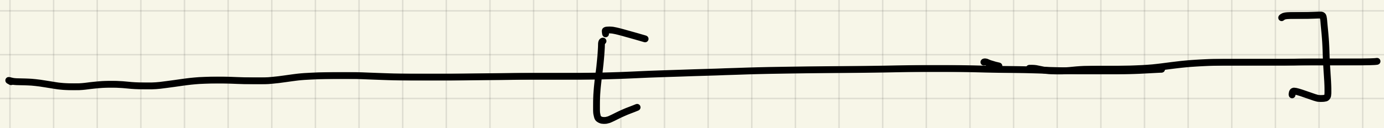
20 dic.

Lemma $\forall \ \varepsilon, a, b \in \mathbb{R}_+ \ \exists$

$t_0 > \max\{a, b\}$ s.t.

$$\sup_{s \in [t_0 - b, t_0]} |u(s)|_{L^{p+1}(\mathbb{R}^d)} \leq \varepsilon$$



Pf

$$u(t) = e^{it\Delta} u_0 - i\int_0^t e^{i(t-s)\Delta} |u|^{p-1} u \, ds$$

$$= e^{it\Delta} u_0 - i\int_0^{t-\tau} e^{i(t-s)\Delta} |u|^{p-1} u \, ds$$

$$- i \int_{t-\tau}^{t} e^{i(t-s)\Delta} |u|^{p-1} u \, ds$$

$$= e^{it\Delta} u_0 + w(t,\tau) + z(t,\tau)$$

Claim $\| e^{it\Delta} u_0 \|_{L^{p+1}} \xrightarrow{t \to +\infty} 0$

. .

Claim $\| w(t,\tau) \|_{L^{p+1}} \leq C \tau^{-\frac{d(p-1) - 2\max\{1, p-1\}}{2(p+1)}}$

$$1 < q < \frac{2(p+1)}{d(p-1)}$$

$\alpha > 0$

$$\| z(t,\tau) \|_{L^{p+1}} \leq C \tau^{\alpha} \left(\int_{t-\tau}^{t} \| u \|_{L^{p+1}}^{pq'} \, ds \right)^{\frac{1}{q'}}$$

$$p q' > p+1 \iff \frac{1}{q'} < \frac{p}{p+1}$$

$$1 - \frac{1}{q} < \frac{p}{p+1} \qquad *$$

$$q < \frac{2(p+1)}{d(p-1)}$$

$$\frac{1}{q} > \frac{d(p-1)}{2(p+1)} \textcircled{=} \frac{d}{2} - \frac{d}{p+1} = d \, \frac{p+1-2}{2(p+1)}$$

$$\frac{1}{q} > \frac{d}{2} - \frac{d}{p+1}$$

$$\frac{1}{q'} = 1 - \frac{1}{q} < 1 - \frac{d}{2} + \frac{d}{p+1} = \frac{2-d}{2} + \frac{d}{p+1}$$

$$= \frac{(2-d)(p+1) + 2d}{2(p+1)} =$$

$$= \frac{2(p+1) - d(p+1) + 2d}{2(p+1)} =$$

$$= \frac{p}{p+1} + \frac{2 - d(p+1) + 2d}{2(p+1)}$$

$$\frac{1}{q'} < \frac{p}{p+1} + \frac{\overbrace{2 - d(p+1) + 2d}^{<0}}{2(p+1)}$$

$$\frac{1}{q'} < \frac{p}{p+1}$$

$$2 + 2d - d(p+1) < 0$$

$$\frac{2+2d}{d} < p+1$$

$$p > 1 + \frac{4}{d}$$

$$\frac{2}{d} + 2 < p + 1 \iff p > 1 + \frac{2}{d}$$

$$pq' > p+1$$

$$\alpha > 0$$

$$|z(t,\tau)|_{L^{p+1}} \leq C \quad \tau^{\alpha} \left(\int_{t-\tau}^{t} |u|_{L^{p+1}}^{pq'} \, ds \right)^{\frac{1}{q'}}$$

$$|u|_{L^{p+1}}^{pq'} = |u|_{L^{p+1}}^{p+1} \quad |u|_{L^{p+1}}^{pq'-p-1}$$

$$2 < p+1 < d^{*}+1 \qquad H^{1}(\mathbb{R}^{d}) \hookrightarrow L^{d^{*}+1}(\mathbb{R}^{d})$$

$$\frac{1}{p+1} = \frac{\beta}{2} + \frac{(1-\beta)}{d^{*}+1} \qquad \beta \in (0,1)$$

$$|u|_{L^{p+1}} \leq |u|_{L^{2}}^{\beta} \; |u|_{L^{d^{*}+1}}^{1-\beta}$$

$$\lesssim |u|_{L^{2}}^{\beta} \; |u|_{H^{1}}^{1-\beta}$$

$$\leq \qquad C(u_0)$$

$$|u|_{L^{p+1}}^{pq'} \leq |u|_{L^{p+1}}^{p+1} \; C$$

$$|z(t,\tau)|_{L^{p+1}} \leq C \quad \tau^{\alpha} \left(\int_{t-\tau}^{t} |u|^{p+1}_{L^{p+1}} \, ds \right)^{\frac{1}{q'}}$$

$$\leq C \tau^{\alpha} \left(\int_{t-\tau}^{t} ds \int_{|x| \geq s \lg s} |u|^{p+1} \, dx \right)^{\frac{1}{q'}} +$$

$$+ \tau^{\alpha} \left(\int_{t-\tau}^{t} ds \int_{|x| \leq s \lg s} |u|^{p+1} \, dx \right)^{\frac{1}{q'}}$$

$$|z(t,\tau)|_{L^{p+1}} \leq C \quad \tau^{\alpha + \frac{1}{q'}} \left(\sup_{t-\tau \leq s \leq t} |u|^{p+1}_{L^{p+1}(|x| \geq s \lg s)} \right)^{\frac{1}{q'}}$$

$$+ C \tau^{\alpha} \left(\int_{t-\tau}^{t} ds \int_{|x| \leq s \lg s} |u|^{p+1} \, dx \right)^{\frac{1}{q'}}$$

$$|w(t,\tau)|_{L^{p+1}} \leq C \quad \tau^{\frac{-d(p-1) + 2\max(1, p-1)}{2(p+1)}} < \frac{\varepsilon}{4}$$

For $\exists \quad t_1 > \max\{a, b\}$

s.t. for $t \geq t_1$

$$|e^{tA}u_0|_{L^{p+1}} + C\,\tau^{d+\frac{1}{q'}}\left(\sup_{t-\tau\le s\le t} |u|^{p+1}_{L^{p+1}(|x|\le s\lg s)}\right)^{\frac{1}{q'}}$$

$$< \frac{\varepsilon}{4}$$

We have already proved that

$$\boxed{\begin{array}{l} \forall\, \varepsilon>0\quad t>1 \quad \text{an } \tau>0 \quad \exists \\ t_0 > \max\{t, 2\tau) \quad \text{s.t.} \\ \displaystyle\int_{t_0-2\tau}^{t_0} \int_{|x|\le s\lg s} |u|^{p+1}\,dx\,ds \le \varepsilon \end{array}}$$

so $\exists\ t_2 > t_1 + 2\tau$ s.t. for

$t \in [t_2-\tau, t_2 \cancel{-\tau})$

$$C\,\tau^{d}\left(\int_{t-\tau}^{t} ds \int_{|x|\le s\lg s} |u|^{p+1}\,dx\right)^{\frac{1}{q'}}$$

$$\le C\,\tau^{d}\left(\int_{t_2-2\tau}^{t_2} \int_{|x|\le s\lg s} |u|^{p+1}\,dx\right)^{\frac{1}{q'}} < \frac{\varepsilon}{4}$$

$$|u|_{L^{p+1}_{[t_2-\tau,\, t_2]}} \le |e^{it\Delta}u_0|_{L^{p+1}} + |w|_{L^{p+1}}$$

$$+ |z|_{L^{p+1}}$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} +$$

$$\underbrace{+ C\tau^{\alpha}\left(\int_{t-\tau}^{t} ds \int_{|x|\le s \lg s} |u|^{p+1} dx\right)^{\frac{1}{q'}}}_{< \frac{\varepsilon}{4}}$$

$$< \varepsilon \qquad \forall\ t \in [t_2-\tau, t_2]$$

$$t_0 = t_2 > \max\{a,b\},\ \tau \ge b$$

<u>Lemma</u> $\forall\ \varepsilon, a, b \in \mathbb{R}_+ \quad \exists$

$t_0 > \max\{a, b\}$, t

$$\sup_{s\in[t_0-b,\, t_0]} |u(s)|_{L^{p+1}(\mathbb{R}^d)} \le \varepsilon$$

Finally we show $\lim_{t\to+\infty} |u(t)|_{L^{p+1}} = 0$

$\varepsilon > 0 \qquad t \qquad \tau_\varepsilon$

$$|u(t)|_{L^{p+1}} \leq |e^{i\Delta t} u_0|_{L^{p+1}}$$

$$+ C \tau_\varepsilon^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}}$$

$$+ |z(t,\tau_\varepsilon)|_{L^{p+1}}$$

$$|e^{it\Delta} u_0|_{L^{p+1}} < \frac{\varepsilon}{4} \qquad t > t_\varepsilon$$

$$C \tau_\varepsilon^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}} = \frac{\varepsilon}{4}$$

$$|z(t,\tau_\varepsilon)|_{L^{p+1}} = \left| \int_{t-\tau_\varepsilon}^{t} e^{i(t-s)\Delta} |u|^{p-1} u \, ds \right|_{L^{p+1}}$$

$$\leq \int_{t-\tau_\varepsilon}^{t} | e^{i(t-s)\Delta} |u|^{p-1} u |_{L^{p+1}} \, ds$$

$$\lesssim \int_{t-\tau_\varepsilon}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} |u(s)|_{L^{p+1}}^{p} \, ds$$

$$\leq \tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}} \sup_{s\in[t-\tau_\varepsilon,t]} |u(s)|^p_{L^{p+1}}$$

We choose $t_0 > \max\{t_1, \tau_\varepsilon\}$

so that

$$\sup_{s\in[t_0-\tau_\varepsilon,t_0]} |u(s)|^p_{L^{p+1}} \leq \varepsilon^p$$

$$t_\varepsilon = \sup\{t \geq \underline{t_0} : |u(s)|_{L^{p+1}} \leq \varepsilon \ \forall\ s \in [t-\tau_\varepsilon, t]\}$$

$t_\varepsilon = +\infty$ $\quad t_\varepsilon^0$

Let $\quad t_\varepsilon < \infty$

$$\Rightarrow \quad |u(t_\varepsilon)|_{L^{p+1}} = \varepsilon$$

$$\sup_{s\in[t-\tau_\varepsilon,t]} |u(s)|_{L^{p+1}}$$

$$\sup_{s\in[t_\varepsilon-\tau_\varepsilon,t_\varepsilon]} |u(s)|_{L^{p+1}} = \varepsilon$$

$$\varepsilon = \|u(t_\varepsilon)\|_{L^{p+1}} \leq \|e^{i\Delta t_\varepsilon} u_0\|_{L^{p+1}}$$

$$+ C\,\tau_\varepsilon^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}}$$

$$+ \|z(t_\varepsilon, \tau_\varepsilon)\|_{L^{p+1}}$$

$$< \frac{\varepsilon}{2} + C\,\tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}} \sup_{s\in[t_\varepsilon-\tau_\varepsilon, t_\varepsilon]} \|u(s)\|_{L^{p+1}}^{p}$$

$$\varepsilon < \left(\frac{1}{2} + C\,\tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}} \varepsilon^{p-1}\right) \varepsilon$$

If

$$C\,\tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}} \varepsilon^{p-1} < \frac{1}{2}$$ I get a

contradiction

$$\frac{1}{2C} \leq \tau_\varepsilon^{1-\frac{d(p-1)}{2(p+1)}} \varepsilon^{p-1}$$

$$2C\,\tau_\varepsilon^{-\frac{d(p-1)-2\max\{1,p-1\}}{2(p+1)}} = \varepsilon$$

$$\frac{1}{2C} \le C_1 \tau_\varepsilon^{1 - \frac{d(p-1)}{2(p+1)} - \frac{d(p-1)^2 - 2(p-1)\max\{1,p-1\}}{2(p+1)}}$$

$$2(p+1) - d(p-1) - d(p-1)^2 + 2(p-1)\max\{1,p-1\}$$

$$= (p-1)\left[2\max\{1,p-1\} - d - d(p-1)\right] + 2(p+1)$$

$$= (p-1)\left[2\max\{1,p-1\} + 2 - d(p-1)\right] - d(p-1) \underbrace{- 2(p-1) + 2(p+1)}_{4}$$

$\boxed{p-1 \le 1}$

$$(p-1)\left[2 + 2 - d(p-1)\right] + 4 - d(p-1)$$

$$= (p-1)\left(4 - d(p-1)\right) + 4 - d(p-1)$$

$$= p\left(4 - d(p-1)\right) < 0$$

$$4 \le d(p-1) \qquad p > 1 + \frac{4}{d}$$

$p-1 > 1$

$$(p-1)\left[2\overset{p-1}{\cancel{\max\{1,p-1\}}} + 2 - d(p-1)\right] - d(p-1) + 4$$

$$(p-1)\left[2(p-1)+2-d(p-1)\right]-d(p-1)+4$$

$$=(p-1)\left[2-(d-2)(p-1)\right]-d(p-1)+4 \leq$$

$d \geq 4$

$$\leq (p-1)\left[2-2(p-1)\right]-4(p-1)+4$$

$p-1 \geq 1$

$$= -2(p-1)p-4(p-2) < 0$$