}$

$p(r) = r^2 - 3r + 2$

$p'(r) = 2r - 3$

$L[y] = e^x$ $\qquad$ $p(r) = (r-1)^2$

$L[y] = y'' - 2y' + y$ $\qquad$ $= r^2 - 2r + 1$

$L[y] = 0 \qquad y_h = ?$

$L[e^{rx}] = p(r)\, e^{rx} = (r-1)^2\, e^{rx} = 0 \qquad \boxed{r=1}$

$L[xe^{rx}] = x\, \underbrace{p(r)}^{(r-1)^2}\, \overbrace{e^{rx}}^{e^x} + \overbrace{p'(r)}^{2(r-1)}\, e^{rx} = 0$

$\to x(r-1)^2 e^{rx} + 2(r-1)e^{rx} = 0$

per $r=1$ ho una soluzione $x e^x$

$y_h = A e^x + B x e^x$

$y_p = ? \qquad y_p = C\, x^2 e^x$

$L[x^2 e^{rx}] = (x^2 e^{rx})'' - 2(x^2 e^{rx})' + x^2 e^{rx}$

$= x^2 \underbrace{(r^2 - 2r + 1)}_{p(r)} e^{rx} +$

$+ \left(2 (x^2)' r - 2 (x^2)'\right) e^{rx} + 2 e^{rx}$

$L[x^2 e^{rx}] = x^2 p(r) e^{rx} + 2x \underbrace{(2r-2)}_{p'(r)} e^{rx} + 2 e^{rx}$

$L[C x^2 e^{rx}] = C\, x^2 p(r) e^{rx} + C\, 2x\, p'(r) e^{rx} + 2C e^{rx}$

$= e^x$

$r = 1$

$\to C x^2 \underbrace{p(1)}_{0} + C\, 2x \underbrace{p'(1)}_{0} + 2C = 1$

$C = \frac{1}{2}$

$y_p = \frac{1}{2} x^2 e^x$