Theorem 0.1 (Riesz–Thorin). *Let T be a linear map from $L^{p_0}(\mathbb{R}^d)\cap L^{p_1}(\mathbb{R}^d)$ to $L^{q_0}(\mathbb{R}^d)\cap L^{q_1}(\mathbb{R}^d)$ satisfying*

$$\|Tf\|_{L^{q_j}} \le M_j \|f\|_{L^{p_j}} \text{ for } j=0,1.$$

Then for $t\in(0,1)$ and for p_t and q_t defined by

$$\frac{1}{p_t}=\frac{1-t}{p_0}+\frac{t}{p_1} \quad , \quad \frac{1}{q_t}=\frac{1-t}{q_0}+\frac{t}{q_1}$$

we have

$$\|Tf\|_{L^{q_t}} \le (M_0)^{1-t}(M_1)^t\|f\|_{L^{p_t}} \text{ for } f\in L^{p_0}(\mathbb{R}^d)\cap L^{p_1}(\mathbb{R}^d).$$

Proof. First of all notice that if $f\in L^a\cap L^b$ with $a<b$ then $f\in L^c$ for any $c\in(a,b)$. To see this recall Hölder

$$\|fg\|_{L^r}\le\|f\|_{L^p}\|g\|_{L^q} \text{ for } \frac{1}{r}=\frac{1}{p}+\frac{1}{q}$$

Then, since $\frac{1}{c}=\frac{t}{a}+\frac{1-t}{b}$ for $t\in(0,1)$ from $|f|=|f|^t|f|^{1-t}$ we have

$$\|f\|_{L^c}=\||f|^t|f|^{1-t}\|_{L^c}\le\||f|^t\|_{L^{\frac{a}{t}}}\||f|^{1-t}\|_{L^{\frac{b}{1-t}}}=\|f\|^t_{L^a}\|f\|^{1-t}_{L^b}.$$

For $p_t=p_0=p_1=\infty$ (in fact we can repeat a similar argument for $p_t=p_0=p_1$ any fixed value in $[1,\infty]$) we then have

$$\|Tf\|_{L^{q_t}}\le\|Tf\|^t_{L^{q_1}}\|Tf\|^{1-t}_{L^{q_0}}\le(M_0)^{1-t}(M_1)^t\|f\|_{L^\infty}.$$

So let us suppose $p_t<\infty$. Then it is enough to prove

$$\left|\int Tfg dx\right|\le(M_0)^{1-t}(M_1)^t\|f\|_{L^{p_t}}\|g\|_{L^{q_t'}}=(M_0)^{1-t}(M_1)^t$$

considering only $\|f\|_{L^{p_t}}=\|g\|_{L^{q_t'}}=1$ for simple functions $f=\sum_{j=1}^m a_j\chi_{E_j}$ where we can take the E_j to be finite measure sets mutually disjoint. If $q_t'<\infty$ we can also reduce to simple functions $g=\sum_{k=1}^N b_k\chi_{F_k}$ where the F_j are finite measure sets mutually disjoint. The case $q_t'=\infty$ reduces to the case $p_t=\infty$ by duality. In fact, see Remark 16 p. 44 [2],

$$\|T\|_{\mathcal{L}(L^{p_t},L^1)}=\|T^*\|_{\mathcal{L}(L^\infty,L^{p_t'})}.$$

Notice that if both $p_0<\infty$ and $p_1<\infty$ and since we are treating $q_0=q_1=1$, then $\|T\|_{\mathcal{L}(L^{p_j},L^1)}=\|T^*\|_{\mathcal{L}(L^\infty,L^{p_j'})}\le M_j$ and so one reduces to the case $p_t=\infty$. If, say, $p_0=\infty$, then $\|T\|_{\mathcal{L}(L^{p_1},L^1)}=\|T^*\|_{\mathcal{L}(L^\infty,L^{p_1'})}\le M_1$ since $p_1<\infty$, but $\|T\|_{\mathcal{L}(L^{p_0},L^1)}=\|T^*\|_{\mathcal{L}(L^\infty,(L^\infty)')}\le M_0$, so in other words, we don't get a Lebesgue space. However, the issue is to bound for $f\in L^{p_0}\cap L^\infty$ a $T^*f\in L^1\cap(L^\infty)'=L^1$ where $\|T^*f\|_{(L^\infty)'}=\|T^*f\|_{L^1}$, so that one can still apply the above argument used for $p_t=\infty$.

Let us turn to the case $p_t < \infty$ and $q_t' < \infty$. For $a_j = e^{\mathrm{i}\theta_j}|a_j|$ and $b_k = e^{\mathrm{i}\psi_k}|b_k|$ the polar representations, set

$$f_z := \sum_{j=1}^{m} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{\mathrm{i}\theta_j}\chi_{E_j} \text{ with } \alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}$$

$$g_z := \sum_{k=1}^{N} |b_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{\mathrm{i}\psi_k}\chi_{F_k} \text{ with } \beta(z) := \frac{1-z}{q_0} + \frac{z}{q_1}.$$

Notice that since we are assuming $q_t' < \infty$, then $q_t > 1$ and so $\beta(t) = \frac{1}{q_t} < 1$, so that g_z is well defined. Similarly, since $p_t < \infty$ we have $\alpha(t) = \frac{1}{p_t} > 0$, so also f_z is well defined.
We consider now the function

$$F(z) = \int T f_z g_z dx.$$

Our goal is to prove $|F(t)| \leq M_0^{1-t} M_1^t$.
$F(z)$ is holomorphic in $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \leq \operatorname{Re} z \leq 1$. Boundedness follows from estimates like

$$||a_j|^{\frac{\alpha(z)}{\alpha(t)}}| = |a_j|^{\frac{\alpha(\operatorname{Re} z)}{\alpha(t)}} \text{ which is bounded for } 0 \leq \operatorname{Re} z \leq 1.$$

We have $F(t) = \int T f g dx$ since $f_t = f$ and $g_t = g$.
By the 3 lines lemma, see below, which yields $|F(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ if the two estimates below are true, our theorem is a consequence of the following two inequalities

$$|F(z)| \leq M_0 \text{ for } \operatorname{Re} z = 0 \; ;$$
$$|F(z)| \leq M_1 \text{ for } \operatorname{Re} z = 1 \; .$$

For $z = \mathrm{i}y$ we have for $p_0 < \infty$

$$|f_{\mathrm{i}y}|^{p_0} = \sum_{j=1}^{m} \left| |a_j|^{\frac{\alpha(\mathrm{i}y)}{\alpha(t)}} \right|^{p_0} \chi_{E_j} = \sum_{j=1}^{m} \left| |a_j|^{\frac{\frac{1}{p_0} + \mathrm{i}y\left(\frac{1}{p_1} - \frac{1}{p_0}\right)}{\frac{1}{p_t}}} \right|^{p_0} \chi_{E_j}$$
$$= \sum_{j=1}^{m} \left| |a_j|^{\mathrm{i}y p_t\left(\frac{1}{p_1} - \frac{1}{p_0}\right)} |a_j|^{\frac{p_t}{p_0}} \right|^{p_0} \chi_{E_j} = \sum_{j=1}^{m} |a_j|^{p_t} \chi_{E_j} = |f|^{p_t}.$$

This implies

$$\|f_{\mathrm{i}y}\|_{p_0} = \left(\int_{\mathbb{R}^d} |f_{\mathrm{i}y}|^{p_0} dx \right)^{\frac{1}{p_0}} = \left(\int_{\mathbb{R}^d} |f|^{p_t} dx \right)^{\frac{1}{p_0}} = 1. \tag{0.1}$$

Notice that we have also $\|f_{\mathrm{i}y}\|_\infty = 1$ when $p_0 = \infty$.
Proceeding similarly, using $1 - \beta(z) = \frac{1-z}{q_0'} + \frac{z}{q_1'}$, for $z = \mathrm{i}y$ and $q_0' < \infty$ we have

$$|g_{\mathrm{i}y}|^{q_0'} = \sum_{k=1}^{N} \left| |b_k|^{\frac{1-\beta(\mathrm{i}y)}{1-\beta(t)}} \right|^{q_0'} \chi_{F_k} = \sum_{k=1}^{N} \left| |b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'} - \frac{1}{q_0'}\right)}{\frac{1}{q_t'}}} |b_k|^{\frac{\frac{1}{q_0'}}{\frac{1}{q_t'}}} \right|^{q_0'} \chi_{F_k} = \sum_{j=1}^{N} |b_k|^{q_t'} \chi_{F_k} = |g|^{q_t'}.$$

This implies

$$\|g_{\mathrm{i}y}\|_{q_0'} = \left(\int_{\mathbb{R}^d} |g_{\mathrm{i}y}|^{q_0'} dx\right)^{\frac{1}{q_0'}} = \left(\int_{\mathbb{R}^d} |g|^{q_t'} dx\right)^{\frac{1}{q_0'}} = 1. \tag{0.2}$$

Notice that we have also $\|g_{\mathrm{i}y}\|_\infty = 1$ when $q_0' = \infty$.
Then

$$|F(\mathrm{i}y)| \le \|Tf_{\mathrm{i}y}\|_{q_0}\|g_{\mathrm{i}y}\|_{q_0'} \le M_0\|f_{\mathrm{i}y}\|_{p_0}\|g_{\mathrm{i}y}\|_{q_0'} = M_0.$$

By a similar argument

$$|f_{1+\mathrm{i}y}|^{p_1} = |f|^{p_t}$$
$$|g_{1+\mathrm{i}y}|^{q_1'} = |g|^{q_t'}.$$

Indeed by $\alpha(1+\mathrm{i}y) = \frac{1+\mathrm{i}y}{p_1} - \frac{\mathrm{i}y}{p_0}$

$$|f_{1+\mathrm{i}y}|^{p_1} = \sum_{j=1}^m \left||a_j|^{\frac{\alpha(1+\mathrm{i}y)}{\alpha(t)}}\right|^{p_1} \chi_{E_j} = \sum_{j=1}^m \left||a_j|^{\frac{\frac{1}{p_1}+\mathrm{i}y\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}{\frac{1}{p_t}}}\right|^{p_1} \chi_{E_j}$$
$$= \sum_{j=1}^m \left||a_j|^{\frac{p_t}{p_1}}\right|^{p_1} \chi_{E_j} = \sum_{j=1}^m |a_j|^{p_t}\chi_{E_j} = |f|^{p_t}$$

and by $1-\beta(1+\mathrm{i}y) = \frac{1+\mathrm{i}y}{q_1'} - \frac{\mathrm{i}y}{q_0'}$

$$|g_{1+\mathrm{i}y}|^{q_1'} = \sum_{k=1}^N \left||b_k|^{\frac{1-\beta(1+\mathrm{i}y)}{1-\beta(t)}}\right|^{q_1'} \chi_{F_k} = \sum_{k=1}^N \left||b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'}-\frac{1}{q_0'}\right)}{\frac{1}{q_t'}}} |b_k|^{\frac{\frac{1}{q_1'}}{\frac{1}{q_t'}}}\right|^{q_1'} \chi_{F_k} = \sum_{j=1}^N |b_k|^{q_t'}\chi_{F_k} = |g|^{q_t'}.$$

Finally

$$|F(1+\mathrm{i}y)| \le \|Tf_{1+\mathrm{i}y}\|_{q_1}\|g_{1+\mathrm{i}y}\|_{q_1'} \le M_1\|f_{1+\mathrm{i}y}\|_{p_1}\|g_{1+\mathrm{i}y}\|_{q_1'} = M_1\|f\|_{p_t}^{\frac{p_t}{p_1}}\|g\|_{q_t'}^{\frac{q_t'}{q_1'}} = M_1.$$

□

Here we have used the following lemma.

Lemma 0.2 (Three Lines Lemma). *Let $F(z)$ be holomorphic in the strip $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \le \operatorname{Re} z \le 1$ and such that*

$$|F(z)| \le M_0 \text{ for } \operatorname{Re} z = 0\ ,$$
$$|F(z)| \le M_1 \text{ for } \operatorname{Re} z = 1\ .$$

Then we have $|F(z)| \le M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ for all $0 < \operatorname{Re} z < 1$.

Proof. Let us start with the special case $M_0 = M_1 = 1$ and set $B := \|F\|_{L^\infty}$. Set $h_\epsilon(z) := (1+\epsilon z)^{-1}$ with $\epsilon > 0$. Since $\mathrm{Re}(1+\epsilon z) = 1+\epsilon x \geq 1$ it follows $|h_\epsilon(z)| \leq 1$ in the strip. Furthermore $\mathrm{Im}(1+\epsilon z) = \epsilon y$ implies also $|h_\epsilon(z)| \leq |\epsilon y|^{-1}$. Consider now the two horizontal lines $y = \pm B/\epsilon$ and let R be the rectangle $0 \leq x \leq 1$ and $|y| \leq B/\epsilon$. In $|y| \geq B/\epsilon$ we have

$$|F(z)h_\epsilon(z)| \leq \frac{B}{|\epsilon y|} \leq \frac{B}{|\epsilon B/\epsilon|} = 1.$$

On the other hand by the maximum modulus principle

$$\sup_R |F(z)h_\epsilon(z)| = \sup_{\partial R} |F(z)h_\epsilon(z)| \leq 1,$$

where on the horizontal sides the last inequality follows from the previous inequality and on the vertical sides follows from $|F(z)| \leq 1$ for $\mathrm{Re}\, z = 0, 1$ and from $|h_\epsilon(z)| \leq 1$.
Hence in the whole strip $0 \leq x \leq 1$ we have $|F(z)h_\epsilon(z)| \leq 1$ for any $\epsilon > 0$. This implies

$$\lim_{\epsilon \searrow 0} |F(z)h_\epsilon(z)| = |F(z)| \leq 1$$

in the whole strip $0 \leq x \leq 1$.
In the general case $(M_0, M_1) \neq (1,1)$ set $g(z) := M_0^{1-z} M_1^z$. Notice that

$$g(z) = e^{(1-z)\log M_0} e^{z \log M_1} \Rightarrow |g(z)| = M_0^{1-x} M_1^x \Rightarrow$$
$$\min(M_0, M_1) \leq |g(z)| \leq \max(M_0, M_1).$$

So $F(z)g^{-1}(z)$ satisfies the hypotheses of the case $M_0 = M_1 = 1$ and so $|F(z)| \leq |g(z)| = M_0^{1-\mathrm{Re}\, z} M_1^{\mathrm{Re}\, z}$

□

Recall the formula

$$e^{-\varepsilon \frac{|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2\varepsilon}}\, dx \text{ for any } \varepsilon > 0. \tag{0.3}$$

1 Schrödinger equations

For $u_0 \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ the linear homogeneous Schrödinger equation is

$$\mathrm{i}u_t + \triangle u = 0\,, u(0,x) = u_0(x). \tag{1.1}$$

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + \mathrm{i}|\xi|^2 \widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{u}_0(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-\mathrm{i}t|\xi|^2}\widehat{u}_0(\xi)$. We have $e^{-\mathrm{i}t|\xi|^2} = \widehat{G}(t,\xi)$ with $G(t,x) = (2t\mathrm{i})^{-\frac{d}{2}} e^{\frac{\mathrm{i}|x|^2}{4t}}$. This follows from the following generalization of (0.3) for $\mathrm{Re}\, z > 0$

$$e^{-z\frac{|\xi|^2}{2}} = (2\pi z)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2z}}\, dx.$$

This formula follows from the fact that both sides are holomorphic in $\mathrm{Re}\, z > 0$ and coincide for $z \in \mathbb{R}_+$. Then taking the limit $z \to 2\mathrm{i}$ for $\mathrm{Re}\, z > 0$ and using the continuity of $\mathcal{F}$ in $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ we get

$$e^{-\mathrm{i}|\xi|^2} = (4\pi \mathrm{i})^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{\frac{\mathrm{i}|x|^2}{4}} dx.$$

Then $u(t,x) = (2\pi)^{-\frac{d}{2}} G(t,\cdot) * u_0(x)$. In particular, for $u_0 \in L^p(\mathbb{R}^d, \mathbb{C})$ for $p \in [1,2]$ and by Reisz's interpolation defines for any $t > 0$ an operator which we denote by

$$e^{\mathrm{i}\triangle t} u_0(x) = (4\pi \mathrm{i} t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{\mathrm{i}|x-y|^2}{4t}} u_0(y) dy \qquad (1.2)$$

which is s.t. $e^{\mathrm{i}\triangle t} : L^p(\mathbb{R}^d, \mathbb{C}) \to L^{p'}(\mathbb{R}^d, \mathbb{C})$ for $p \in [1,2]$ and $p' = \frac{p}{p-1}$ with $\|e^{\mathrm{i}\triangle t} u_0\|_{L^{p'}} \le (4\pi t)^{-d(\frac{1}{2}-\frac{1}{p'})} \|u_0\|_{L^p}$ by Riesz interpolation.

Remark 1.1. Notice that for no $p \neq 2$ and $t > 0$ we have that $e^{\mathrm{i}\triangle t}$ defines a bounded operator $L^p(\mathbb{R}^d, \mathbb{C}) \to L^p(\mathbb{R}^d, \mathbb{C})$, see [9].

Remark 1.2. Notice that $e^{\triangle t} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded operator for all $1 \le p \le q \le \infty$.

Notice that (1.1) is time reversible. and if $u(t,x) = e^{\mathrm{i}\triangle t} u_0(x)$, then $v(t,x) = \overline{u(-t,x)} = e^{\mathrm{i}\triangle t} \overline{u_0}(x)$ is a solution.

Let now $u(t,x) = e^{\mathrm{i}\triangle t} u_0(x)$, and for $\mathbf{v}, D \in \mathbb{R}^d$ consider $v_0(x) = e^{\mathrm{i}\frac{\mathbf{v}}{2}\cdot x} u_0(x - D)$. Then

$$v(t,x) := e^{\mathrm{i}\triangle t} v_0(x) = e^{\frac{\mathrm{i}}{2}\mathbf{v}\cdot x - \mathrm{i}\frac{\mathbf{v}^2}{4} t} u(t, x - t\mathbf{v} - D).$$

In the sequel, given $v, w \in L^2(\mathbb{R}^d, \mathbb{C})$ we will use the notation

$$\langle v, w \rangle = \mathrm{Re} \int_{\mathbb{R}^d} v(x) \overline{w}(x) dx. \qquad (1.3)$$

In the sequel we will reinterpret the equation

$$\mathrm{i} u_t + \triangle u = f\,, \quad u(0) = u_0 \in H^1(\mathbb{R}^d) \qquad (1.4)$$

in the integral form

$$u(t) = e^{\mathrm{i}t\triangle} u_0 - \mathrm{i} \int_0^t e^{\mathrm{i}(t-t')\triangle} f(t') dt'. \qquad (1.5)$$

To understand this formula we will need Strichartz's inequalities.
We say that a pair (q,r) is *admissible* when

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \qquad (1.6)$$

$$2 \le r \le \frac{2d}{d-2} \ (2 \le r \le \infty \text{ if } d = 1,\ 2 \le r < \infty \text{ if } d = 2). \qquad (1.7)$$

Remark 1.3. The pair $(\infty, 2)$ is always admissible. The *endpoint* $(2, \frac{2d}{d-2})$ is admissible for $d \geq 3$ but the point $(2, \infty)$ is not for $d = 2$. The equality (1.6) needs to be true by the parabolic scaling $u(t,x) \rightsquigarrow u(\lambda^2 t, \lambda x)$, which preserves the set of solutions to (1.1).

We have the following important result.

Theorem 1.4 (Strichartz's estimates)**.** *The following facts hold.*

(1) For every $u_0 \in L^2(\mathbb{R}^d)$ we have $e^{\mathrm{i}\triangle t} u_0 \in L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cap C^0(\mathbb{R}, L^2(\mathbb{R}^d))$ for every admissible (q, r). Furthermore, there exists a C s.t.

$$\|e^{\mathrm{i}\triangle t} u_0\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C\|u_0\|_{L^2}. \tag{1.8}$$

(2) Let I be an interval and let $t_0 \in \overline{I}$. If (γ, ρ) is an admissible pair and $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))$ then for any admissible pair (q, r) the function

$$\mathcal{T} f(t) = \int_{t_0}^{t} e^{\mathrm{i}\triangle(t-s)} f(s) ds \tag{1.9}$$

belongs to $L^q(I, L^r(\mathbb{R}^d)) \cap C^0(\overline{I}, L^2(\mathbb{R}^d))$ and there exists a constant C independent of I and f s.t.

$$\|\mathcal{T} f\|_{L^q(I, L^r(\mathbb{R}^d))} \leq C\|f\|_{L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))}. \tag{1.10}$$

2 Keel and Tao's proof of Strichartz estimates

We will follow the argument by Keel and Tao [8]. We will assume that (X, dx) is a measurable space and that H is a Hilbert space. We consider a family of operators $U(t) : H \to L^2(X)$. We assume the following two hypotheses.

(1) There exists a $C > 0$ s.t.

$$\|U(t) f\|_{L^2} \leq C\|f\|_H \text{ for all } f \in H;$$

(2) there exist a $\sigma > 0$ and a $C > 0$ s.t. for all $t \neq s$ and all $g \in L^1(X)$ we have

$$\|U(t)(U(s))^* g\|_{L^\infty} \leq C|t-s|^{-\sigma}\|g\|_{L^1}.$$

We say that a pair (q, r) is σ*−admissible* when

$$\begin{aligned} &\frac{2}{q} + \frac{2\sigma}{r} = \sigma \\ &r, q \geq 2 \text{ and } (q, r, \sigma) \neq (2, \infty, 1). \end{aligned} \tag{2.1}$$

Particularly important, for $\sigma > 1$, is the point $P = \left(2, \frac{2\sigma}{\sigma - 1}\right)$.

Notice that (1) implies $\|U^*(t)F\|_{L^2} \leq C\|F\|_{L^2}$ by duality and that $\langle U(t)h, f\rangle_{L^2(X)} = \langle h, (U(t))^* f\rangle_H$ [1]

Theorem 2.1 (Keel and Tao's Strichartz estimates)**.** *If $U(t)$ satisfies (1) and (2), and if furthermore there exists an appropriate scaling operator in X and H, then we have*

(3)
$$\|U(t)u_0\|_{L^q(\mathbb{R},L^r(X))} \leq C_{q,r}\|u_0\|_H.$$

(4)
$$\|\int_{\mathbb{R}} (U(s))^* F(s) ds\|_H \leq C\|F\|_{L^{q'}(\mathbb{R},L^{r'}(X))}.$$

(5)
$$\|\int_{t>s} U(t)(U(s))^* F(s) ds\|_{L^q(\mathbb{R},L^r(X))} \leq C_{q,r,\widetilde{q},\widetilde{r}}\|F\|_{L^{\widetilde{q}'}(\mathbb{R},L^{\widetilde{r}'}(X))}.$$

for all admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$.

(3) is called the homogeneous estimate and (5) the non–homogeneous estimate or also the retarded estimate. (3) and (4) are equivalent by duality. The scaling operators are used only in Sect. 2.2.

2.1 Proof of the nonendpoint homogeneous estimate

We consider the case $(q,r) \neq P$. The proof of this case predates the paper by Keel and Tao.
It is elementary that (4) is by duality and hypothesis (1) equivalent

$$\left|\int_{\mathbb{R}^2} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, dtds\right| \leq C\|F\|_{L^{q'}(\mathbb{R},L^{r'}(X))}\|G\|_{L^{q'}(\mathbb{R},L^{r'}(X))}.$$

So we have to prove the above estimate. Furthermore, it is enough to prove the above bound for

$$T(F,G) := \int_{t>s} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, dtds. \tag{2.2}$$

By (1) we know that (3) holds for $q = \infty$ and $r = 2$. So pointwise

$$\begin{aligned}|\langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H| &= \left|\langle U(t)(U(s))^* F(s), G(t)\rangle_{L^2(X)}\right| \\ &\leq \|U(t)(U(s))^* F(s)\|_{L^2(X)}\|G(t)\|_{L^2(X)} \leq C^2\|F(s)\|_{L^2(X)}\|G(t)\|_{L^2(X)}.\end{aligned}$$

[1] Notice that since $h \to \langle U(t)h, f\rangle_{L^2(X)}$ is continuous, an element $f^* \in H$ remains defined such that $\langle U(t)h, f\rangle_{L^2(X)} = \langle h, f^*\rangle_H$. The map $f \to f^*$ is linear, bounded and $(U(t))^* f := f^*$.

Furthermore

$$|\langle (U(s))^*F(s),(U(t))^*G(t)\rangle_H| = \left|\langle U(t)(U(s))^*F(s),G(t)\rangle_{L^2(X)}\right| \le \|U(t)(U(s))^*F(s)\|_{L^\infty(X)}\|G(t)\|_{L^1(X)}$$
$$\le C|t-s|^{-\sigma}\|F(s)\|_{L^1(X)}\|G(t)\|_{L^1(X)}.$$

From the Riesz–Thorin Interpolation Theorem, see Theorem 0.1, we have (omitting the constant) for any $r\in[2,\infty]$

$$\|U(t)(U(s))^*F(s)\|_{L^r(X)} \lesssim |t-s|^{-\sigma\left(1-\frac{2}{r}\right)}\|F(s)\|_{L^{r'}(X)} = |t-s|^{-1-\beta(r,r)}\|F(s)\|_{L^{r'}(X)}$$
$$\text{where } \beta(r,\widetilde{r}) := \sigma-1-\frac{\sigma}{r}-\frac{\sigma}{\widetilde{r}}.$$

Then we conclude

$$|\langle (U(s))^*F(s),(U(t))^*G(t)\rangle_H| \lesssim |t-s|^{-1-\beta(r,r)}\|F(s)\|_{L^{r'}(X)}\|G(t)\|_{L^{r'}(X)}.$$

For $\frac{1}{q'}-\frac{1}{q} = -\beta(r,r)$, using the Hardy,Littlewood Sobolev inequality, see Theorem **??**, which requires $q>q'$,

$$|T(F,G)| \lesssim \|\int_{\mathbb{R}}|t-s|^{-1-\beta(r,r)}\|F(s)\|_{L^{r'}(X)}ds\|_{L^q(\mathbb{R})}\|G\|_{L^{q'}(\mathbb{R},L^{r'}(X))} \lesssim \|F\|_{L^{q'}(\mathbb{R},L^{r'}(X))}\|G\|_{L^{q'}(\mathbb{R},L^{r'}(X))}.$$

Notice that $\frac{1}{q'}-\frac{1}{q} = -\beta(r,r)$ means

$$1-\frac{2}{q} = -\sigma+1+2\frac{\sigma}{r} \Leftrightarrow \frac{2}{q}+\frac{2\sigma}{r} = \sigma$$

and $-\beta(r,r)>0$ means

$$r<\frac{2\sigma}{\sigma-1}.$$

2.2 Proof of the endpoint homogeneous estimate

Here we consider the endpoint case $(q,r)=P=(2,\frac{2\sigma}{\sigma-1})$, when $\sigma>1$.

The introduction of a scaling operator will simplify considerably the discussion. We will denote it by D_λ for $\lambda>0$. We assume the following:

1. there exist operators $D_\lambda : H\to H$ s.t. $\langle D_\lambda f, D_\lambda g\rangle_H = \lambda^{-\sigma}\langle f,g\rangle_H$
2. there exist operators $D_\lambda : L^r(X)\to L^r(X)$ s.t $\|D_\lambda f\|_{L^r(X)} = \lambda^{-\frac{\sigma}{r}}\|f\|_{L^r(X)}$
3. in all cases $D_\lambda^{-1} = D_{\lambda^{-1}}$ and $D_\lambda^* = \lambda^{-\sigma}D_{\lambda^{-1}}$.

Notice that for $\sigma=\frac{d}{2}$, $H=L^2(\mathbb{R}^d)$ and $X=\mathbb{R}^d$ with $L^r(X)$ the standard Lebesgue spaces, then $D_\lambda f(x) := f(\lambda^{\frac{1}{2}}x)$ satisfies the desired requirements. Notice that we used the same notation for dilation operators in H and $L^r(X)$, but they are distinct operators.

Lemma 2.2. *Let the function $t \to U(t)$ satisfy (1) and (2) in Sect. 2. Then $t \to D_\lambda U(\lambda t) D_{\lambda^{-1}}$ satisfies (1) and (2) in Sect. 2 with exactly the same constants C.*

Proof. Indeed

$$\|D_\lambda U(\lambda t) D_{\lambda^{-1}} f\|_{L^2} = \lambda^{-\frac{\sigma}{2}} \|U(\lambda t) D_{\lambda^{-1}} f\|_{L^2} \le C\lambda^{-\frac{\sigma}{2}} \|D_{\lambda^{-1}} f\|_H = C\|f\|_H$$

and from $(D_\lambda U(\lambda s) D_{\lambda^{-1}})^* = D_\lambda (U(\lambda s))^* D_{\lambda^{-1}}$,

$$\begin{aligned} &\|D_\lambda U(\lambda t) D_{\lambda^{-1}} (D_\lambda U(\lambda s) D_{\lambda^{-1}})^* f\|_{L^\infty} \|D_\lambda U(\lambda t)(U(\lambda s))^* D_{\lambda^{-1}} f\|_{L^\infty} \\ &= \|U(\lambda t)(U(\lambda s))^* D_{\lambda^{-1}} f\|_{L^\infty} \le C\lambda^{-\sigma} |t-s|^{-\sigma} \|D_{\lambda^{-1}} f\|_{L^1} = C|t-s|^{-\sigma} \|f\|_{L^1}. \end{aligned}$$

□

After the above preliminary on scaling operators, expand

$$T(F,G) = \sum_{j\in\mathbb{Z}} T_j(F,G) \text{ where } T_j(F,G) := \int_{t-2^j > s > t-2^{j+1}} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H \, dtds. \tag{2.3}$$

We will prove

$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \lesssim \|F\|_{L^2 L^{r'}} \|G\|_{L^2 L^{r'}}. \tag{2.4}$$

We will prove the following.

Lemma 2.3. *For a fixed constant C dependent only on the constants in (1) –(2) Sect. 2. we have*

$$|T_j(F,G)| \le C 2^{-j\beta(a,b)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}. \tag{2.5}$$

with $(1/a, 1/b)$ in a sufficiently small, but fixed neighborhood of $(1/r, 1/r)$, dependent only on σ.

Proof. Notice that

$$\begin{aligned} T_j(F,G) &= \int_{t-2^j > s > t-2^{j+1}} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H \, dtds \\ &= 2^{2j} 2^{j\sigma} \int_{t-1 > s > t-2} \left\langle D_{2^j} (U(2^j s))^* D_{2^{-j}} D_{2^j} F(2^j s), D_{2^j} (U(2^j t))^* D_{2^{-j}} D_{2^j} G(2^j t) \right\rangle_H dtds. \end{aligned}$$

Suppose now that we have (2.4) in the particular case $j = 0$. Then we have

$$\begin{aligned} |T_j(F,G)| &\le C 2^{2j} 2^{j\sigma} \|D_{2^j} F(2^j s)\|_{L^2 L^{a'}} \|D_{2^j} G(2^j t)\|_{L^2 L^{b'}} = C 2^{2j} 2^{j\sigma} 2^{-j\left(1+\frac{\sigma}{a'}+\frac{\sigma}{b'}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} \\ &= C 2^{j\left(2+\sigma-1-2\sigma+\frac{\sigma}{a}+\frac{\sigma}{b}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} = C 2^{j\left(1-\sigma+\frac{\sigma}{a}+\frac{\sigma}{b}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} = C 2^{-j\beta(a,b)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} \end{aligned}$$

where we recall $\beta(a,b) = \sigma - 1 - \frac{\sigma}{a} - \frac{\sigma}{b}$.

So we have reduced to the case $j=0$. Next we do another reduction. We claim that to prove the case $j=0$ it is enough to assume that F and G are supported in time intervals of length 1. Indeed, assuming this case, then we have

$$|T_0(F,G)| \le \sum_{n\in\mathbb{Z}} \left| \int_{n+1>t>n} dt \int_{t-1>s>t-2} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, ds \right|$$

$$\le C \sum_{n\in\mathbb{Z}} \|F\|_{L^2((n,n+1),L^{a'})} \|G\|_{L^2((n-2,n),L^{b'})} \le C \left(\sum_{n\in\mathbb{Z}} \|F\|^2_{L^2((n,n+1),L^{a'})} \right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} \|G\|^2_{L^2((n-2,n),L^{a'})} \right)^{\frac{1}{2}}$$

$$= C\sqrt{2} \left(\sum_{n\in\mathbb{Z}} \|F\|^2_{L^2((n,n+1),L^{a'})} \right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} \|G\|^2_{L^2((n-1,n),L^{b'})} \right)^{\frac{1}{2}} = C\sqrt{2} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}.$$

Hence, in the rest of the proof we will assume that F and G are supported in time intervals of length 1. To prove (2.5) for $j=0$ we consider three cases:

(i) $a=b=\infty$;

(ii) $2 \le a < r$ and $b=2$;

(iii) $a=2$ and $2 \le b < r$.

Then the desired result follows by interpolation.

Let us start with (i). The proof is elementary and straightforward, because we have

$$|T_0(F,G)| \le \int dt \int_{t-1>s>t-2} | \langle U(t)(U(s))^* F(s), G(t)\rangle_{L^2(X)} | ds$$

$$\le C \int dt \int_{t-1>s>t-2} |t-s|^{-\sigma} \|F(s)\|_{L^1} \|G(t)\|_{L^1} \le C \int dt \int_{t-1>s>t-2} \|F(s)\|_{L^1} \|G(t)\|_{L^1}$$

$$\le C \|F\|_{L^1 L^1} \|G\|_{L^1 L^1} \le C \|F\|_{L^2 L^1} \|G\|_{L^2 L^1}.$$

Let us now consider (ii). Here we will use the Strichartz estimates in Sect. 2.1. We have

$$|T_0(F,G)| \le \int | \left\langle \int_{t-1>s>t-2} (U(s))^* F(s) ds, (U(t))^* G(t) \right\rangle_H | dt$$

$$\le \int \left\| \int_{t-1>s>t-2} (U(s))^* F(s) ds \right\|_H \|(U(t))^* G(t)\|_H dt$$

$$\le \sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s) ds \right\|_H \int \|(U(t))^* G(t)\|_H dt$$

$$\le C \|G\|_{L^1 L^2} \sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s) ds \right\|_H ,$$

where we used (1) in Sect. 2. Now, using the non endpoint Strichartz estimates in Sect. 2.1 (notice here $2 \le a < r$) we have, for $(q(a),a)$ admissible,

$$\sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s) ds \right\|_H \le C \|F\|_{L^{q(a)'} L^{a'}} \le C \|F\|_{L^2 L^{a'}}.$$

This proves (ii) and by symmetry yields also (iii). □

Now we need to show that (2.5) implies (2.4). Obviously, we cannot just take $a = b = r$ and sum up, since $\beta(r, r) = 0$. To give an intuition on how to overcome this problem, Keel and Tao consider functions of the form

$$F(t) = 2^{-\frac{k}{r'}} f(t)\chi_{E(t)}(x) \text{ and } G(s) = 2^{-\frac{\widetilde{k}}{r'}} g(s)\chi_{\widetilde{E}(s)}(x), \tag{2.6}$$

with scalar functions $f(t), g(s)$ and $E(t)$ resp. $\widetilde{E}(s)$ sets of size 2^k resp. $2^{\widetilde{k}}$. Applying (2.5) we obtain

$$\begin{aligned}
|T_j(F,G)| &\le C2^{-j\left(\sigma-1-\frac{\sigma}{a}-\frac{\sigma}{b}\right)}2^{-\frac{k}{r'}}2^{\frac{k}{a'}}2^{-\frac{\widetilde{k}}{r'}}2^{\frac{\widetilde{k}}{b'}}\|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{-j\left(\frac{2\sigma}{r}-\frac{\sigma}{a}-\frac{\sigma}{b}\right)}2^{-(k+\widetilde{k})\left(\cancel{1}-\frac{1}{r}\right)+\cancel{(k+\widetilde{k})}-\frac{k}{a}-\frac{\widetilde{k}}{b}}\|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{-j\left(\frac{2\sigma}{r}-\frac{\sigma}{a}-\frac{\sigma}{b}\right)+k\left(\frac{1}{r}-\frac{1}{a}\right)+\widetilde{k}\left(\frac{1}{r}-\frac{1}{b}\right)}\|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{(k-j\sigma)\left(\frac{1}{r}-\frac{1}{a}\right)+\left(\widetilde{k}-j\sigma\right)\left(\frac{1}{r}-\frac{1}{b}\right)}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned} \tag{2.7}$$

Notice now that we can adjust (a, b) s.t. for a fixed small $\varepsilon > 0$ the last term equals

$$C2^{-\varepsilon|k-j\sigma|-\varepsilon\left|\widetilde{k}-j\sigma\right|}\|f\|_{L^2}\|g\|_{L^2} \tag{2.8}$$

whose sum for $j \in \mathbb{Z}$ is finite.

To convert the above intuition in a proof we consider the following preliminary lemma.

Lemma 2.4. *Let $p \in (0, \infty)$. Then any $f \in L^p_x$ can be written as*

$$f = \sum_{k\in\mathbb{Z}} c_k \chi_k$$

where meas(suppχ_k) $\le 2\ 2^k$, $|\chi_k| \le 2^{-\frac{k}{p}}$ and $\|c_k\|_{\ell^p} \le 2^{\frac{1}{p}}\|f\|_{L^p}$.

Proof. Consider the distribution function $\lambda(\alpha) = \text{meas}(\{|f(x)| > \alpha\})$. Then for each k consider

$$\alpha_k := \inf_{\lambda(\alpha)<2^k} \alpha\,, \quad c_k := 2^{\frac{k}{p}}\alpha_k\,, \quad \chi_k := \frac{1}{c_k}\chi_{(\alpha_{k+1},\alpha_k]}(|f|)f.$$

Notice that $\{\alpha_k\}_{k\in\mathbb{Z}}$ is decreasing in k (since, the larger k, the larger is the set $\{\alpha : \lambda(\alpha) < 2^k\}$).
We show the desired properties. We have

$$\text{supp}\chi_k \subseteq \{x : \alpha_{k+1} < |f(x)| \le \alpha_k\} \subseteq \{x : |f(x)| > \alpha_{k+1}\}.$$

Then we get the 1st inequality:

$$\text{meas}(\text{supp}\chi_k) \le \text{meas}(\{x : |f(x)| > \alpha_{k+1}\}) = \lim_{\alpha\to\alpha_{k+1}^+}\lambda(\alpha) = \sup\{\lambda(\alpha) : \alpha > \alpha_{k+1}\}) \le 2^{k+1}.$$

Next, by $|f(x)| \leq \alpha_k$ in $\text{supp}\chi_k$, we have

$$|\chi_k(x)| \leq 2^{-\frac{k}{p}} \frac{|f(x)|}{\alpha_k} \leq 2^{-\frac{k}{p}}.$$

Let now $\lim_{k\to+\infty} \alpha_k = \inf_{k\in\mathbb{Z}} \alpha_k = \underline{\alpha}$ and $\lim_{k\to-\infty} \alpha_k = \sup_{k\in\mathbb{Z}} \alpha_k = \overline{\alpha}$. Then we claim that $\underline{\alpha} = 0$ and that $|f(x)| \leq \overline{\alpha}$ a.e. Indeed, suppose that $|f(x)| > \overline{\alpha}$ on a set of positive measure. There there is $\alpha > \overline{\alpha}$ with $\lambda(\alpha) > 2^k$ for some $k \in \mathbb{Z}$. Then $\alpha_k \geq \alpha > \overline{\alpha}$, which is a contradiction. On the other hand, suppose we have $0 < \alpha < \underline{\alpha}$. Then $\lambda(\alpha) = \infty$, since otherwise $\lambda(\alpha) < 2^k$ for a k, and then $\alpha \geq \alpha_k \geq \underline{\alpha} > \alpha$, getting a contradiction. But by Chebyshev's inequality,

$$\infty > \|f\|_{L^p}^p \geq \alpha^p \lambda(\alpha),$$

hence getting a contradiction. The above claim and the obvious fact that for any x we have $|f(x)| \in (\alpha_{k+1}, \alpha_k]$ for at most one k, prove $f = \sum_{k\in\mathbb{Z}} c_k\chi_k$ (the claim guarantees the existence of one such k).
We have $\|f\|_{L^p} \leq 2^{\frac{1}{p}} \|c_k\|_{\ell^p}$ by

$$\|f\|_{L^p}^p = \int |f|^p dx = \int \sum_{k\in\mathbb{Z}} |c_k|^p |\chi_k|^p dx = \sum_{k\in\mathbb{Z}} |c_k|^p \int |\chi_k|^p dx \leq \sum_{k\in\mathbb{Z}} |c_k|^p 2^{-k} \text{meas}(\text{supp}\chi_k)$$
$$\leq 2 \sum_{k\in\mathbb{Z}} |c_k|^p$$

Next we have

$$\sum_{k\in\mathbb{Z}} |c_k|^p = \sum_{k\in\mathbb{Z}} 2^k \alpha_k^p = \int_{\mathbb{R}_+} \alpha^p \left(\sum 2^k \delta(\alpha - \alpha_k)\right) d\alpha = \int_{\mathbb{R}_+} \alpha^p (-F'(\alpha)) d\alpha$$

where

$$F(\alpha) := \sum_{k\in\mathbb{Z}} 2^k H(\alpha_k - \alpha) = \sum_{\alpha_k > \alpha} 2^k \leq \sum_{2^k \leq \lambda(\alpha)} 2^k \leq 2\lambda(\alpha).$$

Then, integrating by parts and using (**??**),

$$\sum_{k\in\mathbb{Z}} |c_k|^p = p \int_{\mathbb{R}_+} \alpha^{p-1} F(\alpha) d\alpha \leq 2p \int_{\mathbb{R}_+} \alpha^{p-1} \lambda(\alpha) d\alpha = 2\|f\|_{L^p}^p,$$

so that $\|c_k\|_{\ell^p} \leq 2^{\frac{1}{p}} \|f\|_{L^p}$.

□

Furthermore we have the following.

Lemma 2.5. *Let $1 \leq q, r < \infty$ and let $f \in L^q(I, L_x^r)$ with I an interval. Then we can write the expansion of Lemma 2.4*

$$f = \sum_{k\in\mathbb{Z}} c_k(t)\chi_k(t) \tag{2.9}$$

with $t \to \{c_k(t)\}$ a map in $L^q(I, \ell^r)$.

Proof. Formally this follows immediately from

$$\|\{c_k(t)\}\|_{L^q(I,\ell^r)} = \|\|\{c_k(t)\}\|_{\ell^r}\|_{L^q(I)} \le 2^{\frac{1}{p}}\|\|f\|_{L^r_x}\|_{L^q(I)}.$$

However one needs to argue that the function $t \to \{c_k(t)\}$ is measurable. By a density argument it is enough to consider the case of simple functions $f = \sum_{j=1,\dots,n} \chi_{E_j}(t) g_j(x)$ with E_j mutually disjoints sets. Then $\lambda(t,\alpha) = \text{meas}(\{|f(t,x)| > \alpha\}) = \sum_{j=1,\dots,n} \chi_{E_j}(t)\lambda_j(\alpha)$ with λ_j the distribution function of g_j. Then $\alpha_k(t) = \sum_{j=1,\dots,n} \chi_{E_j}(t)\alpha_k^{(j)}$ with $\alpha_k^{(j)}$ defined like in Lemma 2.4 for each g_j. Then

$$\{c_k(t)\} = \sum_{j=1,\dots,n} \chi_{E_j}(t)\{c_k^{(j)}\} \text{ for } c_k^{(j)} = 2^{\frac{k}{p}}\alpha_k^{(j)}.$$

This is measurable in t.

□

Consider now the

$$F(t) = \sum_{k\in\mathbb{Z}} f_k(t)\chi_k(t)\,, \quad G(s) = \sum_{k\in\mathbb{Z}} g_k(s)\widetilde{\chi}_k(s). \tag{2.10}$$

By (2.6)–(2.8) e have

$$\sum_j |T_j(F,G)| \le \sum_{j,k,\widetilde{k}} |T_j(f_k\chi_k, g_{\widetilde{k}}\widetilde{\chi}_{\widetilde{k}})| \le C\sum_{j,k,\widetilde{k}} 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|}\|f_k\|_{L^2_t}\|g_{\widetilde{k}}\|_{L^2_t}$$
$$= C\sum_{k,\widetilde{k}}\left(\sum_j 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|}\right)\|f_k\|_{L^2_t}\|g_{\widetilde{k}}\|_{L^2_t}.$$

We claim that for a fixed $C = C(\sigma,\varepsilon)$

$$\sum_j 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|} \le C2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|). \tag{2.11}$$

To prove this inequality, it is not restrictive to assume $k \le \widetilde{k}$. Then the summation on the left can be rewritten as

$$\sum_{j\sigma\le k} 2^{2\varepsilon j\sigma-\varepsilon(k+\widetilde{k})} + \sum_{k<j\sigma\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} + \sum_{\widetilde{k}<j\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon j\sigma}.$$

Then (here $[t]\in\mathbb{Z}$ is the integer part of $t\in\mathbb{R}$, defined by $[t]\le t<[t]+1$)

$$\sum_{j\sigma\le k} 2^{2\varepsilon j\sigma-\varepsilon(k+\widetilde{k})} = 2^{-\varepsilon(k+\widetilde{k})}\sum_{j\le\left[\frac{k}{\sigma}\right]} 2^{2\varepsilon j\sigma} = 2^{-\varepsilon(k+\widetilde{k})}\sum_{j=0}^{\infty} 2^{2\varepsilon\sigma\left(\left[\frac{k}{\sigma}\right]-j\right)} = C_{\varepsilon\sigma}\ 2^{-\varepsilon(k+\widetilde{k})+2\varepsilon\sigma\left[\frac{k}{\sigma}\right]}$$
$$\le C_{\varepsilon\sigma}2^{-\varepsilon(k+\widetilde{k})+2\varepsilon\sigma\frac{k}{\sigma}} = C_{\varepsilon\sigma}2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon\sigma}2^{-\varepsilon|k-\widetilde{k}|} \text{ where } C_{\varepsilon\sigma} = \frac{1}{1-2^{-2\varepsilon\sigma}}.$$

We have

$$\sum_{\widetilde{k}<j\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon j\sigma} \le 2^{\varepsilon(k+\widetilde{k})} \sum_{j\ge\left[\frac{\widetilde{k}}{\sigma}\right]+1} 2^{-2\varepsilon j\sigma} = 2^{\varepsilon(k+\widetilde{k})} \sum_{j=0}^{\infty} 2^{-2\varepsilon\sigma\left(\left[\frac{\widetilde{k}}{\sigma}\right]+1+j\right)} = C_{\varepsilon\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon\sigma\left(\left[\frac{\widetilde{k}}{\sigma}\right]+1\right)}$$

$$\le C_{\varepsilon\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon\sigma\frac{\widetilde{k}}{\sigma}} = C_{\varepsilon\sigma} 2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon\sigma} 2^{-\varepsilon|k-\widetilde{k}|}.$$

Finally

$$\sum_{k<j\sigma\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} = 2^{-\varepsilon(\widetilde{k}-k)} \sum_{\left[\frac{k}{\sigma}\right]+1\le j\sigma\le\left[\frac{\widetilde{k}}{\sigma}\right]} 1 = 2^{-\varepsilon(\widetilde{k}-k)} \left(\left[\frac{\widetilde{k}}{\sigma}\right] - \left[\frac{k}{\sigma}\right] - 1\right) \le \sigma^{-1} 2^{-\varepsilon(\widetilde{k}-k)}(\widetilde{k}-k)$$

Hence (2.11) is proved. From this we conclude that for a fixed C

$$\sum_j |T_j(F,G)| \le C \sum_{k,\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|f_k\|_{L_t^2}\|g_{\widetilde{k}}\|_{L_t^2}$$

$$\le C\|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})} \left\| \{\sum_{\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|g_{\widetilde{k}}\|_{L_t^2}\} \right\|_{\ell^2(\mathbb{Z})}$$

$$\le C\left(\sum_k 2^{-\varepsilon|k|}(1+|k|)\right) \|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}\|\{\|g_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}$$

where we used Lemma **??**. So, using $r' \le 2$,

$$\sum_j |T_j(F,G)| \le C'\|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}\|\{\|g_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})} = C'\|\|\{f_k\}\|_{\ell^2(\mathbb{Z})}\|_{L_t^2}\|\|\{g_k\}\|_{\ell^2(\mathbb{Z})}\|_{L_t^2}$$

$$\le C''\|\|\{f_k\}\|_{\ell^{r'}(\mathbb{Z})}\|_{L_t^2}\|\|\{g_k\}\|_{\ell^{r'}(\mathbb{Z})}\|_{L_t^2} \le C'''\|\|F\|_{L_x^{r'}}\|_{L_t^2}\|\|G\|_{L_x^{r'}}\|_{L_t^2}$$

which completes the proof of (2.4).

2.3 Proof of the non homogeneous estimate

We need to prove that for all admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$ we have

$$|T(F,G)| \le C_{q,r,\widetilde{q},\widetilde{r}}\|F\|_{L^{q'}(\mathbb{R},L^{r'}(X))}\|G\|_{L^{\widetilde{q}'}(\mathbb{R},L^{\widetilde{r}'}(X))}. \tag{2.12}$$

We have already proved that this is true for $(q,r)=(\widetilde{q},\widetilde{r})$. Furthermore, proceeding like in Lemma 2.3

$$|T(F,G)| \le \int \left| \left\langle \int_{t>s} (U(s))^* F(s)ds, (U(t))^* G(t) \right\rangle_H \right| dt$$

$$\le \int \| \int_{t>s} (U(s))^* F(s)ds\|_H \|(U(t))^* G(t)\|_H dt \le \sup_t \| \int_{t>s} (U(s))^* F(s)ds\|_H \int \|(U(t))^* G(t)\|_H dt$$

$$\le C\|G\|_{L^1L^2} \sup_t \| \int_{t>s} (U(s))^* F(s)ds\|_H,$$

Then, by (4) in Theorem 2.1 (that is the dual homogenous estimates, which are already proved) for any admissible pair (q,r)

$$\sup_t \| \int_{t>s} (U(s))^* F(s) ds \|_H = \sup_t \| \int_{\mathbb{R}} (U(s))^* F(s) \chi_{(-\infty,t)}(s) ds \|_H \leq C \| F \chi_{(-\infty,t)} \|_{L^{q'}(\mathbb{R},L^{r'})} \leq C \| F \|_{L^{q'}(\mathbb{R},L^{r'})}.$$

So (2.12) holds for $(\widetilde{q},\widetilde{r}) = (\infty, 2)$ and any admissible pair (q,r). Obviously, symmetrically (2.12) holds for $(q,r) = (\infty,2)$ and any admissible pair $(\widetilde{q},\widetilde{r})$. Finally, let us consider (q,r) and $(\widetilde{q},\widetilde{r})$ not in one of the cases already covered. Then it is not restrictive to assume that $(\widetilde{q},\widetilde{r}) = (a_{t_0}, b_{t_0})$ for $t_0 \in (0,1)$ where

$$\left(\frac{1}{a_t}, \frac{1}{b_t}\right) = t\left(\frac{1}{q}, \frac{1}{r}\right) + (1-t)\left(\frac{1}{\infty}, \frac{1}{2}\right).$$

In the cases $t = 0, 1$ the inequality holds, because these are cases considered above. By a generalization of Riesz–Thorin, Theorem 0.1, the inequality holds also for the intermediate t's. □

2.4 Improved non–homogeneous Strichartz estimates

While the homogeneous Strichartz estimates (1.8) are optimal, the non–homogeneous Strichartz estimates (1.10) as described in Claim 2 of Theorem 1.4 are not optimal.
We say that a pair (q,r) is *acceptable* when

$$\frac{1}{q} < \frac{d}{2} - \frac{d}{r} \tag{2.13}$$

$$2 \leq r \leq \infty \text{ and } 2 \leq r < \infty. \tag{2.14}$$

Remark 2.6. Admissible pairs are acceptable, but the viceversa is not necessarily true.

We state without proof the following theorem from [7]

Theorem 2.7 (Inhomogeneous Strichartz estimates). *Statement 2 in Theorem 1.4 is true for any pairs (q,r) and (γ,ρ) which are acceptable, satisfy*

$$\frac{1}{q} + \frac{1}{\gamma} = \frac{d}{2}\left(1 - \frac{1}{r} - \frac{1}{\rho}\right) \tag{2.15}$$

and the following conditions:

- *if $d = 1$ no further conditions;*
- *if $d = 2$, $r < \infty$ and $\rho < \infty$*
- *if $d \geq 3$ we distinguish two cases.*

1. The non–sharp case

$$\frac{1}{q} + \frac{1}{\gamma} < 1, \tag{2.16}$$

$$\frac{d-2}{d}\ \frac{1}{r} \le \frac{1}{\rho} \ \textit{and}\ \frac{d-2}{d}\ \frac{1}{\rho} \le \frac{1}{r} \tag{2.17}$$

2. The sharp case

$$\frac{1}{q} + \frac{1}{\gamma} = 1, \tag{2.18}$$

$$\frac{d-2}{d}\ \frac{1}{r} < \frac{1}{\rho} \ \textit{and}\ \frac{d-2}{d}\ \frac{1}{\rho} < \frac{1}{r} \ \textit{and} \tag{2.19}$$

$$\frac{1}{r} \le \frac{1}{q} \ \textit{and}\ \frac{1}{\rho} \le \frac{1}{\gamma}. \tag{2.20}$$

3 The semilinear Schrödinger equation

There is a vast literature on semilinear Schrödinger equations. For a survey, with a concise discussion of some physical motivations, we refer to [14]. Here though, we consider only the mathematical formalism and only the pure power semilinear Schrödinger equations

$$\begin{cases} \mathrm{i}u_t = -\triangle u + \lambda |u|^{p-1}u \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{3.1}$$

for $\lambda \in \{1,-1\}$ and $p > 1$. Here $p < d^*$ with $d^* = \infty$ for $d = 1,2$ and $d^* = \frac{d+2}{d-2}$ for $d \ge 3$. We collect here a number of facts needed later.

Lemma 3.1. *We have the following facts.*

(1) For $1 < p < d^*$ we have the Gagliardo–Nirenberg inequality:

$$\|u\|_{L^{p+1}(\mathbb{R}^d)} \le C_p \|\nabla u\|^{\alpha}_{L^2(\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^2(\mathbb{R}^d)} \ \textit{for}\ \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}. \tag{3.2}$$

(2) The map $u \to |u|^{p-1}u$ is a locally Lipschitz from $H^1(\mathbb{R}^d)$ to $H^{-1}(\mathbb{R}^d)$.

(3) For $u \in W^{1,p+1}(\mathbb{R}^d, \mathbb{C})$ we have $\nabla(|u|^{p-1}u) = p|u|^{p-1}\nabla u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \nabla \overline{u}$ and belonging to $L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C})$.

Proof. For (1) see Theorem **??**.

We turn (2). By (3.2) we know that $u \to |u|^{p-1}u$ maps $H^1(\mathbb{R}^d) \to L^{p+1}(\mathbb{R}^d) \to L^{\frac{p+1}{p}}(\mathbb{R}^d)$. Furthermore this map is locally Lipschitz:

$$\begin{aligned} &\||u|^{p-1}u - |v|^{p-1}v\|_{L^{\frac{p+1}{p}}} \le C\|(|u|^{p-1} + |v|^{p-1})(u-v)\|_{L^{\frac{p+1}{p}}} \\ &\le C'(\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}})\|u-v\|_{L^{p+1}} \end{aligned}$$

where we have used, for $w = v - u$,

$$|u|^{p-1}u - |v|^{p-1}v = \int_0^1 \frac{d}{dt}\left(|u+tw|^{p-1}(u+tw)\right)dt =$$
$$\int_0^1 |u+tw|^{p-1}dtw + \int_0^1 (u+tw)\frac{d}{dt}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-1}{2}}dt = \int_0^1 |u+tw|^{p-1}dtw +$$
$$\sum_{j=1}^{2}\int_0^1 (u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-3}{2}}2(u_j+tw_j)dtw_j$$

which from $|u+tw| \le |u|+|v|$ for $t \in [0,1]$ and

$$\left|(u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-3}{2}}2(u_j+tw_j)w_j\right| \le (p-1)|u+tw|^{p-1}|w|$$

yields

$$\left||u|^{p-1}u - |v|^{p-1}v\right| \le p(|u|+|v|)^{p-1}|u-v| \le p2^{p-1}(|u|^{p-1}+|v|^{p-1})|u-v|,$$

where in the last step we used, for $|u| \ge |v|$,

$$(|u|+|v|)^{p-1} \le 2^{p-1}|u|^{p-1} \le 2^{p-1}(|u|^{p-1}+|v|^{p-1}).$$

Next, we show that we have an embedding $L^{\frac{p+1}{p}}(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d)$. Indeed, this is equivalent to $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$ with in turn is a consequence of (3.2).

We turn (3). First of all we claim that if $G \in C^1(\mathbb{C},\mathbb{C})$ with $G(0) = 0$ and $|\nabla G| \le M < \infty$, then $\nabla(G(u)) = \partial_u G(u)\nabla u + \partial_{\overline{u}}G(u)\nabla\overline{u}$ in the sense of distributions. This claim can be proved like Proposition 9.5 in [2] and we skip the proof here.
Let us now consider an increasing function $g \in C^\infty(\mathbb{R}_+,\mathbb{R})$ s.t.

$$g(s) = \begin{cases} s^{\frac{p-1}{2}} & \text{for } 0 \le s \le 1 \\ 2^{\frac{p-1}{2}} & \text{for } s \ge 2 \end{cases}$$

and let us define $G_m(u) = m^{p-1}g\left(\frac{|u|^2}{m^2}\right)u$ for $m \in \mathbb{N}$. Then, by the claim, for all $\varphi \in C_c^\infty(\mathbb{R}^d,\mathbb{C})$ and all $u \in W^{1,p+1}(\mathbb{R}^d,\mathbb{C})$ we have

$$-\int G_m(u)\ \partial_j\varphi = \int \left(\partial_u G_m(u)\partial_j u + \partial_{\overline{u}}G_m(u)\partial_j\overline{u}\right)\ \varphi. \tag{3.3}$$

Let us take now the limit for $m \to \infty$. We have

$$\int G_m(u)\ \partial_j\varphi = \int |u|^{p-1}u\ \partial_j\varphi - \int_{|u|\ge m}|u|^{p-1}u\ \partial_j\varphi + \int_{|u|\ge m}G_m(u)\ \partial_j\varphi.$$

Now we have

$$\int_{|u|\geq m} |u|^{p-1}u \ \partial_j\varphi \xrightarrow{m\to\infty} 0 \text{ by Dominated Convergence}$$

since $\chi_{\{|u|\geq m\}}(x) \xrightarrow{m\to\infty} 0$ a.e. by Chebyshev's inequality. Similarly

$$\begin{aligned}
&\left|\int_{|u|\geq m} G_m(u)\ \partial_j\varphi\right| \leq \int_{|u|\geq m} |G_m(u)\ \partial_j\varphi| \leq 2^{p-1}\int_{|u|\geq m} m^{p-1}|u|\ \partial_j\varphi| \\
&\leq 2^{p-1}\int_{|u|\geq m} |u|^p|\partial_j\varphi| \xrightarrow{m\to\infty} 0
\end{aligned}$$

Next, we consider the limit of the r.h.s. of (3.3). For $G(u) = |u|^{p-1}u$ we have

$$\begin{aligned}
&\int (\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j\overline{u})\ \varphi = \int (\partial_u G(u)\partial_j u + \partial_{\overline{u}} G(u)\partial_j\overline{u}) \\
&- \int_{|u|\geq m} (\partial_u G(u)\partial_j u + \partial_{\overline{u}} G(u)\partial_j\overline{u})\ \varphi + \int_{|u|\geq m} (\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j\overline{u})\ \varphi.
\end{aligned}$$

Then, like before, the terms of the 2nd line converge to 0 as $m \to \infty$ and so we conclude that all $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ and all $u \in W^{1,p+1}(\mathbb{R}^d, \mathbb{C})$ we have

$$-\int |u|^{p-1}u\ \partial_j\varphi = \int \left(p|u|^{p-1}\partial_j u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \partial_j\overline{u}\right)\ \varphi.$$

The fact of belonging to $L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C})$ follows immediately from Hölder inequality.

□

Important are the following quantities:

$$\begin{aligned}
E(u) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx \\
P_j(u) &= \frac{1}{2}\operatorname{Im}\int_{\mathbb{R}^d} \partial_j u\,\overline{u} dx \\
Q(u) &= \int_{\mathbb{R}^d} |u|^2 dx.
\end{aligned} \tag{3.4}$$

Here $E(u)$ is the energy, $P_j(u)$ for $j = 1, ..., d$ are the linear momenta and $Q(u)$ is the mass or charge.

Remark 3.2. Notice that $Q, P_j \in C^\infty(H^1(\mathbb{R}^d), \mathbb{R})$ while $E \in C^1(H^1(\mathbb{R}^d), \mathbb{R})$. We will show that the above quantities are conserved for solutions in $H^1(\mathbb{R}^d, \mathbb{C})$. Here E is the hamiltonian. The system is invariant under the transformation $u \to e^{\mathrm{i}\vartheta}u$ for $\vartheta \in \mathbb{R}$ and the transformations $u(x_1, ...x_{j-1}, x_j, x_{j+1}, ..., x_d) \to u(x_1, ...x_{j-1}, x_j - \tau, x_{j+1}, ..., x_d)$ for $\tau \in \mathbb{R}$. The related Noether invariants are Q and P_j.

Remark 3.3. Notice that if $u(t,x)$ solves the equation (3.1) then also $\tau^{\frac{2}{p}}u(\tau^2 t,\tau x)$ solves the equation (3.1), with initial value $\tau^{\frac{2}{p}}u_0(\tau\cdot)$. Notice that

$$\|\tau^{\frac{2}{p}}u_0(,\tau\cdot)\|_{\dot H^{s_p}(\mathbb{R}^d)} = \|u_0\|_{\dot H^{s_p}(\mathbb{R}^d)} \text{ for } s_p = \frac{d}{2}-\frac{2}{p}.$$

$\dot H^{s_p}(\mathbb{R}^d)$ is the critical space for the equation (3.1) and equation (3.1) is critical for $\dot H^{s_p}(\mathbb{R}^d)$. Equation (3.1) is supercritical for $\dot H^{s}(\mathbb{R}^d)$ with $s < s_p$. In practice when an equation is critical or supercritical the well–posedness is either hard to prove or not true.

3.1 The local existence

We will consider the following integral formulation of (3.1):

$$u(t) = e^{\mathrm{i}t\triangle}u_0 - \mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds. \tag{3.5}$$

Proposition 3.4 (Local well posedness in $L^2(\mathbb{R}^d)$)**.** *For any $p \in (1, 1+4/d)$ and any $u_0 \in L^2(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of* (3.5) *with*

$$u \in C([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q}+\frac{d}{p+1}=\frac{d}{2}. \tag{3.6}$$

Furthermore, there exists a (decreasing) function $T(\cdot) : [0,+\infty) \to (0,+\infty]$ such that the above T satisfies $T \geq T(\|u_0\|_{L^2}) > 0$.
Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T',T'], L^2(\mathbb{R}^d)) \cap L^q([-T',T'], L^{p+1}(\mathbb{R}^d)) \tag{3.7}$$

and is Lipschitz.

Finally, we have $u \in L^a([-T,T], L^b(\mathbb{R}^d))$ for all admissible pairs (a,b).

Remark 3.5. We will prove later that for $p \in (1, 1+2/d)$ that we can take $T = \infty$ always.

Proof. The proof is a fixed point argument. We set for an $a > 0$ to be fixed below

$$E(T,a) = \Big\{v \in C([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d)) :$$
$$\|v\|_T := \|v\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} + \|v\|_{L^q([-T,T],L^{p+1}(\mathbb{R}^d))} \leq a\Big\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5). Our first aim is to show that for $T = T(\|u_0\|_{L^2})$ sufficiently small, then $\Phi : E(T,a) \to E(T,a)$ is a contraction.
By Strichartz's estimates

$$\|\Phi(u)\|_T \leq c_0\|u_0\|_{L^2} + c_0\||u|^{p-1}u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})}$$
$$= c_0\|u_0\|_{L^2} + c_0\|u\|^p_{L^{pq'}([-T,T],L^{p+1})}$$

We will see in a moment that

$$p \in (1, 1 + 4/d) \Longleftrightarrow pq' < q. \tag{3.8}$$

Assuming this for a moment, by Hölder we conclude that for a $\theta > 0$

$$\|\Phi(u)\|_T \leq c_0 \|u_0\|_{L^2} + c_0 (2T)^\theta \|u\|^p_{L^q([-T,T], L^{p+1})} \leq c_0 \|u_0\|_{L^2} + c_0 (2T)^\theta a^p.$$

So for $c_0 (2T)^\theta a^{p-1} < 1/2$, which can be obtained by picking T small enough, we have

$$\|\Phi(u)\|_T \leq c_0 \|u_0\|_{L^2} + \frac{a}{2} \leq a$$

if $a \geq 2c_0 \|u_0\|_{L^2}$. Hence $\Phi(E(T,a)) \subseteq E(T,a)$. Let us fix here an $a > 2c_0 \|u_0\|_{L^2}$. Now let us show that Φ is a contraction for T small enough. We have

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\|_T \leq c_0 \||u|^{p-1}u - |v|^{p-1}v\|_{L^{q'}([-T,T], L^{\frac{p+1}{p}})} \\
&\leq c_0 C \|(\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}})\|u - v\|_{L^{p+1}}\|_{L^{q'}(-T,T)} \\
&\leq c_0 C (\|u\|^{p-1}_{L^q([-T,T], L^{p+1}} + \|v\|^{p-1}_{L^q([-T,T], L^{p+1})})\|u - v\|_{L^\rho([-T,T], L^{p+1})}
\end{aligned}$$

where $\frac{p-1}{q} + \frac{1}{\rho} = \frac{1}{q'}$. Since we are still assuming (3.8), we must have $\rho < q$, for $\rho \geq q$ would imply $pq' \geq q$, contrary to (3.8). Then by Hölder and for an appropriate $\theta > 0$

$$\|\Phi(u) - \Phi(v)\|_T \leq c_0 C 2a^{p-1} T^\theta \|u - v\|_{L^q([-T,T], L^{p+1})} \leq c_0 C 2a^{p-1} T^\theta \|u - v\|_T.$$

So, for $c_0 C 2a^{p-1} T^\theta < 1$, where $a > 2c_0 \|u_0\|_{L^2}$, we obtain that Φ is a contraction and we obtain the existence and uniqueness of the solution.

Next, let us prove (3.8). Obviously $pq' < q$ is equivalent to $p/q < 1 - 1/q$, in turn to $(p+1)/q < 1$, that is to $1/q < 1/(p+1)$. But $1/q = d/4 - d/(2p+2)$, so the last inequality is equivalent to

$$d/4 < \left(\frac{d}{2} + 1\right)/(p+1) \Leftrightarrow p + 1 < \frac{2d+4}{d} = 2 + \frac{4}{d}$$

and this yields the desired result.

We have proved the existence of a $T = T(\|u_0\|_{L^2})$ with the desired properties. Then there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ such that for any $v_0 \in V$ we have $a > 2c_0 \|u_0\|_{L^2}$. Then there is a corresponding solution $v(t)$ in $C([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d))$. Let now $T' \in (0,T)$ to be fixed. Using the equation and proceeding like above,

$$\begin{aligned}
&\|u - v\|_{T'} \leq c_0 \|u_0 - v_0\|_{L^2} + c_0 C (2T')^\theta \left(\|u\|^{p-1}_{T'} + \|v\|^{p-1}_{T'}\right) \|u - v\|_{T'} \\
&\leq c_0 \|u_0 - v_0\|_{L^2} + c_0 C (2T')^\theta 2 \left((2c_0 \|v_0\|_{L^2})^{p-1} + (2c_0 \|u_0\|_{L^2})^{p-1}\right) \|u - v\|_{T'}.
\end{aligned}$$

Adjusting T', we can assume that it satisfies (recall $a > 2c_0 \max\{\|v_0\|_{L^2}, \|u_0\|_{L^2}\}$)

$$4c_0 C(2T')^\theta a^{p-1} < 1/2.$$

Notice that here $T' = T'(\|u_0\|_{L^2})$. Renaming $T = T'$, from the above we get

$$\|u - v\|_T \leq 2c_0\|u_0 - v_0\|_{L^2}$$

and this gives the desired Lipschitz continuity.

Finally, the last statement follows from (3.5) and the Strichartz Estimates.

□

Proposition 3.6 (Local well posedness in $H^1(\mathbb{R}^d)$)**.** *For any $p \in (1, d^*)$ and any $u_0 \in H^1(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of (3.5) with*

$$u \in C([-T,T], H^1(\mathbb{R}^d)) \cap L^q([-T,T], W^{1,p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{3.9}$$

Furthermore, there exists a (decreasing) function $T(\cdot) : [0, +\infty) \to (0, +\infty]$ such that the above T satisfies $T \geq T(\|u_0\|_{H^1}) > 0$.
Moreover, for any $T' \in (0, T)$ there exists a neighborhood V of u_0 in $H^1(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T', T'], L^2(\mathbb{R}^d)) \cap L^q([-T', T'], W^{1,p+1}(\mathbb{R}^d))$$

and is Lipschitz.

Finally, we have $u \in L^a([-T,T], W^{1,b}(\mathbb{R}^d))$ for all admissible pairs (a, b).

Proof. The proof is similar to that of Proposition 3.4. The proof is a fixed point argument.This time we set

$$E^1(T,a) = \Big\{v \in C([-T,T], H^1(\mathbb{R}^d)) \cap L^q([-T,T], W^{1,p+1}(\mathbb{R}^d)) : \\ \|v\|_T^{(1)} := \|v\|_{L^\infty([-T,T],H^1(\mathbb{R}^d))} + \|v\|_{L^q([-T,T],W^{1,p+1}(\mathbb{R}^d))} \leq a\Big\}$$

and, as before, use $\Phi(u)$ for the r.h.s. of (3.5). We need to show that by taking T sufficiently small then $\Phi : E^1(T,a) \to E^1(T,a)$ and is a contraction. The argument is similar to the one in Proposition 3.4 and is based on the Strichartz estimates. We will only consider some of the estimates. By Lemma 3.1 and Strichartz's estimates, we have

$$\begin{aligned} \|\nabla\Phi(u)\|_T &\leq c_0\|u_0\|_{H^1} + c_0\||u|^{p-1}\nabla u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \\ &= c_0\|u_0\|_{L^2} + c_0\|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})}\|\nabla u\|_{L^q([-T,T],L^{p+1})}. \end{aligned} \tag{3.10}$$

where $\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'}$. Notice that if $\beta < q$, we can proceed exactly like in Proposition 3.4. However this works only for $p \in (1, 1 + 4/d)$, which is not necessarily true here. Instead, using the Sobolev Embedding we bound

$$\|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})} \lesssim \|u\|^{p-1}_{L^\beta([-T,T],H^1)} \leq (2T)^{\frac{p-1}{\beta}}\|u\|^{p-1}_{L^\infty([-T,T],H^1)} \leq (2T)^{\frac{p-1}{\beta}}(\|u\|_T^{(1)})^{p-1}.$$

So, inserting this in the previous inequality we get

$$\|\nabla\Phi(u)\|_T \le c_0\|u_0\|_{H^1} + c_0(2T)^{\frac{p-1}{\beta}}(\|u\|_T^{(1)})^p. \tag{3.11}$$

Here it is important to remark that the admissible pair $(q, p+1)$ is s.t. $q > 2$. Indeed, for $d = 1, 2$ it is always true that, if $p+1 < \infty$, then the q in (3.29) is $q > 2$. On the other hand, for $d \ge 3$ recall that

$$p + 1 < d^* + 1 = \frac{d+2}{d-2} + 1 = \frac{2d}{d-2}.$$

And so again, since $(q, p+1)$ differs from the endpoint admissible pair $(2, \dfrac{2d}{d-2})$, we necessarily have $q > 2$ also if $d \ge 3$.
In turn, the fact that $q > 2$ implies that the β in the above formulas is $\beta < \infty$. This implies that we can pick T small enough s.t. $(2T)^{\frac{p-1}{\beta}} a^{p-1} < 1/2$, which from (3.11) yields $\|\Phi(u)\|_T^{(1)} \le c_1\|u_0\|_{H^1} + a/2 \le a$ for $a > 2c_1\|u_0\|_{H^1}$. From these arguments, it is easy to conclude that there exists a $T(\|u_0\|_{H^1})$ s.t. for $T \in (0, T(\|u_0\|_{H^1}))$ we have $\Phi\left(E^1(T,a)\right) \subseteq E^1(T,a)$. Proceeding similarly and like in Proposition 3.4, it can be shown that there exists a $T_1(\|u_0\|_{H^1})$ s.t. for $T \in (0, T_1(\|u_0\|_{H^1}))$ the map Φ is a contraction inside $E^1(T,a)$. The Lipschitz continuity in terms of the initial data can be shown like in Proposition 3.4 and the last statement follows from the Strichartz estimates.

□

Proposition 3.7 (Conservation laws)**.** *Let $u(t)$ be a solution* (3.5) *as in Proposition 3.6. Then all the three quantities in* (3.4) *are constant in t.*

Proof. For $u \in C((-T_2, T_1), H^1(\mathbb{R}^d))$ a maximal solution of (3.5) we will show that there exists $[-T, T] \subset (-T_2, T_1)$ where $E(u(t)) = E(u(0))$, $Q(u(t)) = Q(u(0))$ and $P_j(u(t)) = P_j(u(0))$. In fact this shows that $E(u(t))$, $Q(u(t))$ and $P_j(u(t))$ are locally constant in t. Since these functions are continuous in t, the set of $t \in (-T_2, T_1)$ where $E(u(t)) = E(u(0))$ is closed in $(-T_2, T_1)$; on the other hand, it is also open in $(-T_2, T_1)$ since $E(u(t))$ is locally constant, and hence we have $E(u(t)) = E(u(0))$ for all $t \in (-T_2, T_1)$. Similarly $Q(u(t)) = Q(u(0))$ and $P_j(u(t)) = P_j(u(0))$ for all $t \in (-T_2, T_1)$.

Step 1: truncations of the NLS. For $\varphi \in C_c^\infty(\mathbb{R}, [0,1])$ a function with $\varphi = 1$ near 0 and with support contained in the ball $B_{\mathbb{R}^d}(0, r_0)$, consider [2] the operators $\mathbf{Q}_n = \varphi(\sqrt{-\triangle}/n)$. The truncations $\mathbf{Q}_n(|u|^{p-1}u)$ are locally Lipschitz functions from $H^1(\mathbb{R}^d)$ into itself as they are compositions $H^1(\mathbb{R}^d) \overset{|u|^{p-1}u}{\to} H^{-1}(\mathbb{R}^d) \overset{\mathbf{Q}_n}{\to} H^1(\mathbb{R}^d))$ of a locally Lipschitz function, Lemma 3.1, and of bounded linear maps.

[2]Notice that using everywhere the projections $\mathbf{P}_n = \chi_{[0,n]}(\sqrt{-\triangle})$ would be a bad choice for this proof. Difficulties would arise from the fact proved by C.Feffermann [6] that $\mathbf{P}_n$ for $d \ge 2$ is bounded from $L^p(\mathbb{R}^d)$ into itself only if $p = 2$. On the other hand it is elementary that the $\mathbf{Q}_n$ are of the form $\rho_{\frac{1}{n}} *$ for a $\rho \in \mathcal{S}(\mathbb{R}^d)$ and so are uniformly bounded from $L^p(\mathbb{R}^d)$ into itself for all p and form a sequence converging strongly to the identity operator.

We consider the following truncations of the NLS

$$\begin{cases} \mathrm{i}u_{nt} = -\mathbf{P}_{nr_0}\triangle u_n + \lambda \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n) \text{ for } (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u_n(0) = \mathbf{Q}_n u_0. \end{cases} \tag{3.12}$$

By the theory of ODE's, there exists a maximal solution $u_n(t) \in C^1(-T_1(n), T_2(n)), H^1(\mathbb{R}^d))$ of (3.12) . Furthermore, if $T_2(n) < \infty$ then we must have blow up

$$\lim_{t\nearrow T_2(n)} \|u_n(t)\|_{H^1} = +\infty \text{ if } T_2(n) < \infty \tag{3.13}$$

with a similar blow up phenomenon if $T_1(n) < \infty$.
To get bounds on this sequence of functions we consider invariants of motion. The following will be proved later.

Claim 3.8. The following functions are invariants of motion of (3.12):

$$\begin{aligned} &E_n(v) := \frac{1}{2}\|P_{nr_0}\nabla v\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n v|^{p+1}dx \\ &P_j(v) \text{ with } j = 1, ..., d, \\ &Q(v). \end{aligned} \tag{3.14}$$

We assume Claim 3.8 and proceed. It is easy to check that $u_n = \mathbf{P}_{nr_0}u_n$. We claim that $T_1(n) = T_2(n) = \infty$. Indeed by $Q(u_n(t)) = Q(\mathbf{Q}_n u_0) \le Q(u_0)$ we have

$$\|u_n(t)\|_{H^1} = \|\mathbf{P}_{nr_0}u_n(t)\|_{H^1} \le nr_0\|u_n(t)\|_{L^2} = nr_0\|\mathbf{Q}_n u_0\|_{L^2} \le nr_0\|u_0\|_{L^2}. \tag{3.15}$$

Let us now fix M such that $\|u_0\|_{H^1} < M$ and let us set

$$\theta_n := \sup\{\tau > 0 : \|u_n(t)\|_{H^1} < 2M \text{ for } |t| < \tau.\} \tag{3.16}$$

Our main focus is now to prove that there exists a fixed $T(M) > 0$ s.t. $\theta_n \ge T(M)$ for all n.
First of all we prove that $u_n \in C^{0,\frac{1}{2}}((-\theta_n, \theta_n), L^2)$ with a fixed Hölder constant $C(M)$. By interpolation

$$\begin{aligned} &\|u_n(t) - u_n(s)\|_{L^2} \lesssim \|u_n(t) - u_n(s)\|_{H^1}^{\frac{1}{2}}\|u_n(t) - u_n(s)\|_{H^{-1}}^{\frac{1}{2}} \\ &\le \sqrt{2}\|u_n\|_{L^\infty((-\theta_n,\theta_n),H^1)}^{\frac{1}{2}}\|u_{nt}\|_{L^\infty((-\theta_n,\theta_n),H^{-1})}^{\frac{1}{2}}\sqrt{|t-s|} \\ &\le C(M)\sqrt{|t-s|} \text{ for } t, s \in (-\theta_n, \theta_n) \end{aligned} \tag{3.17}$$

Now we want to prove

$$\|u_n(t)\|_{H^1}^2 \le \|u_0\|_{H^1}^2 + C(M)t^b \text{ for some fixed } b > 0 \text{ and for } t \in (-\theta_n, \theta_n). \tag{3.18}$$

From $E_n(u_n(t)) = E_n(\mathbf{Q}_n u_0)$ and $Q(u_n(t)) = Q(\mathbf{Q}_n u_0)$ we get

$$\|u_n(t)\|_{H^1}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1}dx = \|\mathbf{Q}_n u_0\|_{H^1}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^n} |\mathbf{Q}_n^2 u_0|^{p+1}dx.$$

Hence using Hölder and Gagliardo–Nirenberg

$$\begin{aligned}
&\|u_n(t)\|_{H^1}^2 \le \|u_0\|_{H^1}^2 + \frac{2}{p+1}\int_{\mathbb{R}^d}\Big| \ |\mathbf{Q}_n u_n(t)|^{p+1} - |\mathbf{Q}_n^2 u_0|^{p+1}\Big|\, dx \\
&\le \|u_0\|_{H^1}^2 + C\int_{\mathbb{R}^d}(|\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p)|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0|dx \\
&\le \|u_0\|_{H^1}^2 + C\||\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p\|_{L^{\frac{p+1}{p}}}\|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0\|_{L^{p+1}} \\
&\le \|u_0\|_{H^1}^2 + C_1\left(\|\mathbf{Q}_n u_n(t)\|_{L^{p+1}}^p + \|\mathbf{Q}_n^2 u_0\|_{L^{p+1}}^p\right)\|u_n(t) - \mathbf{Q}_n u_0\|_{\dot H^1}^\alpha \|u_n(t) - \mathbf{Q}_n u_0\|_{L^2}^{1-\alpha}
\end{aligned}$$

Then by (3.17) with $s = 0$, the Sobolev Embedding Theorem and (3.16) we get (3.18). Now for $T(M)$ defined s.t. $C(M)T(M)^b = 2M^2$ (for the $C(M)$ in (3.18)) from (3.18) we get

$$\|u_n(t)\|_{L^\infty([-T(M),T(M)],H^1)} \le \sqrt{3}M. \tag{3.19}$$

Since $\sqrt{3}M < 2M$ this obviously means that $T(M) < \theta_n$ since, if we had $\theta_n \le T(M)$ then, by the fact that $u_n \in C^1(\mathbb{R}, H^1)$, the definition of θ_n in (3.16) would be contradicted. Hence we have

$$\|u_n\|_{L^\infty([-T(M),T(M)],H^1)} < 2M \tag{3.20}$$

This completes step 1, up to Claim 3.8.

The proof of Claim 3.8 is rather elementary and involves applying to (3.12) $\langle \ , u_{nt}\rangle$, $\langle \ , \mathrm{i}u_n\rangle$ and $\left\langle \ , \partial_{x_j} u_n\right\rangle$ and integration by parts. We will do this now, but then we will discuss also the fact that Claim 3.8 is just a consequence of the fact that (3.12) is a hamiltonian system with hamiltonian E_n and that the invariance of Q resp. P_j just due to Nöther principle and the invariance with respect to multiplication by $e^{\mathrm{i}\vartheta}$ resp. translation.

Indeed, applying $\langle \cdot, u_{nt}\rangle$ to (3.12)

$$\begin{aligned}
0 &= -\langle \mathbf{P}_{nr_0}\triangle u_n, u_{nt}\rangle + \lambda\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), u_{nt}\rangle \\
&= -\langle \triangle u_n, u_{nt}\rangle + \lambda\langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathbf{Q}_n u_{nt}\rangle = \frac{d}{dt}E_n(u_n).
\end{aligned}$$

Notice furthermore that, by $u_n = \mathbf{P}_{nr_0}u_n$, we have

$$E_n(u_n) = \frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d}|\mathbf{Q}_n u_n|^{p+1}dx.$$

Similarly when we apply $\langle \cdot, \mathrm{i}u_n\rangle$ to (3.12) we get

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2} = -\langle \mathbf{P}_{nr_0}\triangle u_n, \mathrm{i}u_n\rangle + \lambda\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle. \tag{3.21}$$

We have to show that r.h.s. are equal to 0. We observe that the the 1st term is 0 because the bounded operator $\mathrm{i}\mathbf{P}_{nr_0}\triangle$ of $L^2(\mathbb{R}^d)$ into itself is antisymmetric: $(\mathrm{i}\mathbf{P}_{nr_0}\triangle)^* = -\mathrm{i}\mathbf{P}_{nr_0}\triangle$. For the 2nd term we use

$$\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle = \langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}\mathbf{Q}_n u_n\rangle = \lambda\,\mathrm{Re}\,\mathrm{i}\int_{\mathbb{R}^d}|\mathbf{Q}_n u_n|^{p+1}dx = 0.$$

This yields $\frac{d}{dt}Q(u_n(t)) = 0$. In a similar fashion we can prove $\frac{d}{dt}P_j(u_n(t)) = 0$.

These computations obscure somewhat the following simple facts. First of all, (3.12) and, in a somewhat formal sense also (3.1), is a hamiltonian system. First of all, the symplectic form is

$$\Omega(X,Y) := \langle \mathrm{i}X, Y\rangle \tag{3.22}$$

where

$$\langle f, g\rangle = \mathrm{Re}\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx. \tag{3.23}$$

Notice that Ω satisfies the following definition for $X = L^2(\mathbb{R}^d,\mathbb{C})$ or $X = H^1(\mathbb{R}^d,\mathbb{C})$.

Definition 3.9. Let X be a Banach space on $\mathbb{R}$ and let X' be its dual. A strong symplectic form is a 2-form ω on X s.t. $d\omega = 0$ (i.e. ω is closed) and s.t. the map $X \ni x \to \omega(x,\cdot) \in X'$ is an isomorphism.

Definition 3.10 (Gradient)**.** Let $F \in C^1(L^2(\mathbb{R}^d,\mathbb{C}),\mathbb{R})$. Then the gradient $\nabla F \in C^0(L^2(\mathbb{R}^d,\mathbb{C}), L^2(\mathbb{R}^d,\mathbb{C}))$ is defined by

$$\langle \nabla F(u), Y\rangle = dF(u)Y \text{ for all } u, Y \in L^2(\mathbb{R}^d,\mathbb{C}).$$

Notice that

$$\begin{aligned}\langle \nabla E_n(u), Y\rangle &= \frac{d}{dt}\left(\frac{1}{2}\|\mathbf{P}_{nr_0}\nabla(u+tY)\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d}|\mathbf{Q}_n(u+tY)|^{p+1}dx\right)\Big|_{t=0} \\ &= \left\langle -\mathbf{P}_{nr_0}\triangle u + \lambda\mathbf{Q}_n(|\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u), Y\right\rangle.\end{aligned} \tag{3.24}$$

We are interested in hamiltonian vector fields.

Definition 3.11 (Hamiltonian vector field)**.** Let ω be a strong symplectic form on the Banach space X and $F \in C^1(X,\mathbb{R})$. We define the Hamiltonian vector field X_F with respect to ω by

$$\omega(X_F(u), Y) := dF(u)Y \text{ for all } u, Y \in X.$$

From $\Omega(X_F, Y) = \langle \mathrm{i}X_F, Y\rangle = \langle \nabla F, Y\rangle$ we conclude $X_F = -\mathrm{i}\nabla F$. Then from (3.24) it is straightforward to conclude that (3.12) is a hamiltonian system with hamiltonian E_n.

Definition 3.12 (Poisson bracket)**.** Let ω be a strong symplectic form in a Banach space X and let $F, G \in C^1(X,\mathbb{R})$. Then the Poisson bracket $\{F, G\}$ is given by

$$\{F,G\}(u) := \omega(X_F(u), X_G(u)) = dF(u)X_G(u).$$

So, for Ω we have $\{F,G\} = \langle \nabla F, -\mathrm{i}\nabla G\rangle = \langle \mathrm{i}\nabla F, \nabla G\rangle$. Now notice that if $F \in C^1(X,\mathbb{R})$ then

$$\frac{d}{dt}\left(F(u_n(t))\right) = \langle \nabla F(u_n(t)), \dot{u}_n(t)\rangle = \langle \nabla F(u_n(t)), -\mathrm{i}\nabla E_n(u_n(t))\rangle = \{F, E_n\}|_{u_n(t)} \tag{3.25}$$

Notice now that the map $u \to e^{\mathrm{i}\vartheta} u$ leaves E_n invariant. In particular the last assertion implies that

$$\begin{aligned}0 &= \frac{d}{d\vartheta} E_n(u)\Big|_{\vartheta=0} = \frac{d}{d\vartheta} E_n(e^{\mathrm{i}\vartheta} u)\Big|_{\vartheta=0} \\ &= \langle \nabla E_n(u), \mathrm{i}u\rangle = \langle \nabla E_n(u), \mathrm{i}\nabla Q(u)\rangle = \langle \mathrm{i}\nabla Q(u), \nabla E_n(u)\rangle = \{Q, E_n\}|_u\end{aligned}$$

But then, since $\{Q, E_n\} = 0$, by (3.25) we obviously have $\frac{d}{dt}(Q(u_n(t))) = 0$.

Let us consider now, for $\{\overrightarrow{e}_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$, the transformation $(\tau_{\lambda \overrightarrow{e}_j} F)(x) := F(x - \lambda \overrightarrow{e}_j)$. Obviously E_n is invariant by this transformation and

$$\begin{aligned}0 &= \frac{d}{d\lambda} E_n(u)\Big|_{\lambda=0} = \frac{d}{d\lambda} E_n(\tau_{\lambda \overrightarrow{e}_j} u)\Big|_{\lambda=0} \\ &= -\langle \nabla E_n(u), \partial_j u\rangle = \langle \nabla E_n(u), \mathrm{i}\nabla P_j(u)\rangle = \langle \mathrm{i}\nabla P_j(u), \nabla E_n(u)\rangle = \{P_j, E_n\}|_u\end{aligned}$$

But then, since $\{P_j, E_n\} = 0$, by (3.25) we obviously have $\frac{d}{dt}(P_j(u_n(t))) = 0$.
The above argument gives a link between group actions and invariants.

Step 2: Convergence $u_n \to u$. Let us consider $I := [-T, T] \subseteq [-T(M), T(M)] \cap (-T_2, T_1)$. Obviously we have

$$u_n(t) = e^{\mathrm{i}t\triangle} \mathbf{Q}_n u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle} \mathbf{Q}_n(|\mathbf{Q}_n u_n(s)|^{p-1} \mathbf{Q}_n u_n(s)) ds.$$

Taking the difference with (3.5) we obtain

$$\begin{aligned}u(t) - u_n(t) &= e^{\mathrm{i}t\triangle}(1 - \mathbf{Q}_n) u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}(1 - \mathbf{Q}_n)|u(s)|^{p-1} u(s) ds \\ &- \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle} \mathbf{Q}_n \left(|u(s)|^{p-1} u(s) - |\mathbf{Q}_n u(s)|^{p-1} \mathbf{Q}_n u(s)\right) ds \\ &- \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle} \mathbf{Q}_n \left(|\mathbf{Q}_n u(s)|^{p-1} \mathbf{Q}_n u(s) - |\mathbf{Q}_n u_n(s)|^{p-1} \mathbf{Q}_n u_n(s)\right) ds.\end{aligned}$$

Then we have

$$\begin{aligned}&\|u - u_n\|_{L^q(I, W^{1,p+1})} + \|u - u_n\|_{L^\infty(I, H^1)} \le c_0 \|(1 - \mathbf{Q}_n) u_0\|_{H^1} + c_0 \|(1 - \mathbf{Q}_n)|u|^{p-1} u\|_{L^{q'}(I, W^{1,\frac{p+1}{p}})} \\ &+ c_0 \||u|^{p-1} u - |\mathbf{Q}_n u|^{p-1} \mathbf{Q}_n u\|_{L^{q'}(I, W^{1,\frac{p+1}{p}})} \\ &+ c_0 \||\mathbf{Q}_n u|^{p-1} \mathbf{Q}_n u - |\mathbf{Q}_n u_n|^{p-1} \mathbf{Q}_n u_n\|_{L^{q'}(I, W^{1,\frac{p+1}{p}})}.\end{aligned}$$

and so, for a fixed $\vartheta > 0$

$$\|u-u_n\|_{L^q(I,W^{1,p+1})}+\|u-u_n\|_{L^\infty(I,H^1)}\le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1}+c_0\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}$$
$$+c_0C|I|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)}+\|\mathbf{Q}_nu\|^{p-1}_{L^\infty(,H^1)}\right)\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}$$
$$+c_0C|I|^\vartheta\left(\|\mathbf{Q}_nu\|^{p-1}_{L^\infty(I,H^1)}+\|\mathbf{Q}_nu_n\|^{p-1}_{L^\infty(I,H^1)}\right)\|\mathbf{Q}_n(u-u_n)\|_{L^q(I,W^{1,p+1})}$$
$$\le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1}+c_0\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}$$
$$+c_0C|I|^\vartheta 2\|u\|^{p-1}_{L^\infty(I,H^1)}\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}$$
$$+c_0C|2T|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)}+(C(M))^{p-1}\right)\|u-u_n\|_{L^q(I,W^{1,p+1})}.$$

Then, taking T small so that $c_0C|2T|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)}+(C(M))^{p-1}\right)<1/2$ we conclude

$$\|u-u_n\|_{L^q(I,W^{1,p+1})}+\|u-u_n\|_{L^\infty(I,H^1)}\le 2c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1}+$$
$$2c_0\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}+2c_0C|I|^\vartheta 2\|u\|^{p-1}_{L^\infty(I,H^1)}\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}.$$

But now we have r.h.s.$\overset{n\to\infty}{\to}0$. Hence we have proved that there exist $T>0$ s.t.

$$\lim_{n\to+\infty}\|u-u_n\|_{L^\infty([-T,T],H^1)}=0. \tag{3.26}$$

Now, taking the limit for $n\to+\infty$ in $Q(u_n(t))=Q(\mathbf{Q}_nu_0)$ and $P_j(u_n(t))=P_j(\mathbf{Q}_nu_0)$ we obtain $Q(u(t))=Q(u_0)$ and $P_j(u(t))=P_j(u_0)$ for all $t\in[-T,T]$. Similarly, taking the limit for $n\to+\infty$ in $E_n(u_n)=E_n(\mathbf{Q}_nu_0)$ and with a little bit of work, we obtain $E(u(t))=E(u_0)$ for all $t\in[-T,T]$.

□

Corollary 3.13. *Let $u(t)$ be a solution* (3.5) *as in Proposition 3.4. Then $Q(u(t))=Q(u_0)$. In particular, the solutions in in Proposition 3.4 are globally defined.*

Proof. As above it is enough to show that $Q(u(t))=Q(u_0)$ for $t\in[-T,T]$ for some $T>0$. So let us take the T in the statement of Proposition 3.4 and let us take $T'\in(0,T)$. There exists a sequence $u_0^{(n)}\in H^1(\mathbb{R}^d,\mathbb{C})$ with $u_0^{(n)}\overset{n\to\infty}{\to}u_0$ in $L^2(\mathbb{R}^d,\mathbb{C})$. So for $n\gg1$ we have $u_0^{(n)}\in V$, the V in (3.7). In particular, for the corresponding solutions u_n we have $u^{(n)}\overset{n\to\infty}{\to}u$ in $C([-T',T'],L^2(\mathbb{R}^d))$. Then, since $Q(u^{(n)}(t))=Q(u_0^{(n)})$ for $t\in([-T',T']$, taking the limit we obtain $Q(u(t))=Q(u_0)$ for $t\in([-T',T']$. Since $T'\in(0,T)$ is arbitrary and $t\to Q(u(t))$ is continuous, we have $Q(u(t))=Q(u_0)$ for $t\in([-T,T]$. This implies that $t\to Q(u(t))$ is locally constant, and hence it is constant.

□

Remark 3.14. It can be shown that under the hypotheses of Proposition 3.6 there are unique maximal solutions to (3.5) of the type $u \in C^0((-S,T), H^1(\mathbb{R}^d))$ with $T > 0$ and $S > 0$ and with

$$\lim_{t \to T^-} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty \text{ if } T < +\infty \text{ and}$$
$$\lim_{t \to -S^+} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty \text{ if } S < +\infty.$$

Proposition 3.15 (Conservation or regularity)**.** *Let $u \in C^0((-S,T), H^1(\mathbb{R}^d))$ be a maximal solution of* (3.5)*. Suppose that the initial value satisfies $u_0 \in H^2(\mathbb{R}^d)$. Then*

$$u \in C^0((-S,T), H^2(\mathbb{R}^d)). \tag{3.27}$$

□

3.2 The global existence

We start with the following observation.

Lemma 3.16. *Let $u \in C^0((-S,T), H^1(\mathbb{R}^d))$ be a maximal solution as of Proposition 3.6. Then if $T < \infty$ we have*

$$\lim_{t \nearrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty. \tag{3.28}$$

Analogously, $\lim_{t \searrow -S} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$ *if* $S < \infty$.

Remark 3.17. Notice that it is very important for this lemma that $p < d^*$. Indeed, in the *energy* critical case $p = d^*$, the above statement is false.

Proof. Suppose by contradiction that there exists a solution with $T < \infty$ for which there is a sequence $t_j \nearrow T$ s.t. $\|u(t_j)\|_{H^1(\mathbb{R}^d)} \leq M < \infty$. Then by Proposition 3.6 one can extend $u(t)$ beyond $t_j + T(M) > T$ and get a contradiction.

□

Corollary 3.18. *If $\lambda > 0$ the solutions of Proposition 3.6 are globally defined.*

Proof. Indeed if a solution has maximal interval of existence $(-S,T)$ with $T < \infty$, we must have (3.28). But for $\lambda > 0$ we have $\|\nabla u(t)\|_{L^2} \leq 2E(u(t)) = 2E(u_0)$.

□

Corollary 3.19. *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions of Proposition 3.6 are globally defined.*

Proof. We have

$$2E(u(t)) \geq \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{2|\lambda|}{p+1} C_p^{p+1} \|\nabla u(t)\|^{\alpha(p+1)}_{L^2(\mathbb{R}^d)} \|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)} \text{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}.$$

Notice that

$$\alpha(p+1) = \frac{d}{2}(p+1) - d < 2 \Longleftrightarrow (p+1) - 2 < \frac{4}{d} \Longleftrightarrow p < 1 + \frac{4}{d}.$$

But then, if (3.28) happens, we have

$$2E(u_0) = \lim_{t\nearrow T} 2E(u(t)) \geq \lim_{t\nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \left(1 - \frac{2|\lambda|}{p+1} C_p^{p+1} \|\nabla u(t)\|^{\alpha(p+1)-2}_{L^2(\mathbb{R}^d)} \|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)}\right)$$
$$= \lim_{t\nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} = +\infty,$$

which is absurd. □

Corollary 3.20. *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions of Proposition 3.6 are globally defined.*

3.3 Local existence for the L^2 critical case

We consider now equation (3.5) for $p = 1 + \frac{4}{d}$. Notice that in this case $(p+1, p+1)$ is an admissible pair.

Theorem 3.21. *For any $u_0 \in L^2(\mathbb{R}^d)$ there exists a unique maximal solution of (3.5) with $p = 1 + \frac{4}{d}$ with*

$$u \in C([0, T^*), L^2(\mathbb{R}^d)) \cap L^{p+1}_{loc}([0, T^*), L^{p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{3.29}$$

Furthermore, the mass is preserved, we have $u \in L^a([0, T], L^b(\mathbb{R}^d))$ for any admissible pair, if $T \in (0, T^*)$.
There is continuity with respect to the initial data. And finally, if $T^* < \infty$, then

$$\lim_{T\to T^*} \|u\|_{L^a([0,T],L^b(\mathbb{R}^d))} = +\infty \text{ for any admissible pair with } b \geq p+1. \tag{3.30}$$

Proposition 3.22. *There exists a $\delta > 0$ such that if for some $T > 0$ we have*

$$\|e^{\mathrm{i}t\triangle} u_0\|_{L^{p+1}([0,T),L^{p+1}(\mathbb{R}^d))} < \delta,$$

then there exists a unique solution

$$u \in C([0, T], L^2(\mathbb{R}^d)) \cap L^{p+1}([0, T), L^{p+1}(\mathbb{R}^d)).$$

The mass is constant. Moreover, for any $T' \in (0, T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([0, T'], L^2(\mathbb{R}^d)) \cap L^{p+1}([0, T'], L^{p+1}(\mathbb{R}^d))$$

and is Lipschitz.

Finally, we have $u \in L^a([0, T], L^b(\mathbb{R}^d))$ for all admissible pairs (a, b).

Proof. The proof is a fixed point argument. We set like before

$$E(T,\delta) = \Big\{ v \in L^{p+1}([0,T], L^{p+1}(\mathbb{R}^d)) : \|v\|_{L^{p+1}([0,T],L^{p+1}(\mathbb{R}^d))} \leq 2\delta \Big\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5).
By Strichartz's estimates

$$\begin{aligned}\|\Phi(u)\|_{L^{p+1}([0,T]\times\mathbb{R}^d))} &< \delta + c_0 \||u|^{p-1}u\|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d))} \\ &= \delta + c_0\|u\|^p_{L^{p+1}([0,T]\times\mathbb{R}^d))} \leq \delta + c_0 2^p\delta^p < 2\delta,\end{aligned}$$

for $\delta > 0$ small enough, so that the map Φ preserves $E(T,\delta)$. Now we show that Φ is a contraction in $E(T,\delta)$. We have

$$\begin{aligned}&\|\Phi(u) - \Phi(v)\|_{L^{p+1}([0,T]\times\mathbb{R}^d))} \leq c_0\||u|^{p-1}u - |v|^{p-1}v\|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d))} \\ &\leq c_0 C\|(|u|^{p-1} + |v|^{p-1})|u-v|\|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d))} \\ &\leq c_0 C\left(\|u\|^{p-1}_{L^{p+1}([0,T]\times\mathbb{R}^d))} + \|v\|^{p-1}_{L^{p+1}([0,T]\times\mathbb{R}^d))}\right)\|u-v\|_{L^{p+1}([0,T]\times\mathbb{R}^d))} \\ &\leq c_0 C 2^{p-1}\delta^{p-1}\|u-v\|_{L^{p+1}([0,T]\times\mathbb{R}^d))},\end{aligned}$$

which is a contraction for $\delta > 0$ small enough. The remaining part is also similar to that in Proposition 3.4. In particular, let us now discuss the conservation of mass. The first observation is that if $u_0 \in H^1(\mathbb{R}^d)$ then we have $u \in C([0,T], H^1(\mathbb{R}^d))$. Ti orove this we observe that $u \in C([0,\tau], H^1(\mathbb{R}^d))$ by Proposition 3.6 and if it is not possible to take $\tau \geq T$, then we will have a maximal interval of existence $u \in C([0,\tau), H^1(\mathbb{R}^d))$ with $\tau \in (0,T)$ and blow up $\|\nabla u(s)\|_{H^1} \xrightarrow{s\to\tau} +\infty$. But, for $s < \tau_1 < \tau$,

$$\|\nabla u\|_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))} < \|\nabla e^{\mathrm{it}\triangle}u_0\|_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))} + c_0\|u\|^{p-1}_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))}\|\nabla u\|_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))}.$$

Now, for s close to τ we will have

$$c_0\|u\|^{p-1}_{L^{p+1}([s,\tau_1]\times\mathbb{R}^d))} < 1/2$$

and so, taking $\tau_1 \xrightarrow{\tau}$

$$\|\nabla u\|_{L^{p+1}([s,\tau]\times\mathbb{R}^d))} \leq 2\|\nabla e^{\mathrm{it}\triangle}u_0\|_{L^{p+1}([s,\tau]\times\mathbb{R}^d))}.$$

and in particular

$$\|\nabla u\|_{L^{p+1}([0,\tau]\times\mathbb{R}^d))} < +\infty.$$

Feeding this back in Strichartz inequality, we have

$$\|\nabla u\|_{L^\infty([0,\tau],L^2(\mathbb{R}^d)))} < \|\nabla u_0\|_{L^2(\mathbb{R}^d)} + c_0\|u\|^{p-1}_{L^{p+1}([0,\tau]\times\mathbb{R}^d))}\|\nabla u\|_{L^{p+1}([0,\tau]\times\mathbb{R}^d))} < +\infty,$$

which excludes the blow up $\|\nabla u(s)\|_{H^1} \xrightarrow{s\to\tau} +\infty$. So we conclude that $u \in C([0,T], H^1(\mathbb{R}^d))$ and that, energy, momenta and mass of $u(t)$ are constant in $[0,T]$. If now $u_0 \notin H^1(\mathbb{R}^d)$, we consider a sequence $u_{0n} \in H^1(\mathbb{R}^d)$ with $u_{0n} \xrightarrow{n\to\infty} u_0$ in $L^2(\mathbb{R}^d)$. For any $T' \in (0,T)$, we have by well posedness that for the corresponding solutions we have $u_n \xrightarrow{n\to\infty} u$ in $C([0,T'], L^2(\mathbb{R}^d))$. Then $Q(u_n) \xrightarrow{n\to\infty} Q(u)$ in $C([0,T'],\mathbb{R})$. Since $Q(u_n)$ are constant functions, also $Q(u)$ is constant in $[0,T']$ for all $T' < T$.

□

Proof of Theorem 3.21. Clearly we have $\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}([0,T),L^{p+1}(\mathbb{R}^d))} \xrightarrow{T\to 0^+} 0$, so we can apply Proposition 3.22 for $T > 0$ sufficiently small. There will be a maximal interval of existence. We now prove the blow up result (3.30). Suppose that it is false, and that there is a maximal solution in $[0,T^*)$ with $T^* < \infty$ and

$$\|u\|_{L^a([0,T^*),L^b(\mathbb{R}^d))} < +\infty \text{ for an admissible pair with } b \geq p+1. \tag{3.31}$$

Then if $b > p+1$, we have

$$\|u\|_{L^{p+1}([0,T^*),L^{p+1}(\mathbb{R}^d))} \leq \|u\|^{\mu}_{L^\infty([0,T^*),L^2(\mathbb{R}^d))}\|u\|^{1-\mu}_{L^a([0,T^*),L^b(\mathbb{R}^d))} \text{ for } \mu = \frac{\frac{1}{p+1}-\frac{1}{b}}{\frac{1}{2}-\frac{1}{b}}.$$

So (3.31) holds also for $b = p+1$. Now, for s close to T^* we have from (3.5)

$$e^{\mathrm{i}(t-s)\triangle}u(s) = u(t) + \mathrm{i}\lambda\int_s^t e^{\mathrm{i}(t-t')\triangle}|u(t')|^{p-1}u(t')dt'.$$

This yields

$$\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} \leq \|u\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} + C\|u\|^p_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} \xrightarrow{s<T\to T^{*-}} 0.$$

So

$$\sup_{s<T<T^*}\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} < \delta/2 \Longrightarrow \|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^{p+1}([s,T^*],L^{p+1}(\mathbb{R}^d))} \leq \delta/2$$

where we used the continuity in T of $T \to \|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))}$. Therefore by the continuity there exists $\varepsilon > 0$ small enough so that $\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^{p+1}([s,T_*+\varepsilon],L^{p+1}(\mathbb{R}^d))} < \delta$. Then the solution u can be extended beyond T^* also in the interval $[0, T_* + \varepsilon]$.

□

Example 3.23. In the case $\lambda = -1$ of the L^2- critical focusing NLS

$$\mathrm{i}u_t = -\triangle u - |u|^{\frac{4}{d}}u \text{ in } \mathbb{R}\times\mathbb{R}^d, \tag{3.32}$$

there are related solutions in $H^1\left(\mathbb{R}^d, [0,+\infty)\right)$to

$$-\Delta\phi + \phi - |\phi|^{p-1}\phi = 0. \tag{3.33}$$

In 1–d they are explicit,

$$\phi(x) = \frac{(\frac{p-1}{2}+1)^{\frac{4}{p-1}}}{\cosh^{\frac{2}{p-1}}(\frac{p-1}{2}x)}. \tag{3.34}$$

For $d \geq 2$ there are many types of solitons. For example, the ones in (3.34) are *ground states*, and they are the only ones in $d = 1$. But in $d \geq 2$ there are also excited states. Notice that if $u(t,x)$ is a solution of (3.32), then also the following is a solution,

$$v(t,x) = t^{-\frac{d}{2}}\overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)e^{\mathrm{i}\frac{x^2}{4t}}.$$

Since now, given a solution $\phi(x)$ of (3.33), then $u(t,x) = e^{\mathrm{i}t+\frac{\mathrm{i}}{2}\mathbf{v}\cdot x-\mathrm{i}\frac{\mathbf{v}^2}{4}t}\phi(x - t\mathbf{v} - D)$ is a solution of (3.32), it follows, choosing $\mathbf{v} = D = 0$, that

$$S(t,x) := t^{-\frac{d}{2}}\phi\left(\frac{x}{t}\right)e^{\mathrm{i}\frac{x^2}{4t}}e^{-\frac{\mathrm{i}}{t}} \text{ so also } S(T-t,x) := (T-t)^{-\frac{d}{2}}\phi\left(\frac{x}{T-t}\right)e^{\mathrm{i}\frac{x^2}{4(T-t)}}e^{-\frac{\mathrm{i}}{T-t}}.$$

Obviously this for $T > 0$ has maximal positive lifespan T. Then, for any admissible pair (q,r) with $r > 2$, we have

$$\|S(T-t,x)\|_{L^r(\mathbb{R}^d)} = (T-t)^{-\frac{d}{2}+\frac{d}{r}}\|\phi\|_{L^r(\mathbb{R}^d)} = (T-t)^{-\frac{2}{q}}\|\phi\|_{L^r(\mathbb{R}^d)} \notin L^q(0,T).$$

3.4 The H^1 critical cases

We consider now equation (3.5) for $p = 1 + \frac{4}{d-2}$. We will consider the admissible pair (γ, ρ)

$$\text{admissible pair } (\gamma,\rho) \text{ given by} \rho = \frac{2d^2}{d^2-2d+4}, \quad \gamma = \frac{2d}{d-2}. \tag{3.35}$$

Notice that it is an admissible pair because

$$\frac{2}{\gamma} + \frac{d}{\rho} = \frac{d}{2}.$$

Indeed

$$\frac{\not{2}}{\frac{\not{2}d}{d-2}} + \frac{\not{d}}{\frac{2d^{\not{2}}}{d^2-2d+4}} = \frac{d-2}{d} + \frac{d^2-2d+4}{2d} = 1 - \frac{2}{d} + \frac{d}{2} - 1 + \frac{2}{d} = \frac{d}{2}$$

Theorem 3.24. *For any $u_0 \in H^1(\mathbb{R}^d)$ there exists a unique maximal solution of* (3.5) *with $p = 1 + \frac{4}{d-2}$ with*

$$u \in C([0,T^*), H^1(\mathbb{R}^d)) \cap C^1([0,T^*), H^{-1}(\mathbb{R}^d)). \tag{3.36}$$

Furthermore, the mass and energy are preserved, we have $u \in L^a([0,T], W^{1,b}(\mathbb{R}^d))$ for any admissible pair, if $T \in (0,T^*)$.

There is continuity with respect to the initial data in the following sense. If $0 < T' < T^*$ and if $u_{0n} \xrightarrow{n\to\infty} u_0$ in $H^1(\mathbb{R}^d)$ then for the corresponding solutions we have we have $u_n \xrightarrow{n\to\infty} u$ in $L^p([0,T'], H^1(\mathbb{R}^d))$ for any $p < \infty$.

And finally, if $T^* < \infty$, then

$$\lim_{T\to T^*} \|u\|_{L^a([0,T],L^b(\mathbb{R}^d))} = +\infty \text{ for any admissible pair with } d > b > 2. \tag{3.37}$$

The proof of Theorem 3.24 is based on Proposition 3.25 below. In the course of the proof, we will consider admissible pairs (a, b) with $b \in (2, d)$ the number $\frac{1}{b^*} := \frac{d-b}{bd} = \frac{1}{b} - \frac{1}{d}$. Then there exists an admissible pair (α, β) such that

$$\frac{1}{\beta'} = \frac{1}{\beta} + \frac{\frac{4}{d-2}}{b^*} \text{ which can be rewritten as}$$
$$1 = \frac{2}{\beta} + \frac{4}{d-2}\left(\frac{1}{b} - \frac{1}{d}\right). \tag{3.38}$$

Here notice that for $b^* = \infty$, that is when $b = d$, then $\beta = 2$, and if $b^* = \frac{2d}{d-2}$, that is in the case $b = 2$, we have $\beta = \frac{2d}{d-2}$, which is the endpoint. So for $b \in (2, d)$ we have the intermediate cases $2 < \beta < \frac{2d}{d-2}$. We claim that the α in (α, β) satisfies

$$\frac{1}{\alpha'} = \frac{1}{\alpha} + \frac{\frac{4}{d-2}}{a} \text{ or, equivalently} \tag{3.39}$$
$$1 = \frac{2}{\alpha} + \frac{4}{d-2}\,\frac{d}{2}\,\left(\frac{1}{2} - \frac{1}{b}\right).$$

So in other words, we need to show

$$\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) = \left(\frac{1}{2} - \frac{d}{d-2}\left(\frac{1}{2} - \frac{1}{b}\right), 1 - \frac{2}{d-2}\left(\frac{1}{b} - \frac{1}{d}\right)\right) \text{ for any } b \in (2, d). \tag{3.40}$$

It is enough to check the endpoints, in fact recall that $\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ lays in a line, so it is enough to prove (3.40) just for two values of b, because then this will imply the equality for all values of b. If $b^* = \infty$, that is when $b = d$, then $\beta = 2$, which implies $\alpha = \infty$, and so (3.39) becomes

$$1 = \frac{\frac{4}{d-2}}{a} = \frac{\frac{4}{d-2}}{\frac{4}{d-2}},$$

which is obviously correct.

Looking at $b = 2$, then as we mentioned, we have the endpoint $(\alpha, \beta) = \left(2, \frac{2d}{d-2}\right)$, which makes (3.39) true because $\alpha' = 2$ and $a = 0$.

It is interesting to check when $(a, b) = (\alpha, \beta)$ we obtain exactly the admissible pair in (3.35). Indeed,

$$1 = \frac{2}{a} + \frac{\frac{4}{d-2}}{a} \Longleftrightarrow a = 2 + \frac{4}{d-2} = \frac{2d}{d-2} = \gamma.$$

Finally, since the map $\frac{1}{a} \to \frac{1}{\alpha}$ in (3.39) is affine and $\frac{1}{\gamma}$ is a fixed point, in any case when $a \neq \alpha$ it follows that γ is in between them, and so also ρ is in between b and β and that

$$\text{there exists a } \theta \in (0,1) \text{ with } \left(\frac{1}{\gamma}, \frac{1}{\rho}\right) = \theta \left(\frac{1}{a}, \frac{1}{b}\right) + (1-\theta)\left(\frac{1}{\alpha}, \frac{1}{\beta}\right). \tag{3.41}$$

Proposition 3.25. *There exists a $\delta > 0$ such that if for some $T > 0$ we have*

$$\|e^{\mathrm{i}t\triangle} u_0\|_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))} < \delta,$$

then there exists a unique solution

$$u \in C([0,T], H^1(\mathbb{R}^d)) \cap L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d)).$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([0,T'], L^2(\mathbb{R}^d)) \cap L^\gamma([0,T'), W^{1,\rho}(\mathbb{R}^d)$$

and is Lipschitz.

Finally, we have $u \in L^a([0,T], W^{1,b}(\mathbb{R}^d))$ for all admissible pairs (a,b) and mass and energy are preserved.

Proof (sketch). The proof is by a contraction argument. We set like before

$$E(T,\delta) = \left\{ v \in L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d)) : \|v\|_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))} \le 2\delta \right\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5). Let us open a small parenthesis now, and let us pick an admissible pair (a,b) with $b \in (2,d)$. Then, for $\frac{1}{b^*} = \frac{d-b}{bd} = \frac{1}{b} - \frac{1}{d}$ and (α, β) admissible like in (3.39), by Strichartz estimates, by the Chain Rule in Lemma 3.1 and by $p - 1 = \frac{4}{d-2}$, we have

$$\begin{aligned}
&\|\Phi(u)\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} \le \|e^{\mathrm{i}t\triangle} u_0\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} + c_0 \|u^{p-1} \langle \nabla \rangle u\|_{L^{\alpha'}([0,T), W^{1,\beta'}(\mathbb{R}^d))} \\
&\le \|e^{\mathrm{i}t\triangle} u_0\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} + c_0 \|u\|^{p-1}_{L^a([0,T], L^{b^*})} \|u\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} \\
&\le \|e^{\mathrm{i}t\triangle} u_0\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))} + c_0' \|u\|^{p-1}_{L^a([0,T], W^{1,b})} \|u\|_{L^\alpha([0,T), W^{1,\beta}(\mathbb{R}^d))}
\end{aligned}$$

So, in the particular case $(a,b) = (\alpha, \beta) = (\rho, \gamma)$, we have

$$\|\Phi(u)\|_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))} \le \|e^{\mathrm{i}t\triangle} u_0\|_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))} + c_0' \|u\|^p_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))}$$

Hence in $E(T,\delta)$ we have

$$\|\Phi(u)\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} < \delta + c_0' 2^p \delta^p < 2\delta,$$

for $\delta > 0$ small enough, so that the map Φ preserves $E(T,\delta)$. In a similar fashion we prove that Φ is a contraction in $E(T,\delta)$. We skip the proof on the conservation of mass, energy and momenta.

Proof of Theorem 3.24. Clearly we have $\|e^{\mathrm{i}t\triangle}u_0\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} \xrightarrow{T\to 0^+} 0$, so we can apply Proposition 3.25 for $T > 0$ sufficiently small. There will be a maximal interval of existence. We now prove the blow up result (3.37). Suppose that it is false, and that there is a maximal solution in $[0,T^*)$ with $T^* < \infty$ and

$$\|u\|_{L^a([0,T^*),W^{1,b}(\mathbb{R}^d))} < +\infty. \tag{3.42}$$

But then

$$\|u\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} \le \|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} + c_0'\|u\|^{p-1}_{L^a([0,T],W^{1,b})}\|u\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))}$$

and the fact that $\|u\|^p_{L^a([s,T],W^{1,b}(\mathbb{R}^d))} \xrightarrow{s<T\to T^{*-}} 0$, implies

$$\|u\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} \le 2\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))}$$

for $s < T < T^*$ with s and T close to T^*. This implies in fact that also

$$\|u\|_{L^\alpha([0,T^*),W^{1,\beta}(\mathbb{R}^d))} < +\infty. \tag{3.43}$$

Then, by

$$e^{\mathrm{i}(t-s)\triangle}u(s) = u(t) + \mathrm{i}\lambda\int_s^t e^{\mathrm{i}(t-t')\triangle}|u(t')|^{p-1}u(t')dt',$$

$$\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} \le \|u\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} + c_0'\|u\|^{p-1}_{L^a([s,T],W^{1,b})}\|u\|_{L^\alpha([s,T),W^{1,\beta}(\mathbb{R}^d))}$$
$$\xrightarrow{s<T\to T^{*-}} 0.$$

Since there exists a $\theta\in[0,1]$ with the following, see (3.41),

$$\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\gamma([s,T_*+\varepsilon],W^{1,\rho}(\mathbb{R}^d))} \le \|e^{\mathrm{i}(t-s)\triangle}u(s)\|^\theta_{L^a([s,T],W^{1,b}(\mathbb{R}^d))}\|e^{\mathrm{i}(t-s)\triangle}u(s)\|^{1-\theta}_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))}$$

it follows that we can arrange $\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\gamma([s,T_*+\varepsilon],W^{1,\rho}(\mathbb{R}^d))} < \delta$, for s close enough to T^* and for $\varepsilon > 0$ arbitrarily small. But then the solution u can be extended beyond T^*.

We skip here the discussion of the well posedness.

□

4 The dispersive equation

Here we will consider dispersive equations

$$\begin{cases} \mathrm{i}u_t = -\triangle u + |u|^{p-1}u \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{4.1}$$

with $1 + 4/d < p < d^*$. In this §we will give a partial proof of the following classical result.

Theorem 4.1 (Scattering)**.** *Consider the unique solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$. Then*

$$u \in L^a(\mathbb{R}, W^{1,b}(\mathbb{R}^d)) \textit{ for any admissible pair} \tag{4.2}$$

and there exist $u_\pm \in H^1(\mathbb{R}^d)$ s.t.

$$\lim_{t\to\pm\infty} \|u(t) - e^{\mathrm{i}t\triangle}u_\pm\|_{H^1(\mathbb{R}^d)} = 0. \tag{4.3}$$

Remark 4.2. Scattering (the *completeness of scattering operators*) refers specifically to (4.3). Notice that for $1 < p \le 1 + 2/d$ Scattering (4.3) is false. For $1 + 2/d < p \le 1 + 4/d$ is an open problem.

Here the key deep statement is (4.2). In fact, (4.2) implies easily (4.3), as we show now in the case +. So, assume (4.2), and in particular let

$$u \in L^q(\mathbb{R}_+, W^{1,p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{4.4}$$

From (3.5) with $\lambda = 1$, we have

$$e^{-\mathrm{i}t\triangle}u(t) = u_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}s\triangle}|u(s)|^{p-1}u(s)ds,$$

so that, for $t_1 < t_2$, we have

$$e^{-\mathrm{i}t_2\triangle}u(t_2) - e^{-\mathrm{i}t_1\triangle}u(t_1) = -\mathrm{i}\int_{t_1}^{t_2} e^{-\mathrm{i}s\triangle}|u(s)|^{p-1}u(s)ds.$$

Then

$$\begin{aligned} \|e^{-\mathrm{i}t_2\triangle}u(t_2) - e^{-\mathrm{i}t_1\triangle}u(t_1)\|_{H^1} &\le \left\| \int_{t_1}^{t_2} e^{-\mathrm{i}s\triangle}|u(s)|^{p-1}u(s)ds \right\|_{H^1} \\ &\le \|u\|^{p-1}_{L^\alpha([t_1,t_2],L^{p+1})}\|u\|_{L^q([t_1,t_2],W^{1,p+1})} \end{aligned} \tag{4.5}$$

where $\frac{p-1}{\alpha} + \frac{1}{q} = \frac{1}{q'}$. We claim that $\alpha > q$. Otherwise $\alpha \le q$ and so

$$\frac{p}{q} \le \frac{1}{q'} \Leftrightarrow p + 1 \le q.$$

So, from $p > 1 + \frac{4}{d}$, $(q, p+1)$ is an admissible pair with both entries $> 2 + \frac{4}{d}$. But $\left(2 + \frac{4}{d}, 2 + \frac{4}{d}\right)$ is an admissible pair, so we get an absurd and we conclude $\alpha > q$.
So, let us consider the pair (α, β) which is admissible (notice that $\alpha > q > 2$ implies $\infty > \alpha > 2$ and so $2 < \beta < p+1$. We claim that

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} \text{ with } \tau \in [0,1]. \tag{4.6}$$

Assuming this, (4.5) can be majorized yielding

$$\|e^{-\mathrm{i}t_2\triangle}u(t_2) - e^{-\mathrm{i}t_1\triangle}u(t_1)\|_{H^1} \le c_0 \|u\|^{p-1}_{L^\alpha([t_1,t_2],W^{1,\beta})} \|u\|_{L^q([t_1,t_2],W^{1,p+1})} \xrightarrow{t_1<t_2\to+\infty} 0.$$

This implies that there exists

$$u_+ = \lim_{t\to+\infty} e^{-\mathrm{i}t\triangle}u(t) \text{ in } H^1(\mathbb{R}^d).$$

Then we have

$$e^{\mathrm{i}t\triangle}u_+ - u(t) = -\mathrm{i}\int_t^\infty e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds.$$

As above,

$$\|e^{\mathrm{i}t\triangle}u_+ - u(t)\|_{H^1} \le \|u\|^{p-1}_{L^\alpha([t,\infty),W^{1,\beta})} \|u\|_{L^q([t,\infty),W^{1,p+1})} \xrightarrow{t\to+\infty} 0,$$

which proves the limit (4.3).
Turning to the proof of (4.6), obviously $\alpha > q$ implies $\beta < p+1$ so that

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d}$$

with $\tau > 0$. Since $2 < \beta < p+1 < +\infty$, for $d = 1, 2$ we have $\tau < 1$. For $d \ge 3$ we have $2 < \beta < p+1 < \frac{2d}{d-2}$. Since $\frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$,

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} > \frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$$

which implies $0 < \tau < 1$ by

$$\frac{1-\tau}{d} > \frac{1}{2} - \frac{1}{\beta}.$$

As we indicated above, in Theorem 4.1, the deep statement is (4.2). The proof is rather complicated. For this we will need the following which we will discuss only for dimension $d \ge 3$.

Theorem 4.3. *Let $d \ge 3$. Then given a solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$ we have*

$$\lim_{t\to\pm\infty} \|u(t)\|_{L^r(\mathbb{R}^d)} = 0 \textit{ for all } 2 < r < \frac{2d}{d-2}. \tag{4.7}$$

Remark 4.4. Notice that it is enough to prove only case $r = p+1$. In fact, for $2 < r < p+1$ there is an exponent $\alpha \in (0,1)$ with

$$\|u(t)\|_{L^r(\mathbb{R}^d)} \leq \|u_0\|^{\alpha}_{L^2(\mathbb{R}^d)} \|u(t)\|^{1-\alpha}_{L^{p+1}(\mathbb{R}^d)}$$

which yields (4.7) while for $p+1 < r < \frac{2d}{d-2}$ there is an exponent $\alpha \in (0,1)$ with

$$\begin{aligned}\|u(t)\|_{L^r(\mathbb{R}^d)} &\leq \|u(t)\|^{\alpha}_{L^{p+1}(\mathbb{R}^d)} \|u(t)\|^{1-\alpha}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq \|u(t)\|^{\alpha}_{L^{p+1}(\mathbb{R}^d)} \|u(t)\|^{1-\alpha}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \\ &\leq \|u(t)\|^{\alpha}_{L^{p+1}(\mathbb{R}^d)} (2E(u_0))^{1-\alpha}\end{aligned}$$

which again yields (4.7).

Theorem 4.3 is deep result and implies (4.2) rather easily as we see now. We will use the following elementary lemma.

Lemma 4.5. *consider a function $f(x) = a - x + bx^{\alpha}$ for $x \geq 0$, $a, b > 0$, $\alpha > 1$. We assume that there are $0 < x_0 < x_1$ s.t. $f(x_0) = f(x_1) = 0$. Let now $\phi \in C(I, [0, +\infty))$ be such that $\phi(t) \leq a + b\phi^{\alpha}(t)$ for all $t \in I$ and that there exists a point $t_0 \in I$ s.t. $\phi(t_0) \leq x_0$. Then $\phi(t) \leq x_0$ for all $t \in I$*

Proof. Since $f(\phi(t)) \geq 0$ for all t, and ϕ is continuous, the image of ϕ is either in $[0, x_0]$ or in $[x_1, +\infty)$. Obviously, the first case needs to occur. □

Proof that Theorem 4.3 implies (4.2) *(sketch).* Consider

$$u(t) = e^{\mathrm{i}(t-S)\triangle} u(S) - \mathrm{i} \int_S^t e^{\mathrm{i}(t-s)\triangle} |u(s)|^{p-1} u(s) ds,$$

Then by the Strichartz estimates

$$\begin{aligned}\|u\|_{L^q((S,t),W^{1,p+1})} &\leq C\|u(S)\|_{H^1} + C \left\| \|u\|^{p-1}_{L^{p+1}_x} \|u\|_{W^{1,p+1}_x} \right\|_{L^{q'}(S,t)} \\ &= C\|u(S)\|_{H^1} + C \left(\int_S^t \|u\|^{(p-1)q'-(q-q')}_{L^{p+1}_x} \|u\|^{q-q'}_{L^{p+1}_x} \|u\|^{q'}_{W^{1,p+1}_x} ds \right)^{\frac{1}{q'}} \\ &\leq C\|u(S)\|_{H^1} + C\|u\|^{p-\frac{q}{q'}}_{L^{\infty}((S,t),L^{p+1}_x)} \|u\|^{\frac{q}{q'}}_{L^q([S,t],W^{1,p+1})}.\end{aligned}$$

Here

$$p - \frac{q}{q'} = p + 1 - q > 0 \Leftrightarrow p > 1 + 4/d.$$

From Theorem 4.3, applied to $r = p+1$, we know $\|u\|^{p-\frac{q}{q'}}_{L^{\infty}((S,t),L^{p+1}_x)} \xrightarrow{S \to +\infty} 0$. Furthermore, using conservation of mass and energy, there is a uniform upper bound for $\|u(S)\|_{H^1}$. There exists a constant $C_0 > 0$ s.t. for any $\epsilon > 0$ there is $S_0 > 0$ such that for any $S_0 < S < t$,

$$\|u\|_{L^q((S,t),W^{1,p+1})} \leq C_0 + \epsilon \|u\|^{\frac{q}{q'}}_{L^q([S,t],W^{1,p+1})}.$$

Picking $\epsilon > 0$ sufficiently small, by Lemma 4.5 we conclude that there exists a fixed constant X_0 s.t.

$$\|u\|_{L^q((S,t),W^{1,p+1})} \le X_0 \text{ for any } S_0 < S < t.$$

In particular we can take $t = \infty$. Since we know that $u \in L^q_{loc}(\mathbb{R}, W^{1,p+1})$, we conclude that $\|u\|_{L^q(\mathbb{R}_+,W^{1,p+1})} < +\infty$. Time reversibility of the NLS, yields the same result for negative times. The Strichartz estimates, yield $u \in L^a(\mathbb{R}, W^{1,b})$ for any admissible pair. Like in (3.10), we have

$$\|u\|_{L^a((S,t),W^{1,b})} \le c_0\|u(S)\|_{H^1} + c_0\|u\|^{p-1}_{L^\alpha((S,t),L^{p+1})}\|u\|_{L^q((S,t),W^{1,p+1})}$$

where $\frac{p-1}{\alpha} + \frac{1}{q} = \frac{1}{q'}$ with here $\alpha > q$ by the discussion in the proof of (3.8). So now let (α, β) be an admissible pair. We have $W^{1,\beta}(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$, so, up to a change of constants, we get

$$\|u\|_{L^a((S,t),W^{1,b})} \le c_0\|u(S)\|_{H^1} + c_0\|u\|^{p-1}_{L^\alpha((S,t),W^{1,\beta})}\|u\|_{L^q((S,t),W^{1,p+1})} \tag{4.8}$$

and in particular

$$\|u\|_{L^\alpha((S,t),W^{1,\beta})} \le c_0\|u(S)\|_{H^1} + c_0\|u\|^{p-1}_{L^\alpha((S,t),W^{1,\beta})}\|u\|_{L^q((S,t),W^{1,p+1})}. \tag{4.9}$$

If in (4.9) we have $p \le 2$, then since $\|u\|_{L^q((S,t),W^{1,p+1})} \xrightarrow{S\to+\infty} 0$ the factor $\|u\|_{L^\alpha((S,t),W^{1,\beta})}$ remains bounded for $t \to +\infty$ if $S \gg 1$. If instead $p - 1 > 1$ we can apply Lemma 4.5. So we conclude that in all cases $\|u\|_{L^\alpha((S,t),W^{1,\beta})}$ remains bounded for $t \to +\infty$ if $S \gg 1$. Inserting this information in (4.8), we get the same conclusion for $\|u\|_{L^a((S,t),W^{1,b})}$.

5 Proof of Theorem 4.3

Lemma 5.1. *Let $p \in [1, \infty)$ and $q < d$ with $0 \le q \le p$. Then we have*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q} dx \le \left(\frac{p}{d-q}\right)^q \|u\|^{p-q}_{L^p(\mathbb{R}^d)}\|\nabla u\|^q_{L^p(\mathbb{R}^d)}. \tag{5.1}$$

Proof. The general case $u \in W^{1,p}(\mathbb{R}^d)$ reduces to the special case $u \in C^\infty_c(\mathbb{R}^d)$. In fact, if (5.1) is valid for all $u \in C^\infty_c(\mathbb{R}^d)$, then for a $u \in W^{1,p}(\mathbb{R}^d)$ with $u \notin C^\infty_c(\mathbb{R}^d)$, we can consider a sequence $C^\infty_c(\mathbb{R}^d) \ni u_n \xrightarrow{n\to+\infty} u$ in $W^{1,p}(\mathbb{R}^d)$. Then, up to subsequence, we have $u_n(x) \xrightarrow{n\to+\infty} u(x)$ for a.a. $x \in \mathbb{R}^d$, see p. 94 [2]. Then, by Fathou's Lemma

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q} dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} \frac{|u_n(x)|^p}{|x|^q} dx$$
$$\le \lim_{n\to\infty} \left(\frac{p}{d-q}\right)^q \|u_n\|^{p-q}_{L^p(\mathbb{R}^d)}\|\nabla u_n\|^q_{L^p(\mathbb{R}^d)} = \left(\frac{p}{d-q}\right)^q \|u\|^{p-q}_{L^p(\mathbb{R}^d)}\|\nabla u\|^q_{L^p(\mathbb{R}^d)}.$$

So we will prove (5.1) for $u \in C_c^\infty(\mathbb{R}^d)$. Let $z(x) := |x|^{-q}x$. Then

$$\nabla \cdot z = \nabla(|x|^{-q}) \cdot x + |x|^{-q}\nabla \cdot x = -q|x|^{-q-1}\frac{x}{|x|} \cdot x + d|x|^{-q} = (d-q)|x|^{-q}.$$

Integrating the identity

$$|u|^p \nabla \cdot z = \nabla \cdot (|u|^p z) - p|u|^{p-1}\nabla|u| \cdot z,$$

we obtain for arbitrary $r > 0$

$$(d-q)\int_{|x|>r} \frac{|u(x)|^p}{|x|^q}dx = \int_{|x|>r} \nabla \cdot (|u|^p z)\,dx - p\int_{|x|>r} |u|^{p-1}\nabla|u| \cdot z dx$$
$$\le -p\int_{|x|>r} |u|^{p-1}\nabla|u| \cdot z dx \le p\int_{|x|>r} \frac{|u|^{p-1}|\nabla u|}{|x|^{q-1}}dx,$$

where we used

$$\int_{|x|>r} \nabla \cdot (|u|^p z)\,dx = -\int_{|x|=r} |u|^p z \cdot \frac{x}{|x|}dS = -\int_{|x|=r} |u|^p |x|^{-q+1}dS \le 0.$$

Using $1 - \frac{1}{q} + \frac{p-q}{pq} + \frac{1}{p} = 1$ and Hölder inequality, we have

$$p\int_{|x|>r} \frac{|u|^{p-1}|\nabla u|}{|x|^{q-1}}dx = p\int_{|x|>r} \frac{|u|^{\frac{p(q-1)}{q}}}{|x|^{q-1}}|u|^{\frac{p-q}{q}}|\nabla u|dx$$
$$\le p\left(\int_{|x|>r} \frac{|u|^p}{|x|^q}dx\right)^{\frac{q-1}{q}} \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p-q}{q}}\|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

This yields

$$\int_{|x|>r} \frac{|u(x)|^p}{|x|^q}dx \le \left(\frac{p}{d-q}\right)^q \|u\|_{L^p(\mathbb{R}^d)}^{p-q}\|\nabla u\|_{L^p(\mathbb{R}^d)}^q$$

and, taking $r \to 0^+$, we obtain (5.1).

□

Lemma 5.2. *For $d \ge 4$ there exists a C_d s.t. we have*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^3}dx \le C_d\|u\|_{H^2(\mathbb{R}^d)}^2. \tag{5.2}$$

Proof. We proceed as above for $q = 3$ and $p = 2$, to obtain

$$(d-3)\int_{|x|>r} \frac{|u(x)|^2}{|x|^3}dx \le -p\int_{|x|>r} |u|^{p-1}\nabla|u| \cdot z dx \le 2\int_{|x|>r} \frac{|u||\nabla u|}{|x|^2}dx$$
$$\le 2\left(\int_{|x|>r} \frac{|u|^2}{|x|^2}dx\right)^{\frac{1}{2}} \left(\int_{|x|>r} \frac{|\nabla u|^2}{|x|^2}dx\right)^{\frac{1}{2}}.$$

In the 2nd line we apply (5.1) for $p = q = 2$ to both u and ∇u, to obtain

$$(d-3)\int_{|x|>r}\frac{|u(x)|^2}{|x|^3}dx \le 2\left(\int_{|x|>r}\frac{|u|^2}{|x|^2}dx\right)^{\frac12}\left(\int_{|x|>r}\frac{|\nabla u|^2}{|x|^2}dx\right)^{\frac12} \le 2\left(\frac{2}{d-2}\right)\|\nabla u\|_{L^2(\mathbb{R}^d)}\|\nabla^2 u\|_{L^2(\mathbb{R}^d)}$$

Then (5.2) follows sending $r \to 0$. □

Let $u_0 \in H^2$. Then $u \in C^0([0,T), H^2)$ by the theory by Kato. Then equation (4.1) holds also in a differential sense as

$$\mathrm{i}u_t = -\triangle u + |u|^{p-1}u \text{ in } \mathcal{D}'\left((0,T), L^2(\mathbb{R}^d,\mathbb{C})\right).$$

Notice that $u \in C^1([0,T), L^2)$. Let us now consider the quadratic form

$$\frac12\left\langle \mathrm{i}\left(\partial_r + \frac{d-1}{2r}\right)u, u\right\rangle. \tag{5.3}$$

Notice that it is well defined and self–adjoint. Then, taking the derivative for $u \in C^0([0,T), H^2)\cap C^1([0,T), L^2)$ we have

$$\frac{d}{dt}2^{-1}\left\langle \mathrm{i}\left(\partial_r + \frac{d-1}{2r}\right)u, u\right\rangle = -\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, \mathrm{i}\dot{u}\right\rangle.$$

which can be proved assuming first $u \in C^\infty([0,T), H^2)$ and then proceeding by a density argument. In our case we get

$$\begin{aligned}&\frac{d}{dt}2^{-1}\left\langle \mathrm{i}\left(\partial_r + \frac{d-1}{2r}\right)u, u\right\rangle = \\ &\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, -\mathrm{i}\dot{u}\right\rangle = -\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, -\triangle u + |u|^{p-1}u\right\rangle.\end{aligned} \tag{5.4}$$

The equality (5.4) is crucial, indeed we will use it to prove

$$\frac{d}{dt}\langle \partial_r u, \mathrm{i}u\rangle \ge (d-1)\frac{p-1}{p+1}\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r}dx, \tag{5.5}$$

which tells us that $u \to \langle \partial_r u, \mathrm{i}u\rangle$ is some sort of Lyapunov functional and is crucial in our argument.

The first observation to obtain (5.5), is that the following is true,

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right)u, \mathrm{i}\dot{u}\right\rangle = \frac12\frac{d}{dt}\langle \partial_r u, \mathrm{i}u\rangle. \tag{5.6}$$

Indeed, notice that

$$\begin{aligned}&\frac12\partial_t \operatorname{Re}(\mathrm{i}u\overline{u}_r) + \frac12\nabla\cdot\left(\frac{x}{r}\operatorname{Re}(\mathrm{i}\dot{u}\overline{u})\right)\\ &= \frac12\operatorname{Re}(\mathrm{i}\dot{u}\overline{u}_r) + \cancel{\frac12\operatorname{Re}(\mathrm{i}u\dot{\overline{u}}_r)} + \frac12\left(\nabla\cdot\frac{x}{r}\right)\operatorname{Re}(\mathrm{i}\dot{u}\overline{u}) + \cancel{\frac12\operatorname{Re}(\mathrm{i}\dot{u}_r\overline{u})} + \frac12\operatorname{Re}(\mathrm{i}\dot{u}\overline{u}_r)\\ &= \operatorname{Re}(\mathrm{i}\dot{u}\overline{u}_r) + \frac{d-1}{2r}\operatorname{Re}(\mathrm{i}\dot{u}\overline{u}),\end{aligned}$$

so that integrating in x we obtain exactly (5.6).
The next step to prove (5.5), is the following inequality.

Claim 5.3. Let $u \in H^2(\mathbb{R}^d, \mathbb{C})$. Then

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle \leq 0. \tag{5.7}$$

Proof. The proof is based on the identity

$$\mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} = \nabla \cdot \mathrm{Re}\left\{\nabla u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} - \nabla \cdot \left\{\frac{x}{2r}|\nabla u|^2\right\} \tag{5.8}$$
$$+ \nabla \cdot \left(\frac{d-1}{4}\ \frac{x}{r^3}|u|^2\right) - \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) - \frac{(d-1)(d-3)}{4r^3}|u|^2,$$

which we check now. We have

$$\nabla \cdot \mathrm{Re}\left\{\nabla u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} = \mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \mathrm{Re}\left\{\partial_j u \partial_j \left(\frac{x_k}{r}\partial_k \overline{u}\right)\right\} + \mathrm{Re}\left\{\partial_j u \partial_j \left(\frac{d-1}{2r}\overline{u}\right)\right\}$$
$$= \mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \frac{x_k}{2r}\partial_k |\nabla u|^2 + \frac{1}{r}|\nabla u|^2 - \mathrm{Re}\left\{\frac{x_k x_j}{r^3}\partial_j u \partial_k \overline{u}\right\} + \frac{d-1}{2r}|\nabla u|^2$$
$$- \frac{d-1}{2}\frac{x_j}{r^3}\,\mathrm{Re}\left\{\partial_j u \overline{u}\right\}$$
$$= \mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \partial_k \left(\frac{x_k}{2r}|\nabla u|^2\right) - |\nabla u|^2 \partial_k \left(\frac{x_k}{2r}\right) + \frac{|\nabla u|^2 - |u_r|^2}{r} + \frac{d-1}{2r}|\nabla u|^2$$
$$- \partial_j \left(\frac{d-1}{4}\frac{x_j}{r^3}|u|^2\right) - \frac{d-1}{4}|u|^2 \partial_j \left(\frac{x_j}{r^3}\right).$$

Now we use

$$\partial_k \left(\frac{x_k}{2r}\right) = \frac{d-1}{2r}$$
$$\partial_j \left(\frac{x_j}{r^3}\right) = \frac{d-3}{r^3},$$

to conclude

$$\nabla \cdot \mathrm{Re}\left\{\nabla u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} =$$
$$= \mathrm{Re}\left\{\triangle u \left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \partial_k \left(\frac{x_k}{2r}|\nabla u|^2\right) + \frac{|\nabla u|^2 - |u_r|^2}{r}$$
$$- \partial_j \left(\frac{d-1}{4}\frac{x_j}{r^3}|u|^2\right) - \frac{(d-1)(d-3)}{4r^3}|u|^2,$$

which is (5.8). Now, applying the Divergence Theorem to (5.8) in $\mathbb{R}^d \backslash D_{\mathbb{R}^d}(0,a)$ and take the limit for $a \to 0$ and Lemma 5.1, we have

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle \leq -\int_{\mathbb{R}^d} \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) dx - \frac{(d-1)(d-3)}{4} \lim_{a\to 0^+} \int_{r\geq a} \frac{|u|^2}{r^3} dx$$
$$-\liminf_{a\to 0^+} \int_{r=a} \left[\mathrm{Re}\left\{u_r\left(\overline{u}_r + \frac{d-1}{2a}\overline{u}\right)\right\} - \frac{|\nabla u|^2}{2} + \frac{d-1}{4}\frac{|u|^2}{a^2}\right] dS.$$

Let us now suppose that $u \in C^\infty(\mathbb{R}^d, \mathbb{C})$. Then

$$\lim_{a\to 0^+} \int_{\partial B(x,a)} |\nabla u|^2 dS = 0$$

Similarly, for $d > 3$ and $u \in C^\infty(\mathbb{R}^d, \mathbb{C})$ we have

$$\lim_{a\to 0^+} \int_{r=a} \frac{|u|^2}{a^2} dS = 0$$

Hence, for $d > 3$ and $u \in C^\infty(\mathbb{R}^d, \mathbb{C})$ we obtain 5.1, we have

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle \leq -\int_{\mathbb{R}^d} \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) dx - \frac{(d-1)(d-3)}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{r^3} dx \leq 0. \tag{5.9}$$

For $u \in H^2(\mathbb{R}^d, \mathbb{C})$ and, $u \notin C^\infty(\mathbb{R}^d, \mathbb{C})$ considered a sequence $u_n \xrightarrow{n\to\infty} u$ in $H^2(\mathbb{R}^d, \mathbb{C})$, we have

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u_n, \triangle u_n \right\rangle = -\int_{\mathbb{R}^d} \frac{1}{r}\left(|\nabla u_n|^2 - |u_{nr}|^2\right) dx - \frac{(d-1)(d-3)}{4} \int_{\mathbb{R}^d} \frac{|u_n|^2}{r^3} dx$$

which in the limit converges to (5.9).
For $d = 3$ then $u \in C^0(\mathbb{R}^3)$ and so

$$\lim_{a\to 0^+} \int_{\partial B(0,a)} |u|^2 \frac{dS}{a^2} = 4\pi |u(0)|^2,$$

so that we obtain

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle = -\int_{\mathbb{R}^3} \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) dx - 2\pi |u(0)|^2.$$

□

The next step to prove inequality (5.5) is the following identity,

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, |u|^{p-1}u \right\rangle = \frac{d-1}{2}\frac{p-1}{p+1} \int \frac{|u|^{p+1}}{r}. \tag{5.10}$$

Indeed

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, |u|^{p-1}u \right\rangle = \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} + \frac{1}{2} \int_{\mathbb{R}^d} (|u|^2)^{\frac{p-1}{2}} \partial_r |u|^2 dx$$
$$= \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} + \frac{1}{2} \frac{2}{p+1} \int_{\mathbb{R}^d} \partial_r (|u|^2)^{\frac{p+1}{2}} dx$$
$$= \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} - \frac{d-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} = \frac{d-1}{2} \frac{p-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r}.$$

So now we can prove (5.5). Indeed, from (5.6), (5.4), (5.6) and (5.10), we obtain

$$-\frac{1}{2}\frac{d}{dt} \langle \partial_r u, \mathrm{i}u \rangle = \left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, -\mathrm{i}\dot{u} \right\rangle = -\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, -\triangle u + |u|^{p-1}u \right\rangle$$
$$\leq -\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, |u|^{p-1}u \right\rangle = -\frac{d-1}{2} \frac{p-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r},$$

which yields (5.5).

Lemma 5.4. *We have*

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} \leq \frac{2}{d-1} \frac{p+1}{p-1} \|u_0\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))} \leq \frac{2^{\frac{3}{2}}}{d-1} \frac{p+1}{p-1} \|u_0\|_{L^2(\mathbb{R}^d)} E(u_0). \tag{5.11}$$

furthermore, we have $u(t) \overset{t\to\infty}{\rightharpoonup} 0$ *in* $H^1(\mathbb{R}^d)$.

Proof. To get (5.11) if $u_0 \in H^2(\mathbb{R}^d)$ it is enough to integrate (5.5). The general case follows by density, because if $H^2 \ni u_{0n} \xrightarrow{n\to+\infty} u_0$ in H^1, we know that for any $T > 0$ for the corresponding solutions $u_n \xrightarrow{n\to+\infty} u$ in $C^0([0,T], H^1)$. Then, by the density argument in Lemma 5.1, we have

$$\int_0^T dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} \leq \liminf_{n\to\infty} \int_0^T dt \int_{\mathbb{R}^d} \frac{|u_n|^{p+1}}{r} \leq \frac{2}{d-1} \frac{p+1}{p-1} \|u_{0n}\|_{L^2(\mathbb{R}^d)} \|\nabla u_n\|_{L^\infty([0,T], L^2(\mathbb{R}^d))}$$
$$\xrightarrow{n\to+\infty} \frac{2}{d-1} \frac{p+1}{p-1} \|u_0\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^\infty([0,T], L^2(\mathbb{R}^d))}.$$

Taking the limit for $T \to +\infty$ we obtain (5.11) with $\mathbb{R}$ replaced by $\mathbb{R}_+$, which by time reversibility yields also the general case.
To get $u(t) \overset{t\to\infty}{\rightharpoonup} 0$ in $H^1(\mathbb{R}^d)$ it is enough to show $\langle u(t), \psi \rangle \xrightarrow{t\to+\infty} 0$ for all $\psi \in C_c^\infty(\mathbb{R}^d)$. We have

$$|\langle u, \psi \rangle| \leq \|\frac{u}{r^{\frac{1}{p+1}}}\|_{L^{p+1}} \|r^{\frac{1}{p+1}} \psi\|_{L^{\frac{p+1}{p}}}$$

so that $|\langle u, \psi \rangle|^{p+1} \in L^1(\mathbb{R})$. On the other hand, from

$$\mathrm{i}u_t = -\triangle u + |u|^{p-1}u \text{ in } \mathcal{D}'\left((0,T), H^{-1}(\mathbb{R}^d, \mathbb{C})\right).$$

we have $u \in BC^1\left(\mathbb{R}, H^{-1}(\mathbb{R}^d, \mathbb{C})\right)$ which implies $|\langle u, \psi\rangle|^{2k} \in BC^1(\mathbb{R})$ for $2k \geq p+2$ and for $s < t$ we have

$$\begin{aligned}\Big| |\langle u(t), \psi\rangle|^{2k} - |\langle u(s), \psi\rangle|^{2k} \Big| &= 2k \int_s^t |\langle u(t'), \psi\rangle|^{2k-1} |\langle \dot{u}(t'), \psi\rangle| dt' \\ &\lesssim C\left(\psi, E(u_0), \|u_0\|_{L^2}\right) \int_s^t |\langle u(t'), \psi\rangle|^{p+1} dt' \xrightarrow{s \to +\infty} 0.\end{aligned}$$

□

Before starting the direct proof of Theorem 4.3 we recall the following lemma.

Lemma 5.5. *There exists a constant $C = C_T$ such that for any $u \in L^2((0,T), H^1(\mathbb{R}^d)) \cap H^1((0,T), H^{-1}(\mathbb{R}^d))$ we have $u \in C^0([0,T], L^2(\mathbb{R}^d))$ with*

$$\|u\|_{L^\infty([0,T], L^2(\mathbb{R}^d))} \leq C\left(\|u\|_{L^2((0,T), H^1(\mathbb{R}^d))} + \|\dot{u}\|_{L^2((0,T), H^{-1}(\mathbb{R}^d))}\right). \tag{5.12}$$

Furthermore we have $\|u(t)\|_{L^2}^2 \in AC([0,T])$ with

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\langle u(t), \dot{u}(t)\rangle. \tag{5.13}$$

Proof. Let us assume additionally that $u \in C^1([0,T], L^2(\mathbb{R}^d))$. Then for any fixed $t_0 \in [0,T]$ we have

$$\begin{aligned}&\|u(t)\|_{L^2}^2 = \|u(t_0)\|_{L^2}^2 + 2\int_{t_0}^t \langle u(s), \dot{u}(s)\rangle\, ds \qquad (5.14)\\ &\leq \|u(t_0)\|_{L^2}^2 + \|u\|_{L^2((0,T), H^1(\mathbb{R}^d))}^2 + \|\dot{u}\|_{L^2((0,T), H^{-1}(\mathbb{R}^d))}^2.\end{aligned}$$

We can choose $\|u(t_0)\|_{L^2}^2 = T^{-1}\int_0^T \|u(s)\|_{L^2}^2 ds$ obtaining (5.12) for $C = \sqrt{1 + T^{-1}}$
The general case is obtained by considering a sequence (u_n) in $C^1([0,T], H^1(\mathbb{R}^d))$ converging to u in $L^2((0,T), H^1(\mathbb{R}^d)) \cap H^1((0,T), H^{-1}(\mathbb{R}^d))$. To get such a sequence, we can extend appropriately u into a function in $L^2(\mathbb{R}, H^1(\mathbb{R}^d)) \cap H^1(\mathbb{R}, H^{-1}(\mathbb{R}^d))$, and then we can consider $u_n = \rho_{\epsilon_n} * u$ with $\epsilon_n \xrightarrow{n \to \infty} 0$. Then this sequence satisfies the desired properties.
Then (5.12) implies that (u_n) is a Cauchy sequence in $C^0([0,T], L^2(\mathbb{R}^d))$. The limit is necessarily u, which satisfies (5.12). Also by a limit, we conclude that u satisfies the equality in (5.14), for any fixed $t_0 \in [0,T]$. This implies $\|u(t)\|_{L^2}^2 \in AC([0,T])$ and formula (5.13).

□

Lemma 5.6. *We have*

$$\int_{|x| \geq t \log t} |u|^{p+1} dx \xrightarrow{t \to +\infty} 0. \tag{5.15}$$

Proof. We consider for $M > 0$

$$\theta_M(x) = \begin{cases} \frac{|x|}{M} \text{ for } |x| \le M \\ 1 \text{ for } |x| \ge M \end{cases}$$

Then $\theta_M \in W^{1,\infty}(\mathbb{R}^d)$ with $\|\nabla\theta_M\|_{L^\infty} \le 1/M$. Now we have $u \in C^0(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. Then, by Lemma 5.5 applied to $\sqrt{\theta_M}u$, $t \to 2^{-1}\langle\theta_M u(t), u(t)\rangle \in AC([-T,T])$ for any $T > 0$ with

$$\frac{d}{dt}2^{-1}\langle\theta_M u(t), u(t)\rangle = \langle\theta_M u(t), \dot{u}(t)\rangle .$$

Since we have $\mathrm{i}\dot{u}(t) = -\triangle u + |u|^{p-1}u$ in $\mathcal{D}'(\mathbb{R}, H^{-1})$, we have

$$\left|\frac{d}{dt}2^{-1}\langle\theta_M u(t), u(t)\rangle\right| = \left|\left\langle\theta_M u(t), \mathrm{i}\triangle u - \mathrm{i}|u|^{p-1}u\right\rangle\right| = |\langle\theta_M u(t), \mathrm{i}\triangle u\rangle| \le \|\nabla u\|_{L^2}\|u\|_{L^2}\|\nabla\theta_M\|_{L^\infty}$$
$$\le \|\nabla u\|_{L^2}\|u\|_{L^2}\|\nabla\theta_M\|_{L^\infty} \le CM^{-1}.$$

Then it follows, for a C independent from M,

$$\langle\theta_M u(t), u(t)\rangle \le CM^{-1}t + \langle\theta_M u_0, u_0\rangle .$$

Setting $M = t\log t$, we obtain by dominated convergence

$$\int_{|x|\ge t\log t}|u(t)|^2dx \le \langle\theta_{t\log t}u(t), u(t)\rangle$$
$$\le \frac{C}{\log t} + \int_{|x|\le t\log t}\frac{|x|}{t\log t}|u_0|^2dx + \int_{|x|\ge t\log t}|u_0|^2dx \xrightarrow{t\to+\infty} 0.$$

Finally

$$\|u(t)\|_{L^{p+1}(|x|\ge t\log t)} \le \|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)}\|u(t)\|^{1-\alpha}_{L^{d^*+1}(\mathbb{R}^d)}$$
$$\le C\|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)}\|\nabla u(t)\|^{1-\alpha}_{L^2(\mathbb{R}^d)} \le C'\|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)} \xrightarrow{t\to+\infty} 0.$$

□

Lemma 5.7. *For any $\epsilon > 0$, $t > 1$ and $\tau > 0$ there exists $t_0 > \max(t, 2\tau)$ s.t.*

$$\int_{t_0-2\tau}^{t_0}\int_{|x|\le s\log s}|u|^{p+1}dxds \le \epsilon. \tag{5.16}$$

Proof. The starting point is Lemma 5.4. We have

$$\infty > \int_{\mathbb{R}}dt\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r} \ge \int_2^\infty\frac{ds}{s\log s}\int_{|x|\le s\log s}|u|^{p+1}dx$$
$$\ge \sum_{k=0}^\infty\int_{t+2k\tau}^{t+2(k+1)\tau}\frac{ds}{s\log s}\int_{|x|\le s\log s}|u|^{p+1}dx$$
$$\ge \sum_{k=0}^\infty\frac{1}{(t+2(k+1)\tau)\log(t+2(k+1)\tau)}\int_{t+2k\tau}^{t+2(k+1)\tau}ds\int_{|x|\le s\log s}|u|^{p+1}dx.$$

From this inequality we derive

$$\liminf_{k\to+\infty}\int_{t+2k\tau}^{t+2(k+1)\tau}ds\int_{|x|\le s\log s}|u|^{p+1}dx=0,$$

because otherwise the series would diverge. Hence for any $\epsilon>0$ there exists k_0 arbitrarily large with

$$\int_{t+2k_0\tau}^{t+2(k_0+1)\tau}ds\int_{|x|\le s\log s}|u|^{p+1}dx<\epsilon.$$

So for $t_0=t+2(k_0+1)\tau$ we obtain (5.16).

□

Lemma 5.8. *For any* $\epsilon,a,b\in\mathbb{R}_+$ *there exists* $t_0>\max(a,b)$ *s.t.*

$$\sup_{s\in[t_0-b,t_0]}\|u(s)\|_{L^{p+1}}\le\epsilon. \tag{5.17}$$

Proof. We have

$$\begin{aligned} u(t)&=e^{\mathrm{i}t\triangle}u_0-\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds\\ &=e^{\mathrm{i}t\triangle}u_0\underbrace{-\mathrm{i}\int_0^{t-\tau}e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds}_{w(t,\tau)}\underbrace{-\mathrm{i}\int_{t-\tau}^{1}e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds}_{z(t,\tau)}\\ &=e^{\mathrm{i}t\triangle}u_0+w(t,\tau)+z(t,\tau). \end{aligned}$$

Now we consider each of the last three terms.

Claim 5.9. We have

$$\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}}\xrightarrow{t\to+\infty}0. \tag{5.18}$$

Proof. Indeed, if $u_0\in L^{\frac{p+1}{p}}$, then

$$\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}}\le Ct^{-d\left(\frac12-\frac{1}{p+1}\right)}\|u_0\|_{L^{\frac{p+1}{p}}}\xrightarrow{t\to+\infty}0.$$

The general case follows from the special one using the fact that $H^1\cap L^{\frac{p+1}{p}}$ is dense in H^1. □

Claim 5.10. There is a constant C independent from t and τ s.t.

$$\|w(t,\tau)\|_{L^{p+1}}\le C\tau^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}}. \tag{5.19}$$

Remark 5.11. The exponent is strictly negative. Indeed, for $p-1 \le 1$ we have

$$0 < d(p-1) - 2\max(1, p-1) = d(p-1) - 2 \Longleftrightarrow p > 1 + \frac{2}{d}.$$

If $p - 1 \ge 1$

$$0 < d(p-1) - 2\max(1, p-1) = (d-2)(p-1) \Longleftrightarrow d \ge 3.$$

Proof. We define

$$q = \begin{cases} \infty \text{ if } p \ge 2 \\ \frac{2}{2-p} \text{ if } p < 2. \end{cases}$$

Then we have, for a dimensional constant C,

$$\|w(t,\tau)\|_{L^q} \le C \int_0^{t-\tau} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|^p_{L^{pq'}} ds.$$

Here we claim

$$d\left(\frac{1}{2} - \frac{1}{q}\right) > 1. \tag{5.20}$$

This is obvious by $d \ge 3$ if $q = \infty$. Otherwise, for $p < 2$

$$d\left(\frac{1}{2} - \frac{1}{q}\right) = d\left(\frac{1}{2} - \frac{2-p}{2}\right) = \frac{d}{2}(p-1) > 1 \Longleftrightarrow p > 1 + \frac{2}{d},$$

where the last inequality follows from $p > 1 + \frac{4}{d}$. So we have, for a dimensional constant C

$$\|w(t,\tau)\|_{L^q} \le C\tau^{-d\left(\frac{1}{2}-\frac{1}{q}\right)+1} \sup_s \|u(s)\|^p_{L^{pq'}}. \tag{5.21}$$

We claim now that $2 \le pq' \le p+1$. Indeed, for $p \ge 2$ we have $q' = 1$ and the claim holds. If $p < 2$ then

$$\frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{2-p}{2} = \frac{p}{2}$$

so that $pq' = 2$. So in all cases we have $H^1 \hookrightarrow L^{pq'}$ and we can uniformly bound the last factor on the right in (5.21).

Next, we claim $\|w(t,\tau)\|_{L^2} \le 2\|u_0\|_{L^2}$, which follows from

$$\begin{aligned} w(t,\tau) &= -\mathrm{i}\int_0^{t-\tau} e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds = e^{\mathrm{i}\tau\triangle}\left(-\mathrm{i}\int_0^{t-\tau} e^{\mathrm{i}(t-\tau-s)\triangle}|u(s)|^{p-1}u(s)ds\right) \\ &= e^{\mathrm{i}\tau\triangle}\left(u(t-\tau) - e^{\mathrm{i}(t-\tau)\triangle}u_0\right) = e^{\mathrm{i}\tau\triangle}u(t-\tau) - e^{\mathrm{i}t\triangle}u_0. \end{aligned}$$

Finally, we claim $p+1 \leq q$. This is obviously the case if $q=\infty$. Otherwise $p<2$, and then

$$q > p+1 \Longleftrightarrow \frac{2}{2-p} > p+1 \Longleftrightarrow 2 > (p+1)(2-p) = 2+p-p^2$$

where the last inequality follows from $p>1$ and so from $p-p^2<0$. Finally by Hölder inequality

$$\|w(t,\tau)\|_{L^{p+1}} \leq \|w(t,\tau)\|_{L^2}^{1-\alpha} \|w(t,\tau)\|_{L^q}^{\alpha} \text{ where } \frac{1}{p+1} = \frac{1-\alpha}{2} + \frac{\alpha}{q}.$$

Notice that $\alpha = \frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}$. So

$$\|w(t,\tau)\|_{L^{p+1}} \leq C\tau^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)+\frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}}. \tag{5.22}$$

We now examine the exponent in (5.22). If $q=\infty$ the exponent equals

$$-(d-2)\left(\frac{1}{2}-\frac{1}{p+1}\right) = -\frac{d(p-1)-2(p-1)}{2(p+1)} = -\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}.$$

In the case $q<\infty$, then

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(-d+\frac{1}{\frac{1}{2}-\frac{1}{q}}\right) &= -\frac{p-1}{2(p+1)}\left(d-\frac{2}{p-1}\right)\\ = -\frac{d(p-1)-2}{2(p+1)} &= -\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}.\end{aligned}$$

So we have proved that the exponent in (5.22) is exactly the one in (5.19), which is then proved.

□

We now consider

$$z(t,\tau) = -\mathrm{i}\int_{t-\tau}^{t} e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds.$$

We have

$$\|z(t,\tau)\|_{L^{p+1}} \lesssim \int_{t-\tau}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} \|u\|_{L^{p+1}}^{p} ds. \tag{5.23}$$

Notice that $p<d^*$, that is $p+1<\frac{2d}{d-2}$ is equivalent to $d\left(\frac{1}{2}-\frac{1}{p+1}\right)<1$. Indeed,

$$\frac{1}{p+1} > \frac{d-1}{2d} = \frac{1}{2}-\frac{1}{d}.$$

We now pick $q \in \left(1, \frac{2(p+1)}{d(p-1)}\right)$. Notice that this implies $qd\left(\frac{1}{2} - \frac{1}{p+1}\right) < 1$. Then

$$\|z(t,\tau)\|_{L^{p+1}} \lesssim \left(\int_{t-\tau}^{t} (t-s)^{-dq\left(\frac{1}{2}-\frac{1}{p+1}\right)} ds\right)^{1/q} \left(\int_{t-\tau}^{t} \|u\|_{L^{p+1}}^{pq'} ds\right)^{\frac{1}{q'}}$$
$$= C\tau^{\alpha} \left(\int_{t-\tau}^{t} \|u\|_{L^{p+1}}^{pq'} ds\right)^{\frac{1}{q'}}$$

for some $\alpha > 0$. Now we claim $q'p > p+1$ or, equivalently, $\frac{1}{q'} < \frac{p}{p+1}$ Indeed

$$\frac{1}{q} > \frac{d}{2} - \frac{d}{p+1} \iff \frac{1}{q'} = 1 - \frac{1}{q} < 1 - \frac{d}{2} + \frac{d}{p+1} \iff \frac{1}{q'} < \frac{2-d}{2} + \frac{d}{p+1}$$
$$= \frac{2(p+1) - (p+1)d + 2d}{2(p+1)} = \frac{p}{p+1} + \frac{2-(p+1)d+2d}{2(p+1)} < \frac{p}{p+1},$$

where the last inequality holds because

$$2 - (p+1)d + 2d = 2 - pd + d < 0 \iff p > 1 + \frac{2}{d},$$

with the latter true because, in our case, $p > 1 + \frac{4}{d}$.
From $q'p > p+1 (> 2)$ and $p < d^*$ it follows that,

$$\|u\|_{L^{p+1}}^{pq'} = \|u\|_{L^{p+1}}^{p+1} \|u\|_{L^{p+1}}^{pq'-p-1} \leq \|u\|_{L^{p+1}}^{p+1} \|u\|_{L^2}^{(pq'-p-1)\beta} \|u\|_{L^{d^*+1}}^{(pq'-p-1)(1-\beta)} \text{ for } \frac{1}{p+1} = \frac{\beta}{2} + \frac{1-\beta}{d^*+1}.$$

So, by the Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{d^*+1}(\mathbb{R}^d)$, we conclude for the solutions of our equation

$$\|u\|_{L^{p+1}}^{pq'} \leq C \|u\|_{L^{p+1}}^{p+1} \|u_0\|_{L^2}^{(pq'-p-1)\beta} (2E(u_0))^{\frac{(pq'-p-1)(1-\beta)}{2}}$$

for a dimensional constant C, related to Sobolev embedding. So, for a constant C which depends on the dimensione and u_0, we have

$$\|z(t,\tau)\|_{L^{p+1}} \leq C\tau^{\alpha} \left(\int_{t-\tau}^{t} \|u\|_{L^{p+1}}^{p+1} ds\right)^{\frac{1}{q'}}$$
$$= C\tau^{\alpha} \left(\int_{t-\tau}^{t} ds \int_{|x| \geq s \log s} |u|^{p+1} dx + \int_{t-\tau}^{t} ds \int_{|x| \leq s \log s} |u|^{p+1} dx\right)^{\frac{1}{q'}}$$
$$\leq 2^{\frac{1}{q'}} C\tau^{\delta + \frac{1}{q'}} \left(\sup_{s \in [t-\tau, t]} \|u(s)\|_{L^{p+1}(|x| \geq s \log s)}^{p+1}\right)^{\frac{1}{q'}} + 2^{\frac{1}{q'}} C\tau^{\delta} \left(\int_{t-\tau}^{t} ds \int_{|x| \leq s \log s} |u|^{p+1} dx\right)^{\frac{1}{q'}}. \tag{5.24}$$

Let us take now $\tau > b$ such that

$$\|w(t,\tau)\|_{L^{p+1}} \le C\tau^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} < \frac{\epsilon}{4}. \tag{5.25}$$

Next, using Lemma 5.6 and Claim 5.9 let us take $t_1 > \max(a,b)$ such that for $t \ge t_1$

$$\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}} + 2^{\frac{1}{q'}}C\tau^{\delta+\frac{1}{q'}}\left(\sup_{s\in[t-\tau,t]}\|u(s)\|^{p+1}_{L^{p+1}(|x|\ge s\log s)}\right)^{\frac{1}{q'}} < \frac{\epsilon}{4}. \tag{5.26}$$

Using Lemma 5.7 there exists $t_2 > t_1 + 2\tau$ such that for $t \in [t_2, t_2 - \tau]$

$$2^{\frac{1}{q'}}C\tau^{\delta}\left(\int_{t-\tau}^{t} ds\int_{|x|\le s\log s}|u|^{p+1}dx\right)^{\frac{1}{q'}} \le 2^{\frac{1}{q'}}C\tau^{\delta}\left(\int_{t_2-2\tau}^{t_2} ds\int_{|x|\le s\log s}|u|^{p+1}dx\right)^{\frac{1}{q'}} < \frac{\epsilon}{4}. \tag{5.27}$$

If we consider now the ϵ, a, b in the statement, we can take $t_0 = t_2$ large enough so that $t_0 > \max(a,b)$ and take $\tau > b$ obtaining (5.17).

□

We now move to complete the proof of Theorem 4.3.
Let us fix $\epsilon > 0$. Pick $t > \tau > 0$. Then, in view of $u(t) = e^{\mathrm{i}t\triangle}u_0 + w(t,\tau) + z(t,\tau)$, we have that by Claims 5.9–5.10 there exists $t_1 \ge 0$ and τ_ϵ with

$$\|u(t)\|_{L^{p+1}} \le \|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}} + C\tau_\epsilon^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} + \|z(t,\tau_\epsilon)\|_{L^{p+1}} < \frac{\epsilon}{2} + \|z(t,\tau_\epsilon)\|_{L^{p+1}},$$

where we chose $\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}} < \frac{\epsilon}{4}$ for $t > t_1$ and

$$C\tau_\epsilon^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} = \frac{\epsilon}{4}, \tag{5.28}$$

where C is a dimensional constant. In turn by (5.23)

$$\|z(t,\tau_\epsilon)\|_{L^{p+1}} \lesssim \int_{t-\tau_\epsilon}^{t}(t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)}\|u\|^p_{L^{p+1}}ds \le C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}}\sup_{s\in[t-\tau_\epsilon,t]}\|u(s)\|^p_{L^{p+1}}.$$

From Lemma 5.8 we know that there exists $t_0 > \max(t_1, \tau_\epsilon)$ s.t.

$$\sup_{s\in[t_0-\tau_\epsilon,t_0]}\|u(s)\|_{L^{p+1}} \le \frac{\epsilon}{4}. \tag{5.29}$$

Consider now

$$t_\epsilon = \sup\{t \ge t_0 : \sup_{s\in[t-\tau_\epsilon,t]}\|u(s)\|_{L^{p+1}} \le \epsilon \text{ for all } t \in [t_0,t]\},$$

where (5.29) guarantees that the set on the right hand side contains at least t_0 and in fact,by the continuity in t of the function $t \to \sup_{s\in[t-\tau_\epsilon,t]} \|u(s)\|_{L^{p+1}}$, a whole interval.
If $t_\epsilon = +\infty$ we will have proved the desired result, because in particular this guarantees that $\|u(s)\|_{L^{p+1}} \leq \epsilon$ for all the $t \geq t_0$ and, since here $\epsilon > 0$ is arbitrarily small.

So, let us suppose that $t_\epsilon < \infty$. Then, by $u \in C^0(\mathbb{R}, H^1)$, we have $\|u(t_\epsilon)\|_{L^{p+1}} = \epsilon$. Then we have

$$\epsilon < \frac{\epsilon}{2} + \|z(t_\epsilon, \tau_\epsilon)\|_{L^{p+1}} \leq \frac{\epsilon}{2} + C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \sup_{s\in[t_\epsilon-\tau_\epsilon,t_\epsilon]} \|u(s)\|^p_{L^{p+1}},$$

so that we conclude

$$\epsilon < \frac{\epsilon}{2} + \left(C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1}\right)\epsilon.$$

We now need to check that it is possible to choose τ_ϵ such that both

$$C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1} < \frac{1}{2} \tag{5.30}$$

and (5.28) are true. This will lead to a contradiction. Suppose that for τ_ϵ which satisfies (5.28) inequality (5.30) is false. This implies

$$\frac{1}{2C} \leq \tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1} = C_1 4^{p-1} \tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}-\frac{d(p-1)^2-2(p-1)\max(1,p-1)}{2(p+1)}}, \tag{5.31}$$

where we substituted ϵ^{p-1} using the equality (5.28). We will show now that the exponent of τ_ϵ is negative, so that taking $\tau_\epsilon \gg 1$ formula (5.31) leads to a contradiction. Taking a unique fraction in the exponent and focusing on the numerator, we have

$$\begin{aligned}
&2(p+1) - d(p-1) - d(p-1)^2 + 2(p-1)\max(1, p-1) \\
&= (p-1)\left(2\max(1, p-1) - d - d(p-1)\right) + 2(p+1) \\
&= (p-1)\left(2\max(1, p-1) + 2 - d(p-1)\right) - d(p-1) - 2(p-1) + 2(p+1) \\
&= (p-1)\left(2\max(1, p-1) + 2 - d(p-1)\right) - d(p-1) + 4.
\end{aligned} \tag{5.32}$$

For $p - 1 \leq 1$ the quantity in line (5.32) becomes

$$(p-1)\left(4 - d(p-1)\right) - d(p-1) + 4 = p\left(4 - d(p-1)\right) < 0$$

by $p > 1 + 4/d$ and this completes the proof for $p - 1 \leq 1$.
For $p - 1 > 1$ the quantity in line (5.32) becomes

$$\begin{aligned}
&(p-1)\left(2(p-1) + 2 - d(p-1)\right) - d(p-1) + 4 \\
&= (p-1)\left(2 - (d-2)(p-1)\right) - d(p-1) + 4.
\end{aligned}$$

For $d \geq 4$

$$\begin{aligned}&(p-1)\left(2-(d-2)(p-1)\right)-d(p-1)+4\\&\leq (p-1)\left(2-2(p-1)\right)-4(p-1)+4=-2(p-1)p-4(p-2)<0.\end{aligned}$$

Finally, for $d=3$ and $p-1>1$ the quantity in line (5.32) becomes, for $\alpha=p-1$,

$$\begin{aligned}&(p-1)\left(2(p-1)+2-3(p-1)\right)-3(p-1)+4\\&=-\alpha^2-\alpha+4=:-q(\alpha).\end{aligned}$$

Now, $q(\alpha)=0$ for $\alpha_\pm=-1/2\pm\frac{\sqrt{17}}{2}$. This means that $q(\alpha)<0$ for $p-1>\frac{\sqrt{17}-1}{2}$. The completion of the proof of Theorem 4.3 for the remaining cases, that is $d=3$ and $2<p\leq\frac{\sqrt{17}+1}{2}$ is not in [4].

□

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