Lezione 21 dicembre 2023

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Visto: base ortonormale $\mathcal{B}$ di $(V, \langle \cdot, \cdot \rangle)$

$\{v_1, \dots, v_n\}$ $\quad \langle v_i, v_j \rangle = 0$

$\| v_i \| = 1$

$1 = \sqrt{\langle v_i, v_i \rangle}$

$\Leftrightarrow \underbrace{\langle v_i, v_i \rangle}_{\geq 0} = 1$

in altre parole

$\langle v_i, v_j \rangle = \delta_{ij}$ di Krocker

Esempi: ① $\mathbb{R}^2 = V$, $\langle \cdot, \cdot \rangle = PS$ standard

$$\left\langle \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\rangle := a_1 b_1 + a_2 b_2$$

cons. $\mathcal{E} = \{e_1, e_2\}$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle e_1, e_1 \rangle = 1 \cdot 1 + 0 \cdot 0 = 1$

$\langle e_2, e_2 \rangle = 0 \cdot 0 + 1 \cdot 1 = 1$

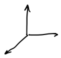
$\langle e_1, e_2 \rangle = 1 \cdot 0 + 0 \cdot 1 = 0$

② $\mathbb{R}^3 = V$, $\langle \cdot, \cdot \rangle = PS$ standard : $\left\langle \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\mathcal{E} = \{e_1, e_2, e_3\}$

$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1$

$\underset{i \neq j}{\langle e_i, e_j \rangle} = 0$



③ $\mathbb{R}^n = V$, $\langle \cdot, \cdot \rangle = PS$ standard

$\mathcal{E} = \{e_1, \dots, e_n\}$ è base ORTONORMALE

④ $\mathbb{C}^n = V$, $\langle \cdot, \cdot \rangle = PS$ standard ; $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$\langle z, w \rangle = \overline{z}_1 \cdot w_1 + \overline{z}_2 \cdot w_2 + \dots + \overline{z}_n \cdot w_n$

Allora $\mathcal{E} = \{e_1, \dots, e_n\}$ è ORTONORM.

Teo: Gram - Schmidt

V sp. euclideo opp. unitario, $W \subseteq V$ ssv

$\dim W = m$; $\dim V = n$

Ogni base ortonorm. di W si estende a base ortonorm. di V.

Dim.: Per induzione sulla CODIMENSIONE di W in V

$$\operatorname{codim}_V W := \underbrace{n - m}_{\in \mathbb{N}} = \dim V - \dim W$$

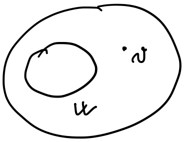
Sia $n-m=0$, ovvero $n=m$ $\Rightarrow$ $W = V$ tesi è vera
(non occorre aggiungere alcun vettore alla base ortonorm. di W)

Supp. vero per $\underbrace{n-m-1}_{\text{cioè per tutti i ssv di dim. } m+1}$ e dim. per $n-m$.

Sia W di dim m, e fisso una base ortonormale $\{w_1, w_2, .., w_m\}$ di W

$n-m>0 \Rightarrow m<n \Rightarrow W \subsetneq V \Rightarrow$

$\exists\ v \in V \setminus W \ (v \in V,\ v \notin W)$



cons. $\tilde{v} = \langle w_1, v\rangle w_1 + \cdots + \langle w_m, v\rangle w_m \in W$

proiezione ortogonale su W

visto: $(v-\tilde{v}) \perp W$

inoltre $v-\tilde{v} \neq 0_V$; se lo fosse, si avrebbe
$v = \tilde{v} \in W$

$\Rightarrow$ posso cons. $\dfrac{v-\tilde{v}}{\|v-\tilde{v}\|} =: w_{m+1}$

Pongo $W' := \mathrm{Span}(w_1, \ldots, w_m, w_{m+1})$

$w_{m+1} \notin \mathrm{Span}(w_1, .., w_m) = W$

$\Rightarrow \dim W' = m+1$

cioè: $\mathrm{codim}_V W' = n-(m+1) = n-m-1$

$\Rightarrow$ applico ipotesi induttiva: $\exists$ un prolungamento di $\underbrace{\{w_1, .., w_m, w_{m+1}\}}_{\text{base di } W'}$ a base ORTONOR. di V

ORTONORMALE per costruzione

$\{w_1, \dots, w_m, w_{m+1}, w_{m+2}, \dots, w_n\}$

$\Longrightarrow$ questo è prolung. ortonorm. di $\{w_1, \dots, w_m\}$.

COR.: Ogni V sp. euclideo o unitario ammette una base ortonormale. $(V \neq \{0_V\})$

Infatti: $v \in V$, $v \neq 0_V$ qualunque

pongo $v_1 := \dfrac{v}{\|v\|}$; Cons. $W = \operatorname{Span}(v_1)$; $\{v_1\}$ è una base ortonormale di W

e poi completo v_1 a base ortonorm. di V

usando il procedimento della dimostrazione di Gram-Schmidt.

OSS.: In una base ortonormale $\mathcal{B} = \{v_1, \dots, v_n\}$

si ha $v = \underbrace{\langle v_1, v\rangle}_{\in \mathbb{K}} v_1 + \langle v_2, v\rangle v_2 + \dots + \langle v_n, v\rangle v_n$

coordinate di v

Infatti: $\boxed{\langle v_j, v\rangle} = \langle v_j, \sum_{i=1}^{n} a_i v_i\rangle = \sum_{i=1}^{n} a_i \underbrace{\langle v_j, v_i\rangle}_{\delta_{ij}} = a_1 \overbrace{\langle v_j, v_1\rangle}^{=0} + a_2 \overbrace{\langle v_j, v_2\rangle}^{=0} + \dots$

$+ a_j \underbrace{\langle v_j, v_j\rangle}_{=1} + \dots + a_n \underbrace{\langle v_j, v_n\rangle}_{=0}$

$= \boxed{a_j}$

$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n = \sum_{i=1}^{n} a_i v_i$

In particolare : se $\mathbb{K} = \mathbb{R}$

$v = \underbrace{\langle v_1, v\rangle}_{} v_1 + \dots + \langle v_n, v\rangle v_n$

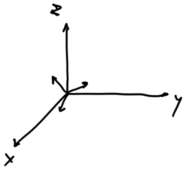
$(\underbrace{\|v_1\|}_{=1} \boxed{\|v\|} \cos\alpha_1) v_1 + \dots + (\underbrace{\|v_n\|}_{=1} \boxed{\|v\|} \cos\alpha_n) v_n$

$$= \|v\| \cdot \underbrace{\left(\cos\alpha_1 \, v_1 + \dots + \cos\alpha_n \, v_n\right)}$$

esercizio: questo vettore ha norma $= 1$

Esempio: $\mathbb{R}^3$ con PS standard

Cons. come primo vettore $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$



Devo vedere se v è versore:

$$\|v\|^2 = \langle v, v \rangle = 1\cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 3$$

$$\|v\| = \sqrt{3}$$

$$\frac{v}{\|v\|} =: v_1 = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

scelgo $w_2 \notin \operatorname{Span}\left(\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}\right)$

es.: $w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\langle v_1, w_2 \rangle = \left\langle \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = 1/\sqrt{3} \neq 0$$

non sono ortogonali; quindi cons.

$$\tilde{w}_2 = \langle v_1, w_2 \rangle v_1$$

$$w_2 - \tilde{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left\langle \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle v_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left(1/\sqrt{3}\right) \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}$$

verifico che: $\left\langle \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} \right\rangle = 0$

infine: $\left\| \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} \right\|^2 = (2/3)^2 + (-1/3)^2 + (-1/3)^2$

$$= 6/9$$

$$\left\| \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} \right\| = \frac{\sqrt{6}}{3}$$

$$\Rightarrow v_2 := \frac{3}{\sqrt{6}} \cdot \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$$

Terzo vettore: scelgo $w_3 \notin \operatorname{Span}(v_1, v_2) = W'$

ad es. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = w_3$

cons. $\underset{\text{proj. ort. su } W'}{\tilde{w}_3} = \langle v_1, w_3 \rangle v_1 + \langle v_2, w_3 \rangle v_2$

poi $w_3 - \tilde{w}_3$; $\dfrac{w_3 - \tilde{w}_3}{\| w_3 - \tilde{w}_3 \|} =: v_3$

Esercizio: fare i conti per trovare v_3 esplicitamente.

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Fatto interessante su $W^\perp$:

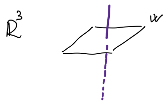
Prop.: V sp. euclideo o unitario

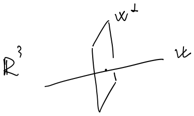
$W \subseteq V$ ssv.

Allora $V = W \oplus W^\perp$

In particolare $\dim V = \dim W + \dim W^\perp$

$\dim W^\perp = \dim V - \dim W$





Dim.: $\{w_1, \dots, w_m\}$ base ortonorm. di W

GS : $\exists$ prolungamento a base ortonormale di V

$\{w_1, \dots, w_m, w_{m+1}, \dots, w_n\}$

$$\forall v \in V, \quad v = \overbrace{\langle w_1, v \rangle w_1 + \dots + \langle w_m, v \rangle w_m}^{= w \in W} +$$

$$+ \underbrace{\langle w_{m+1}, v \rangle w_{m+1} + \dots \langle w_n, v \rangle w_n}_{w' \in W^\perp}$$

infatti:

$$\langle w_1, w' \rangle = \langle w_1, \sum_{j=m+1}^{n} \langle w_j, v \rangle w_j \rangle$$

$$= \sum_{j=m+1}^{n} \langle w_j, v \rangle \underbrace{\langle w_1, w_j \rangle}_{=0} = 0$$

$\vdots$

$\langle w_m, w' \rangle = 0$

$\Longrightarrow W + W^\perp = V$

Infine: $W \cap W^\perp = \{0_V\}$

$W \cap W^\perp = \{ v \in W : \langle w_i, v \rangle = 0 \ \forall i = 1, \dots, m \}$

$v \in W, \quad v = a_1 w_1 + \dots + a_m w_m$

$0 = \langle w_i, v \rangle = \langle w_i, \sum_{j=1}^{m} a_j w_j \rangle = a_i \quad \forall i = 1, \dots, m$

$\Longrightarrow v = 0_V$

Oppure: $v \in W$ e $v \in W^\perp$

$\Longrightarrow \langle v, v \rangle = 0 \Longrightarrow v = 0_V$.

ENDOMORFISMI ORTOGONALI
& UNITARI

Def.: $f: V \to V$, V sp. euclideo o unitario
endomorfismo

si dice ORTOG. (UNITARIO), opp. ISOMETRIE VETTORIALI, o ISOM. LINEARI

TRASF. RIGIDE

se $\forall\, v, w \in V$ si ha

$$\langle v, w\rangle = \langle f(v), f(w)\rangle$$

Oss.: In particolare: $\langle v, v\rangle = \langle f(v), f(v)\rangle$

$$\|v\|^2 = \|f(v)\|^2$$

$$\|v\| = \|f(v)\|$$

$\Rightarrow f$ "conserva le lunghezze"

OSS: Supp. che $f: V \to V$ verifichi

$\forall\, v \in V \;:\; \|v\| = \|f(v)\|$; siano $v, w \in V$

$$\boxed{\langle f(v), f(w)\rangle} = \frac{1}{2}\left(\|f(v)+f(w)\|^2 - \|f(v)\|^2 - \|f(w)\|^2\right)$$

caso $\mathbb{R}$

formula di polarizz.

$$= \frac{1}{2}\left(\|f(v+w)\|^2 - \|f(v)\|^2 - \|f(w)\|^2\right)$$

f AL

$$= \frac{1}{2}\left(\|v+w\|^2 - \|v\|^2 - \|w\|^2\right)$$

f conserva le norme

$$= \boxed{\langle v, w\rangle}$$

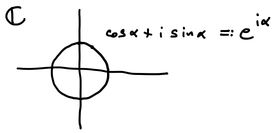
Form. polarizz.

Prop. $f: V \to V$ ORTOG. (UNIT.)

supp. $\lambda \in S_p(f) \Longrightarrow \underbrace{|\lambda|}_{\text{v. ass. opp. modulo}} = 1$

Cioè: caso $\mathbb{R}$: $\lambda = \pm 1$

caso $\mathbb{C}$: $\lambda = e^{i\alpha} = \cos\alpha + i\sin\alpha$



Dim. $\lambda \in S_p(f)$, $\exists\ v \neq 0_V$ t.c.

$f(v) = \lambda v$

$\|v\| = \|f(v)\| = \|\lambda v\| = |\lambda| \cdot \|v\|$

$\underbrace{\|v\|}_{\neq 0} = |\lambda| \cdot \underbrace{\|v\|}_{\neq 0} \Longrightarrow |\lambda| = 1.$

Prop.: Se λ e μ sono autovalori diversi di $f: V \to V$ ort. o unitario allora i relativi autovettori v e w

sono ORTOGONALI :

$\langle v, w \rangle = 0.$

Dim. $f(v) = \lambda v$, $v \neq 0_V$

$f(w) = \mu w$, $w \neq 0_W$

$\langle v, w \rangle = \langle f(v), f(w) \rangle =$

$= \langle \lambda v, \mu w \rangle$

$\overset{\mathbb{C}}{\mid}$

anche $\mathbb{R}$ $= \bar{\lambda} \cdot \mu \cdot \langle v, w \rangle$

$\bar{\lambda}\mu \langle v, w \rangle - \langle v, w \rangle = 0$

$(\bar{\lambda}\mu - 1) \cdot \underbrace{\langle v, w \rangle}_{=0} = 0$

$=0$, opp. $\bar{\lambda} \cdot \mu - 1 = 0$

Supp.

$\boxed{\underbrace{\bar{\lambda} \cdot \mu}_{} = 1}$

$= \lambda^{-1} \cdot \mu = 1 \quad / \cdot \lambda \qquad \Longrightarrow \mu = \lambda$

Ricordo: $\lambda \neq 0$ per Prop. precedente

$\lambda^{-1} = \dfrac{\bar{\lambda}}{\underbrace{|\lambda|^2}_{=1}} = \bar{\lambda}$

$\left(\underset{z \neq 0}{z \cdot \bar{z}} = |z|^2, \quad z \cdot \dfrac{\bar{z}}{|z|^2} = 1 \right)$

Noi abbiamo supp. $\lambda \neq \mu \Longrightarrow \langle v, w \rangle = 0$

MATRICI ORTOGONALI
& UNITARIE

Def.: $A \in M_n(\mathbb{R})$ si dice ORTOGONALE se è invertibile e

$$A^{-1} = {}^tA$$

$A \in M_n(\mathbb{C})$ si dice UNITARIA se è invertibile e

$$A^{-1} = {}^t\overline{A}$$

matrice coniugata

Teorema: V sp. euclideo o unit.
$\mathcal{B}$ base ortonormale; sia $f: V \to V$

Allora: f è endo. ORT. (UNIT.)
$\iff M^{\mathcal{B}}_{\mathcal{B}}(f)$ è ORT. (UNITARIA)

OSS.: f end. ORT o UNITARIO

$\Rightarrow$ f è ISOMORFISMO

infatti: $\|v\| = \|f(v)\|$

in particolare se $f(v) = 0_V$

$\|v\| = \|f(v)\| = \|0_V\| = 0$

$\iff v = 0_V$

$\Rightarrow \ker f = \{0_V\}$; $f: V \to V$

$\Rightarrow f$ isomorfismo.

Dim.: Caso $\mathbb{C}$ (il caso $\mathbb{R}$ segue come caso particolare).

siano $v, w \in V$; $\mathcal{B}$ base ortonormale

siano $x_v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ colonna delle coord. di v in $\mathcal{B}$

$y_w = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ coord. di w in $\mathcal{B}$

$$A := M_{\mathcal{B}}^{\mathcal{B}}(f)$$

$$\boxed{\langle v, w\rangle} = {}^t\bar{x}\cdot \underset{\text{ortonormale}}{M_{\boxed{\mathcal{B}}}}(b)\cdot y = {}^t\bar{x}\cdot \mathbb{I}_n \cdot y \boxed{= {}^t\bar{x}\cdot y}$$

$\langle\cdot,\cdot\rangle = b$ sesquilineare, hermit., def posit.

$$M_{\mathcal{B}}(b) = \begin{pmatrix} b(v_1, v_1) & \cdots & b(v_1, v_n) \\ \vdots & & \vdots \\ b(v_n, v_1) & \cdots & b(v_n, v_n) \end{pmatrix}$$

$$= \begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_n\rangle \\ \vdots & \ddots & \vdots \\ \langle v_n, v_1\rangle & \cdots & \langle v_n, v_n\rangle \end{pmatrix}$$

$$= \mathbb{I}_n$$

Cioè: $\langle v, w\rangle = {}^t\bar{x}\cdot y$

Quindi:

f è ort. (unitario) $\Longleftrightarrow \forall\, v, w : \langle v, w\rangle = \langle f(v), f(w)\rangle$

$$\Longleftrightarrow {}^t\bar{x}\cdot y = {}^t\overline{(A\cdot x)}\cdot(A\cdot y)$$

$$\Longleftrightarrow {}^t\bar{x}\cdot y = {}^t(\bar{A}\cdot \bar{x})\cdot(A\cdot y)$$

$$\Longleftrightarrow {}^t\bar{x}\cdot y = {}^t\bar{x}\cdot {}^t\bar{A}\cdot A\cdot y$$

$${}^t\bar{x}_v\cdot \mathbb{I}_n\cdot y_w = {}^t\bar{x}_v\cdot {}^t\bar{A}\cdot A\cdot y_w \quad \forall\, x_v, y_w \in \mathbb{C}^n$$

in particolare

$${}^t\bar{e}_i\cdot \mathbb{I}_n\cdot e_j = {}^t\bar{e}_i\cdot {}^t\bar{A}\cdot A\cdot e_j$$

$$\|$$

$$\underbrace{{}^t e_i\cdot \mathbb{I}_n\cdot e_j}_{(0 \cdots 1\, 0 - 0)\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0\end{pmatrix}} = \underbrace{{}^t e_i\cdot {}^t\bar{A}\cdot A\cdot e_j} \quad \forall\, i, j$$

$$= \delta_{ij} = ({}^t\bar{A}\cdot A)_{(i,j)} \quad \Longrightarrow {}^t\bar{A}\cdot A = \mathbb{I}_n$$

$$\Longrightarrow {}^t\bar{A} = A^{-1}$$

$${}^t e_i\cdot M\cdot e_j$$

$$\underbrace{\underset{1\times n}{(0 \cdots 1 \cdots 0)}\cdot \underset{n\times n}{M}}\cdot \underset{n\times 1}{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0\end{pmatrix}}$$

$$M_{(i)} \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = m_{ij}$$

ULTIMISSIMA OSS:

Nel caso $\mathbb{R}$: $A^{-1} = {}^tA \iff$ le colonne di A formano una base ORTON. di $\mathbb{R}^n$

Dim.: $A^{-1} = {}^tA \iff {}^tA \cdot A = I_n$

$$\iff ({}^tA)_{(i)} \cdot A^{(j)} = \delta_{ij}$$

$$\iff {}^t(A^{(i)}) \cdot A^{(j)} = \delta_{ij}$$

$$\iff \langle A^{(i)}, A^{(j)} \rangle = \delta_{ij}.$$

Esempio: $R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

$$(R_\alpha)^{(1)} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \quad (R_\alpha)^{(2)} = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

$$\| R_\alpha^{(1)} \| = \sqrt{\cos^2\alpha + \sin^2\alpha} = \sqrt{1} = 1$$

$$\| R_\alpha^{(2)} \| = \sqrt{(-\sin\alpha)^2 + (\cos\alpha)^2} = \sqrt{1} = 1$$

$$\left\langle \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} \right\rangle = \cos\alpha \cdot (-\sin\alpha) + \sin\alpha \cdot \cos\alpha$$

$$= 0$$

Vale anche nel caso complesso:

$A \in M_n(\mathbb{C})$ è UNITARIA $\iff$ le colonne di A formano una base ortonormale di $\mathbb{C}^n$ rispetto al PS standard

$$A^{-1} = {}^t\bar{A}$$