Quantum Computing
2 - Complex Linear Algebra

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Complex linear algebra

The basic mathematical objects in Quantum Mechanics are **state vectors** and **linear operators** (matrices). Because the theory is fundamentally linear, and the probability amplitudes are complex numbers, the mathematics underlying quantum mechanics is **complex linear algebra**.

Vectors

Vectors are members of a complex vector space, or **Hilbert space**, with an associated inner product.

It is important to remember that these abstract mathematical objects represent physical things, and should be considered independent of the particular representations they are given by choosing a particular basis.

State vectors in quantum mechanics are written in **Dirac notation**. The basic object is the **ket-vector** $|\psi\rangle$, which (given a particular basis) can be represented as a *column vector*. The adjoint of a ket-vector is a **bra-vector** $\langle\psi|$, represented as a *row vector*.

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \quad \langle\psi| = (\alpha_1^* \cdots \alpha_N^*) = |\psi\rangle^\dagger$$

If the vector $|\psi\rangle$ is normalized, that means

$$\langle\psi|\psi\rangle = \sum_{j=1}^{N} |\alpha_j|^2 = 1$$

Inner and Outer Products.

Given two vectors $|\psi\rangle$ and $|\varphi\rangle$,

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \quad |\phi\rangle = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix},$$

the **inner product** between them is written

$$\langle\phi|\psi\rangle = (\beta_1^* \cdots \beta_N^*) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \sum_j \beta_j^* \alpha_j$$

The inner product is independent of the choice of basis. $\langle\varphi|\psi\rangle$ is called a *bracket*. Note that $\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$, and for a normalized vector $\langle\psi|\psi\rangle = 1$. If two vectors are orthogonal then $\langle\psi|\varphi\rangle = 0$.

It is also handy to write the **outer product** (also sometimes called a *dyad*):

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} (\beta_1^* \cdots \beta_N^*) = \begin{pmatrix} \alpha_1\beta_1^* & \cdots & \alpha_1\beta_N^* \\ \vdots & \ddots & \vdots \\ \alpha_N\beta_1^* & \cdots & \alpha_N\beta_N^* \end{pmatrix}$$

A dyad $|\psi\rangle\langle\varphi|$ is a **linear operator**. As we shall see, it is common (and often convenient) to write more general operators as linear combinations of dyads.

Orthonormal bases.

An orthonormal basis for an N-dimensional space has N vectors that satisfy

$$\langle i|j\rangle = \delta_{ij}, \quad \sum_{j=1}^{N} |j\rangle\langle j| = \hat{I}.$$

It is normally most convenient to choose a particular orthonormal basis $\{|j\rangle\}$ and work in terms of it. **As long as one works within a fixed basis, one can treat state vectors as column vectors and operators as matrices**.

Linear operators

A **linear operator** *O* transforms states to states such that

$$\hat{O}(a|\psi\rangle + b|\phi\rangle) = a\hat{O}|\psi\rangle + b\hat{O}|\phi\rangle$$

for all states $|\psi\rangle$, $|\varphi\rangle$ and complex numbers *a, b*. Given a choice of basis $\{|j\rangle\}$, an operator can be represented by a **matrix**

$$\hat{O} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix} \equiv [a_{ij}]$$

where

$$\langle i|\hat{O}|j\rangle = a_{ij},$$

are called **matrix elements**. The operator can be written as a sum over outer products

$$\hat{O} = \sum_{ij} a_{ij}|i\rangle\langle j|$$

The matrix representation depends on the choice of basis. We will only be dealing with orthonormal bases in this class.

Three operations on operators are of particular relevance in Quantum Mechanics: the trace, the commutator and the Hermitian conjugation.

The trace.

The trace of an operator is the sum of the diagonal elements:

$$\mathrm{Tr}\{\hat{O}\} = \sum_j \langle j|\hat{O}|j\rangle = \sum_j a_{jj}.$$

A traceless operator has Tr{O} = 0. The trace is *independent of the choice of basis*. If $\{|j\rangle\}$ and $\{|\varphi_k\rangle\}$ are both orthonormal bases, then

$$\mathrm{Tr}\{\hat{O}\} = \sum_j \langle j|\hat{O}|j\rangle = \sum_k \langle \phi_k|\hat{O}|\phi_k\rangle$$

The trace also has the useful *cyclic property:*

$$\mathrm{Tr}\{\hat{A}\hat{B}\} = \mathrm{Tr}\{\hat{B}\hat{A}\}$$

This applies to products of any number of operators:

$$\mathrm{Tr}\{\hat{A}\hat{B}\hat{C}\} = \mathrm{Tr}\{\hat{C}\hat{A}\hat{B}\} = \mathrm{Tr}\{\hat{B}\hat{C}\hat{A}\}$$

This invariance implies that $\mathrm{Tr}\{|\varphi\rangle\langle\psi|\} = \langle\psi|\varphi\rangle$.

The Commutator.

Matrix multiplication is *noncommutative*, in general. That is, in general AB differs from BA. Given two operators A and B the *commutator* is [A,B] = AB – BA.

[A,B] = 0 if and only if a and B commute. Occasionally, one will encounter matrices that *anticommute*: AB = - BA. For example, the Pauli matrices anticommute with each other. In these cases, it is sometimes helpful to define the *anticommutator*:

$$\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}.$$

Hermitian Conjugation.

One of the most important operations in complex linear algebra is *Hermitian conjugation*. The Hermitian conjugate $O^\dagger$ is the *complex conjugate* of the *transpose* of an operator O. If in a particular basis

$$O = [a_{ij}] \quad \text{then} \quad O^\dagger = [a_{ji}{}^*]$$

Hermitian conjugation works similarly to transposition in real linear algebra: $(AB)^\dagger = B^\dagger A^\dagger$. When applied to state vectors, $(|\psi\rangle)^\dagger = \langle\psi|$. Similarly, for dyads $(|\psi\rangle\langle\varphi|)^\dagger = |\varphi\rangle\langle\psi|$.

Note that Hermitian conjugation is *not* linear, but rather is *antilinear*:

$$(a\hat{O})^\dagger = a^*\hat{O}^\dagger, \quad (a|\psi\rangle)^\dagger = a^*\langle\psi|.$$

We now introduce the most relevant type of operators for Quantum Mechanics: normal operators, Hermitian operators, unitary operators, and projection operators.

Normal operators.

A *normal operator* satisfies $O^\dagger O = O\,O^\dagger$. **Operators are *diagonalizable* if and only if they are normal.** That is, for normal O we can always find an orthonormal basis $\{|\varphi_j\rangle\}$ such that

$$\hat{O} = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|, \quad \mathrm{Tr}\{\hat{O}\} = \sum_j \lambda_j$$

and any diagonalizable operator must be normal. This is called the **spectral theroem**.

These values λ_j are the *eigenvalues* of O and $\{|\varphi_j\rangle\}$ the corresponding *eigenvectors*, $O|\varphi_j\rangle = \lambda_j|\varphi_j\rangle$. If O is *nondegenerate*—i.e., all the λ_j are distinct—then the eigenvectors are unique (up to a phase). Otherwise there is some freedom in choosing this *eigenbasis*.

If two normal operators A and B commute, it is possibleto find an eigenbasis which simultaneously diagonalizes *both* of them. (The converse is also true.)

Hermitian operators.

One very useful class of operators are the *Hermitian operators H* that satisfy $H = H^\dagger$. These are the complex analogue of *symmetric* matrices. They are **obviously normal**: $H^\dagger H = H^2 = H\,H^\dagger$. The eigenvalues of a Hermitian matrix are always *real*.

An example are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is easy to check that any 2 x 2 Hermitian matrix can be written as a linear combination of the Pauli matrices and the identity

Consider a 2 x 2 matrix

$$O = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Hemiticity requires

$$O = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = O^{\dagger} = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}$$

Therefore

$$\begin{aligned} \alpha &= a + d \\ \beta &= b - ic \\ \gamma &= b + ic \\ \delta &= a - d \end{aligned}$$

where a, b, c, d are real numbers. Therefore we have

$$\hat{O} = a\hat{I} + b\hat{\sigma}_x + c\hat{\sigma}_y + d\hat{\sigma}_x$$

Unitary Operators.

A *unitary* operator satisfies $U^\dagger U = U U^\dagger = I$. It is clearly a normal operator. All of its eigenvalues have unit norm; that is, $|\lambda_j| = 1$ for all j. This means that

$$\lambda_j = \exp(i\vartheta_j)$$

for real $0 \le \vartheta_j < 2\pi$.

There is a correspondence between Hermitian and unitary operators: for every unitary operator U there is an Hermitian operator H such that

$$U = \exp(iH).$$

(We will clarify what $\exp(iH)$ means shortly.)

A unitary operator is equivalent to a **change of basis**. This is easy to check: if $\{|j\rangle\}$ is an orthonormal basis, then so is $\{U|j\rangle\}$:

$$(\hat{U}|i\rangle)^\dagger(\hat{U}|j\rangle) = \langle i|\hat{U}^\dagger\hat{U}|j\rangle = \langle i|j\rangle = \delta_{ij}$$

For spin-1/2, the most general 2 × 2 unitary can be written (up to a global phase)

$$\hat{U} = \cos(\theta/2)\hat{I} + i\sin(\theta/2)\vec{n}\cdot\hat{\vec{\sigma}}$$

where n is again a real unit three-vector (n_x, n_y, n_z) and sigma = (sigma_x, sigma_y, sigma_z). In the Bloch sphere picture, U is a rotation by ϑ about the axis n.

Orthogonal projectors.

An *orthogonal projector P is* an Hermitian operator that obeys $P^2 = P$. All the eigenvalues of P are either 0 or 1. The *complement* of a projector 1–P is also a projector. Note that $|\psi\rangle\langle\psi|$ is a projector for *any* normalized state vector $|\psi\rangle$. Such a projector is called *one-dimensional*; a projector more generally has dimension d = Tr[P].

In dimension 2, *any* projector can be written in the form

$$\hat{\mathcal{P}} = \left(\hat{I} + \vec{n} \cdot \hat{\vec{\sigma}}\right)/2 = |\psi_{\vec{n}}\rangle\langle\psi_{\vec{n}}|,$$

where n is a (real) unit three-vector (n_x, n_y, n_z) (i.e., with $n_x^2 + n_y^2 + n_z^2 = 1$) and sigma are the three Pauli matrices .In the Bloch sphere picture, n is the direction in space, and P the projector onto the state $|\psi_n\rangle$ that is spin up along that axis.

Operator space.

The space of all operators on a particular Hilbert space of dimension *N* is itself a Hilbert space of dimension N^2; sometimes this fact can be very useful. If A and B operators, so is aB + bB for any complex *a, b*.

One can define an inner product on operator space. The most commonly used one is $(A, B) \equiv \mathrm{Tr}\{A^\dagger B\}$ (the *Frobenius* or **Hilbert-Schmidt inner product**).

It is easy to see that $(A, B) = (B, A)^*$, and $(A, A) \geq 0$ with equality only for $A = 0$. With respect to this inner product, the Pauli matrices together with the identity form an orthogonal basis for all operators on 2D Hilbert space.

A linear transformation on operator space is often referred to as a *superoperator*.

Functions of Operators.

It is common to consider a function of an operator $f(O)$, where f is ordinarily a function on the complex numbers, which is itself an operator.

Suppose that $f(x)$ is defined by a Taylor series: $f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots$. For the operator version, we write

$$f(\hat{O}) = c_0\hat{I} + c_1(\hat{O} - x_0\hat{I}) + c_2(\hat{O} - x_0\hat{I})^2 + \cdots$$

In practice, only the case $x_0 = 0$ is of interest, in which case:

$$f(\hat{O}) = c_0\hat{I} + c_1\hat{O} + c_2\hat{O}^2 + \dots$$

For particular functions and operators, this series can sometimes be summed explicitly. If O is **normal**, we simplify by writing O in diagonal form:

$$\hat{O} = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$$

It is easy to show that

$$\hat{O}^n = \sum_j \lambda_j^n |\phi_j\rangle\langle\phi_j|$$

And therefore

$$f(\hat{O}) = \sum_j \left[\sum_n c_n \lambda_j^n\right] |\phi_j\rangle\langle\phi_j| = \sum_j f(\lambda_j)|\phi_j\rangle\langle\phi_j|$$

In particular, for **projectors**, $P^n = P$ for all n greater or equal to 1. Then

$$\begin{aligned} f(\hat{P}) &= c_0\hat{I} + c_1\hat{P} + c_2\hat{P} + \dots \\ &= c_0\hat{I} \pm c_0\hat{P} + c_1\hat{P} + c_2\hat{P} + \dots \\ &= f(0)[\hat{I} - \hat{P}] + f(1)\hat{P} \end{aligned}$$

For **idempotent** operators obeying **O² = I** (such as the Pauli matrices), one can sum the even and odd term separately. In particular:

$$
\begin{aligned}
e^{i\theta\hat{O}} &= \hat{I} + i\theta\hat{O} - \frac{\theta^2\hat{O}^2}{2} - i\frac{\theta^3\hat{O}^3}{3!} + \frac{\theta^4\hat{O}^4}{4!} + i\frac{\theta^5\hat{O}^5}{5!} - \dots \\
&= \hat{I} + i\theta\hat{O} - \frac{\theta^2}{2}\hat{I} - i\frac{\theta^3}{3!}\hat{O} + \frac{\theta^4}{4!}\hat{I} + i\frac{\theta^5}{5!}\hat{O} - \dots \\
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \dots\right)\hat{I} + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)\hat{O} \\
&= \cos(\theta)\hat{I} + i\sin(\theta)\hat{O}
\end{aligned}
$$

As an application, let a be n real, three-dimensional unit vector and ϑ a real number. Then

$$(\vec{n}\cdot\vec{\sigma})^2 = 1$$

where we used the property of the Pauli matrices

$$(\vec{n}\cdot\vec{\sigma})(\vec{m}\cdot\vec{\sigma}) = (\vec{n}\cdot\vec{m})\hat{I} + i(\vec{n}\times\vec{n})\cdot\vec{\sigma}$$

As such:

$$e^{i\theta\vec{n}\cdot\vec{\sigma}} = \cos(\theta)\hat{I} + i\sin(\theta)\vec{n}\cdot\vec{\sigma}$$

As in the previous example, the most commonly-used function is the exponential

$$\exp(x) = 1 + x + x^2/2! + \cdots,$$

but others also occur from time to time:

$$\cos(x) = 1 - x^2/2 + x^4/4! - \cdots$$

$$\sin(x) = x - x^3/3! + x^5/5! - \cdots$$

$$\log(1+x) = x - x^2/2 + x^3/3 - \cdots$$

Exercise 2.34: Find the square root and logarithm of the matrix

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

Since the matrix is: $4\hat{I} + 3\hat{\sigma}_x$

Then the eigenvectors are those of the Pauli matrix and the eigenvalues equal to 4+3 = 7 and 4-3 = 1. The eigenprojectors are:

$$P_+ = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$P_- = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Therefore:

$$\sqrt{A} = \sqrt{7}P_+ + P_- = \frac{1}{4}\begin{bmatrix} \sqrt{7}+1 & \sqrt{7}-1 \\ \sqrt{7}-1 & \sqrt{7}+1 \end{bmatrix}$$

As a mater of fact, if one takes the square, the original matrix A is recovered.

Tensor products.

The tensor (or Kronecker) product is a way of combining two Hilbert spaces to produce a higher dimensional space. Let $|\psi\rangle$ be a state in a D_1-dimensional Hilbert space H_1 and $|\varphi\rangle$ be a state in a D_2-dimensional Hilbert space H_2. Then we define $|\psi\rangle \otimes |\varphi\rangle$ to be a state in the $D_1 D_2$-dimensional space $H_1 \otimes H_2$. Such a state is called a *product state*. Any state in this larger space can be written as a linear combination of product states

$$|\Psi\rangle = \sum_\ell \alpha_\ell |\psi_\ell\rangle \otimes |\phi_\ell\rangle$$

where $|\psi_\ell\rangle \in H_1$ and $|\varphi_\ell\rangle \in H_2$.

What are the properties of this product?

$$(a|\psi\rangle + b|\psi'\rangle) \otimes |\phi\rangle = a|\psi\rangle \otimes |\phi\rangle + b|\psi'\rangle \otimes |\phi\rangle.$$

$$|\psi\rangle \otimes (a|\phi\rangle + b|\phi'\rangle) = a|\psi\rangle \otimes |\phi\rangle + b|\psi\rangle \otimes |\phi'\rangle.$$

We need also to define bra-vectors, and the inner product:

$$(|\psi\rangle \otimes |\phi\rangle)^\dagger = \langle\psi| \otimes \langle\phi|.$$

$$(\langle\psi'| \otimes \langle\phi'|)(|\psi\rangle \otimes |\phi\rangle) = \langle\psi'|\psi\rangle\langle\phi'|\phi\rangle.$$

If $\{|j\rangle_1\}$ is a basis for H_1 and $\{|k\rangle_2\}$ is a basis for H_2 then $\{|j\rangle_1 \otimes |k\rangle_2\}$ is a basis for $H_1 \otimes H_2$. Given two states in H_1 and H_2

$$|\psi\rangle = \sum_{j=1}^{D_1} \alpha_j |j\rangle_1, \quad |\phi\rangle = \sum_{k=1}^{D_2} \beta_k |k\rangle_2,$$

in terms of this basis

$$|\psi\rangle \otimes |\phi\rangle = \sum_{j,k} \alpha_j \beta_k |j\rangle_1 \otimes |k\rangle_2.$$

Generic states in $H_1 \otimes H_2$ are *not* product states:

$$|\Psi\rangle = \sum_{j,k} t_{jk} |j\rangle_1 \otimes |k\rangle_2.$$

Operator Tensor Products.

If A is an operator on H_1 and B on H_2, we construct a similar product to get a new operator $A \otimes B$ on $H_1 \otimes H_2$. Its properties are similar to tensor products of states:

$$(a\hat{A} + b\hat{A}') \otimes \hat{B} = a\hat{A} \otimes \hat{B} + b\hat{A}' \otimes \hat{B}.$$

$$\hat{A} \otimes (a\hat{B} + b\hat{B}') = a\hat{A} \otimes \hat{B} + b\hat{A} \otimes \hat{B}'.$$

$$(\hat{A} \otimes \hat{B})^\dagger = \hat{A}^\dagger \otimes \hat{B}^\dagger.$$

$$(\hat{A} \otimes \hat{B})(\hat{A}' \otimes \hat{B}') = \hat{A}\hat{A}' \otimes \hat{B}\hat{B}'.$$

$$\mathrm{Tr}\{\hat{A} \otimes \hat{B}\} = \mathrm{Tr}\{\hat{A}\}\mathrm{Tr}\{\hat{B}\}.$$

We can also apply these tensor product operators to tensor product states:

$$(\hat{A} \otimes \hat{B})(|\psi\rangle \otimes |\phi\rangle) = \hat{A}|\psi\rangle \otimes \hat{B}|\phi\rangle.$$

$$(\langle\psi| \otimes \langle\phi|)(\hat{A} \otimes \hat{B}) = \langle\psi|\hat{A} \otimes \langle\phi|\hat{B}.$$

A general operator on $H_1 \otimes H_2$ is *not* a product operator, but can be written as a linear combination of product operators:

$$\hat{O} = \sum_{\ell} \hat{A}_\ell \otimes \hat{B}_\ell.$$

Tensor products play an important role in quantum mechanics! They describe how the Hilbert spaces of subsystems are combined.

Matrix representation.

What does the matrix representation of a tensor product look like? If $|\psi\rangle$ has amplitudes $(\alpha_1, \cdots, \alpha_{D_1})$ and $|\varphi\rangle$ has amplitudes $(\beta_1, \cdots, \beta_{D_2})$, the state $|\psi\rangle \otimes |\varphi\rangle$ in $H_1 \otimes H_2$ is represented as a $D_1 D_2$-dimensional column vector:

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \alpha_1|\phi\rangle \\ \alpha_2|\phi\rangle \\ \vdots \\ \alpha_{D_1}|\phi\rangle \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \vdots \\ \alpha_1\beta_{D_2} \\ \alpha_2\beta_1 \\ \vdots \\ \alpha_{D_1}\beta_{D_2} \end{pmatrix}$$

Example

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 \\ 1 \times 3 \\ 2 \times 2 \\ 2 \times 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \end{bmatrix}$$

Similarly, if $A = [a_{ij}]$ and $B = [b_{ij}]$ then

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_{11}\hat{B} & \cdots & a_{1D_1}\hat{B} \\ \vdots & \ddots & \vdots \\ a_{D_1 1}\hat{B} & \cdots & a_{D_1 D_1}\hat{B} \end{pmatrix}$$

Example

$$\hat{I} \otimes \hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{Z} \otimes \hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\hat{I} \otimes \hat{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{X} \otimes \hat{Y} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

For brevity, $|\psi\rangle \otimes |\varphi\rangle$ is often written $|\psi\rangle|\varphi\rangle$ or even $|\psi\varphi\rangle$. One can similarly condense the notation for operators; but one should be very careful not to confuse $A \otimes B$ with AB.

Example: Two Spins

Given two states in the *Z* basis,

$$|\psi\rangle = \alpha_1 |\uparrow\rangle + \alpha_2 |\downarrow\rangle \text{ and } |\varphi\rangle = \beta_1 |\uparrow\rangle + \beta_2 |\downarrow\rangle$$

we can write

$$|\psi\rangle \otimes |\varphi\rangle = \alpha_1\beta_1 |\uparrow\uparrow\rangle + \alpha_1\beta_2 |\uparrow\downarrow\rangle + \alpha_2\beta_1 |\downarrow\uparrow\rangle + \alpha_2\beta_2 |\downarrow\downarrow\rangle.$$

A general state of two spins would be

$$|\Psi\rangle = t_{11} |\uparrow\uparrow\rangle + t_{12} |\uparrow\downarrow\rangle + t_{21} |\downarrow\uparrow\rangle + t_{22} |\downarrow\downarrow\rangle.$$

As a column vector this is

$$|\Psi\rangle = \begin{pmatrix} t_{11} \\ t_{12} \\ t_{21} \\ t_{22} \end{pmatrix}$$