Advanced Quantum Mechanics

Angelo Bassi

Academic Year 2022-23

Program

1 – Special Relativity
- Lorentz Transformations
- Relativistic kinematics
- Relativistic dynamics

2 – Quantum Mechanics and Relativity
- EPR paradox
- Bell's theorem
- No signalling, teleportation, no cloning

Textbook

Any book on Special Relativity

Simple derivation of the special theory of relativity without the speed of light axiom

O Certik[1,2]

arXiv:0710.3398v1

Newtonian Physics

Classical Mechanics describes the motion of point particles subject to forces = Newtonian Mechanics

1. Law of inertia
2. F = ma
3. Action-reaction law

Newtonian Physics

Question: When are they valid? Always?

Answer: No. They are valid only for inertial reference frames

Question: What is an inertial frame?

Answer: It is a frame where the first law is valid.

Newtonian Physics

Therefore, the first law establishes the reference frames where the other two laws are valid (for this reason, the first law is not a special case of the second, as it might seem at first sight).

The second law can be extended to non-inertial frames, by introducing *fictitious forces* ➔ more complicated dynamics

Absolute Space

Question: Why are inertial frames so special?

Answer (Newton, 1642 – 1727): Because they are those at rest or moving at constant speed with respect to absolute space.

In this way, Newton introduced a new concept in his theory, having a metaphysical character.

Absolute Space

"Absolute space, in its own nature, without regard to anything external, remains always similar and immovable…"

"Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration…"

(Newton, *Principia*)

Absolute Space

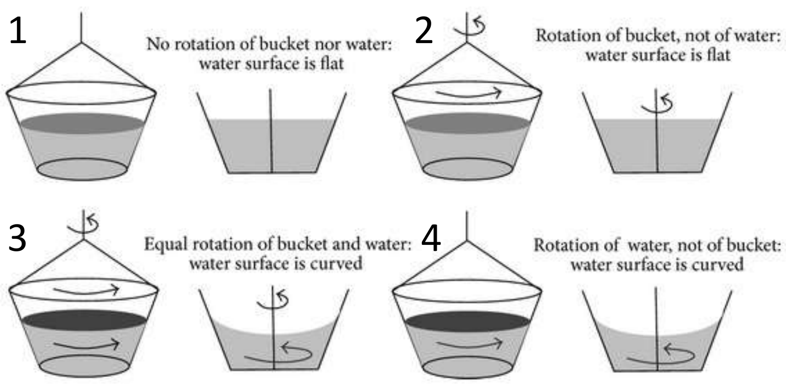
The idea of absolute space and time proved particularly controversial from Newton's times to present. In particular:

Leibnitz (1646 - 1716) thought that space made no sense expect as relative location of bodies; same for time.

Bishop Berkeley (1685 - 1753) had similar ideas.

Absolute Space – the Bucket Experiment

To justify the introduction of these new concepts, Newton devised the bucket experiment



1-3 and 2-4 are examples of same relative motion, yet they are different. The concavity of water is not due to relative motion, but to motion with respect to absolute space.

Mach- back to relative motion

Mach (1838 - 1916) proposed that mechanics is entirely about relative motion. The inertial properties of a body (its mass) are an expression of the interaction with the other bodies in the universe.

According to Mach, *Newton's bucket example illustrates relative motion with respect to the bulk of the universe.*

There is an important piece of evidence in support of Mach's ideas: the fixed stars are not accelerating with respect to absolute space. This is a coincidence according to Newton, a necessity according to Mach.

General Relativity

Einstein (1879 - 1955) was very influenced by Mach's ideas. The story brings to General Relativity, but we will stop here.

It is important to stress that what was thought to be philosophical questions brought to deep changes in our understanding of nature and to new theories.

Principle of Relativity

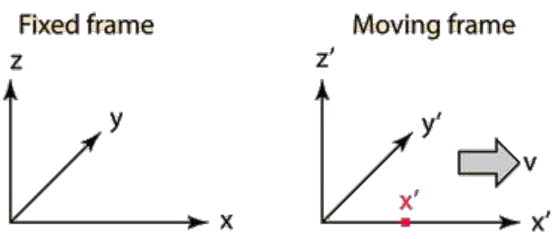
Newton's laws satisfy the principle of relativity, first formulated by Galileo (1564 - 1642)

Physical laws are the same in every inertial frame

In particular, the second law (F = ma) satisfies the principle of relativity. How?

Galilei transformations

To prove invariance, we need to relate two inertial frames. We use Galilei transformations connecting coordinates (t,x,y,z) of an inertial frame O to the coordinates (t',x',y',z') of another inertial frame O'.



$$\boxed{\begin{aligned} t' &= t \\ x' &= x - v \cdot t \\ y' &= y \\ z' &= z \end{aligned}} \qquad \boxed{\begin{aligned} t &= t' \\ x &= x' + v \cdot t' \\ y &= y' \\ z &= z' \end{aligned}}$$

Example

Take two particles interacting through a force depending on the modulus of the distance: $\mathbf{F}(\mathbf{r}_1, \mathbf{r}_2) = \mathbf{F}(|\mathbf{r}_1 - \mathbf{r}_2|)$. Then:

In frame O, for particle 1, we have:

$m_1\mathbf{a}_1 = \mathbf{F}(|\mathbf{r}_1 - \mathbf{r}_2|)$

A trivial calculation shows that in frame O':

$m_1\mathbf{a}_1' = \mathbf{F}(|\mathbf{r}_1' - \mathbf{r}_2'|)$

Why Galilei transformations?

Because they reflect the **geometric properties** of Newtonian spacetime!

Absolute time: $\Delta_t = |t_1 - t_2|$ is the same for every frame. In particular, this implies that simultaneous events in one frame are simultaneous in every frame.

Absolute space: $\Delta_\mathbf{r} = |\mathbf{r}_1 - \mathbf{r}_2|$ is the same for every frame.

Electromagnetism: Maxwell equation (1860s)

(1) $\nabla.\mathbf{E} = \rho/\varepsilon_o$	*Poisson's Equation*
(2) $\nabla.\mathbf{B} = 0$	*No magnetic monopoles*
(3) $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$	*Faraday's Law*
(4) $\nabla \times \mathbf{B} = \mu_o\mathbf{j} + \mu_o\varepsilon_o\partial\mathbf{E}/\partial t$	*Maxwell's Displacement*

Electromagnetism

They imply the wave equation (in vacuum)

$\nabla^2\mathbf{E} - \mu_o\varepsilon_o\partial^2\mathbf{E}/\partial t^2 = 0$ Vector wave equation

This is not invariant under Galilei transformations!

Is it a problem?

Wave equations and Galilei transformations

The wave equation is

$$\boxed{\nabla^2 f - \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = 0} \tag{15}$$

in the inertial frame with coordinates $(\mathbf{x}, t)$ where the medium is at rest and the wave velocity is u (at angular frequency ω). The Galilean coordinate transformation to an inertial frame that moves with velocity $\mathbf{v}$ with respect to the rest frame of the medium is given by eq. (4). The transformations of derivatives with respect to the coordinates are

$$\frac{\partial}{\partial x} = \frac{\partial \mathbf{x}'}{\partial x}\cdot \boldsymbol{\nabla}' + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial t} = \frac{\partial \mathbf{x}'}{\partial t}\cdot \boldsymbol{\nabla}' + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \frac{\partial}{\partial t'} - \mathbf{v}\cdot\boldsymbol{\nabla}', \tag{16}$$

so the wave equation (15) transforms to

$$\boxed{\nabla'^2 f' - \frac{1}{u^2}\frac{\partial^2 f'}{\partial t'^2} + \frac{2}{u^2}\frac{\partial}{\partial t'}\mathbf{v}\cdot\boldsymbol{\nabla}' f' - \frac{(\mathbf{v}\cdot\boldsymbol{\nabla}')^2}{u^2} f' = 0} \tag{17}$$

in the moving frame.

Wave equations and Galilei transformations

Solution of (15)

Consider a wave

$$f = \cos(\mathbf{k}\cdot\mathbf{x} - \omega t), \tag{2}$$

in the inertial frame with coordinates $(\mathbf{x}, t)$ in which the elastic medium that supports the wave is at rest. The phase velocity of this wave has magnitude $u = \omega/k$ and direction $\mathbf{k}$. That is, the phase velocity vector of the wave is

$$\mathbf{u}_p = \frac{\omega}{k}\hat{\mathbf{k}} = \frac{\omega}{k^2}\mathbf{k}. \tag{3}$$

An observer whose velocity is $\mathbf{v}$ with respect to the original frame uses coordinates $(\mathbf{x}', t')$ to describe an event $(\mathbf{x}, t)$ in the original frame obtained by the Galilean transformation

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \qquad t' = t, \tag{4}$$

supposing that the spatial axes in the two frames are parallel.

Wave equations and Galilei transformations

The moving observer sees the wave to have the same amplitude as described by eq. (2), which he describes in terms of $(\mathbf{x}', t')$ as the wave function f',

Solution of (17)

$$f' \equiv \cos(\mathbf{k}' \cdot \mathbf{x}' - \omega' t') = f = \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) = \cos[\mathbf{k} \cdot \mathbf{x}' - (\omega - \mathbf{k} \cdot \mathbf{v})t']. \quad (5)$$

From eq. (5) we see that the wave vector is the same in the moving frame as in the original frame,

$$\mathbf{k}' = \mathbf{k}, \quad (6)$$

so the wavelength is the same in both frames, and the direction of the wave vector is the same in both frames (or the direction of the wave in the moving frame is opposite to that in the original frame if $\mathbf{k} \cdot \mathbf{v} > \omega$). Similarly, the wave frequency in the moving frame is

$$\omega' = |\omega - \mathbf{k} \cdot \mathbf{v}|, \quad (7)$$

which is the well-known Doppler effect for a source at rest and a moving observer.

Wave equations and Galilei transformations

The phase velocity $\mathbf{u}'_p$ of the wave in the moving frame is given by

$$\mathbf{u}'_p = \frac{\omega'}{k'^2}\mathbf{k}' = \frac{\omega'}{k^2}\mathbf{k} = \frac{\omega - \mathbf{k} \cdot \mathbf{v}}{k^2}\mathbf{k} = \mathbf{u}_p - (\mathbf{v} \cdot \hat{\mathbf{u}}_p)\hat{\mathbf{u}}_p = \mathbf{u}_p - \mathbf{v}_\parallel = \mathbf{u}_p - \mathbf{v} + \mathbf{v}_\perp \quad (8)$$

if $\mathbf{k} \cdot \mathbf{v} < \omega$, noting that the components of velocity $\mathbf{v}$ that are parallel and perpendicular to velocity $\mathbf{u}_p$ are $\mathbf{v}_\parallel = (\mathbf{v} \cdot \hat{\mathbf{u}}_p)\hat{\mathbf{u}}_p$ and $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel$, respectively. When $\mathbf{k} \cdot \mathbf{v} > \omega$, the phase velocity in the moving frame is

$$\mathbf{u}'_p = -\frac{\omega'}{k^2}\mathbf{k} = -\frac{\mathbf{k} \cdot \mathbf{v} - \omega}{k^2}\mathbf{k} = \mathbf{u}_p - (\mathbf{v} \cdot \hat{\mathbf{u}}_p)\hat{\mathbf{u}}_p, \quad (9)$$

so the form of the phase velocity transformation is independent of the magnitude of $\mathbf{k} \cdot \mathbf{v}$. However, the transformation of the wave velocity is NOT the same as the transformation of velocity of a particle ($\mathbf{u}' = \mathbf{u} - \mathbf{v}$) if the direction $\mathbf{k}$ of the wave is different from that of the boost $\mathbf{v}$.

If **u** and **v** are parallel, then: **u’** = **u – v.**

Wave equations and Galilei transformations

There is nothing problematic, and there should not be: waves exist in Galilean physics: water and sound waves are an example.

The point here is that waves move in a **medium**, which selects a **preferred reference frame**, that in which the medium is at rest.

The wave equation takes a simple form in the frame in which the medium is at rest; in all other frames it takes a different form.

The **wave** is an **effective effect**. At the fundamental level there is the dynamics of the medium (interactions among molecules), which is **Galilei invariant**.

The medium

Waver waves: water

Sound waves: air

Earthquakes: ground

Electromagnetic waves: aether

To assuming the existence of the aether was the most natural thing to do at the time.

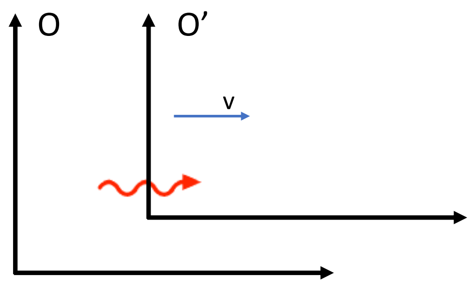
Chasing the aether

If all this is true, one has to look after the aether, which we do not have direct experience of.

First thing to do: Since **only in the reference frame at rest with respect to the aether the speed of light is c**, while in all other inertial frames it is different, one can look for such differences.

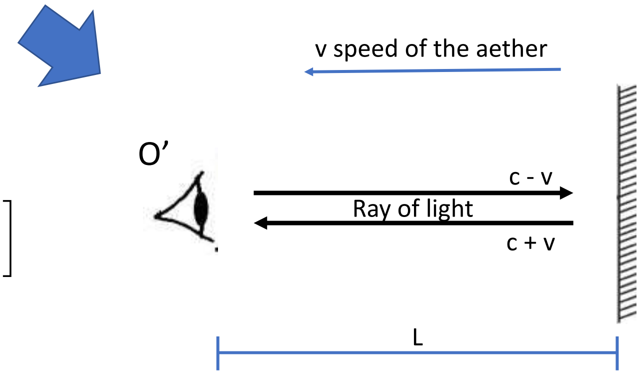
Since the Earth revolves around the Sun, it cannot always be at rest with respect to the aether, therefore we should be able to detect such differences.

Fizeau Experiment



In O, at rest with respect to the aether, light has speed c.

In O’ it has speed c-v.

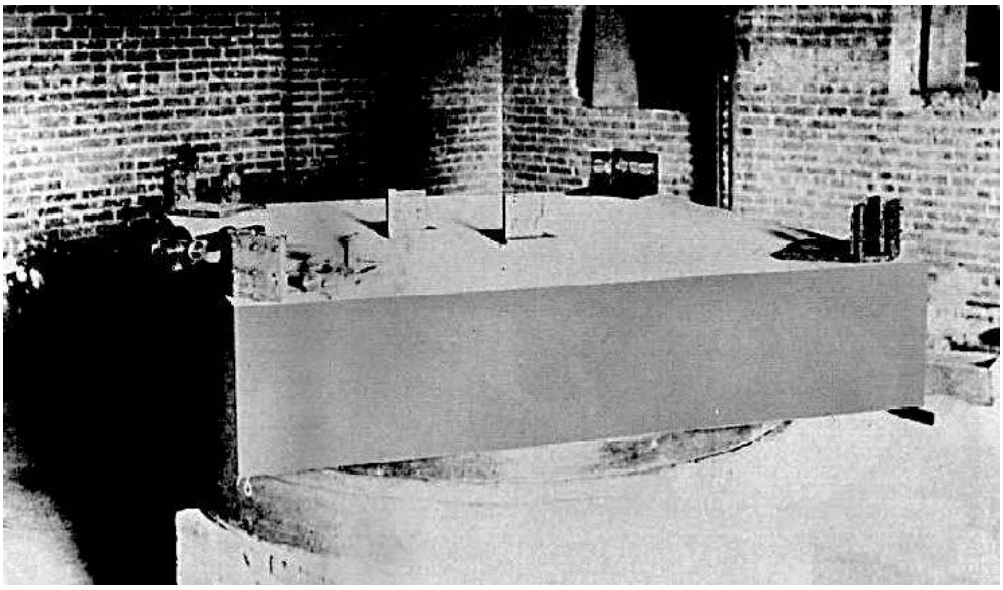


$$T = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2L}{c^2 - v^2}$$

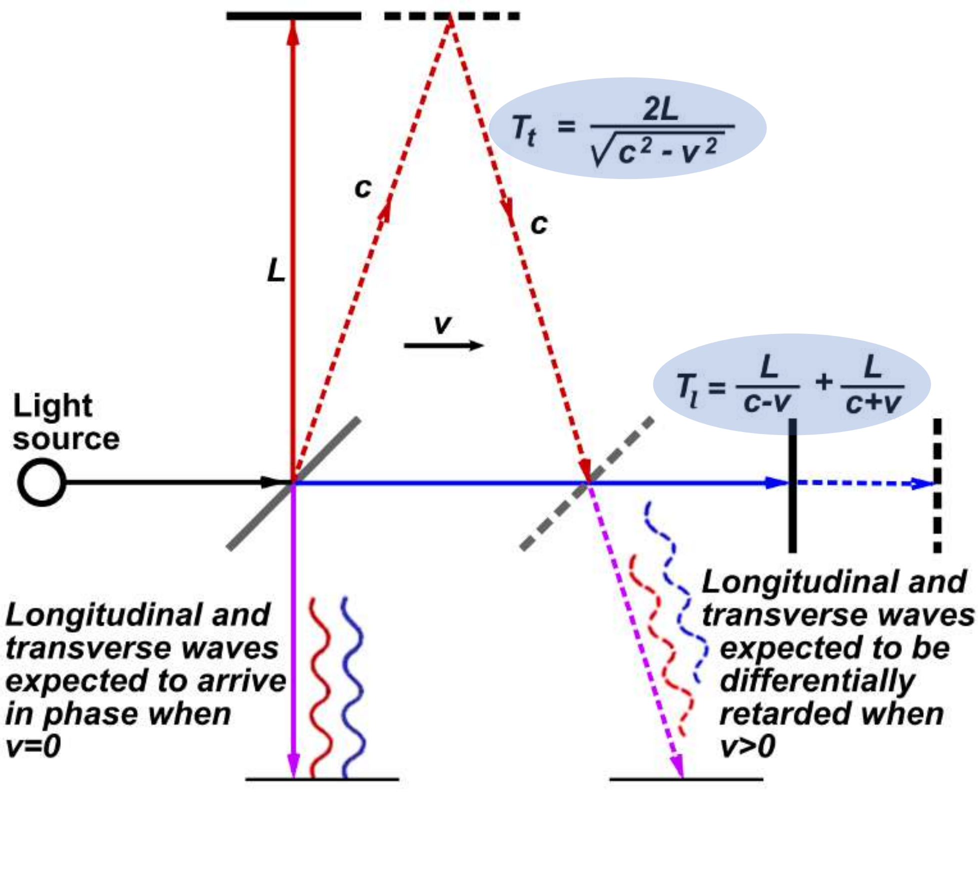
$$= \frac{2L}{c} \cdot \frac{1}{1 - v^2/c^2} = \frac{2L}{c}\left[1 + \frac{v^2}{c^2} \dots\right]$$

Very small to detect

Michelson-Morley Experiment (1887)



Michelson-Morley Experiment



Michelson-Morley Experiment

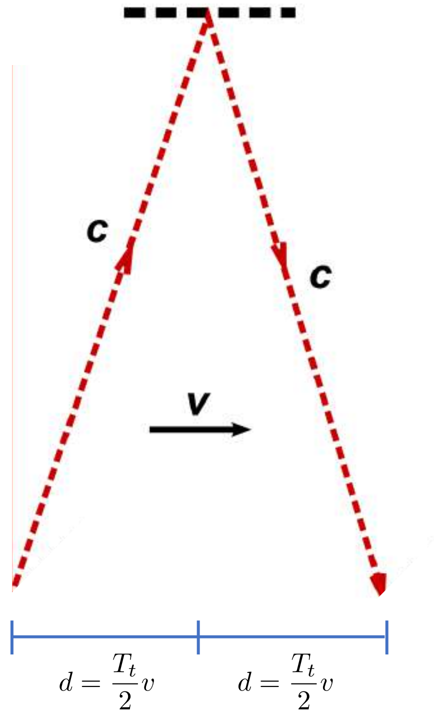
In the aether's rest frame, the lab moves.

The total path is:

$$L_t = 2\sqrt{L^2 + \frac{T_t^2}{4}v^2}$$

Since in this frame the speed of light is c:

$$T_t = \frac{L_t}{c} = \frac{2}{c}\sqrt{L^2 + \frac{T_t^2}{4}v^2} \quad \rightarrow \quad T_t = \frac{2L}{\sqrt{c^2 - v^2}}$$



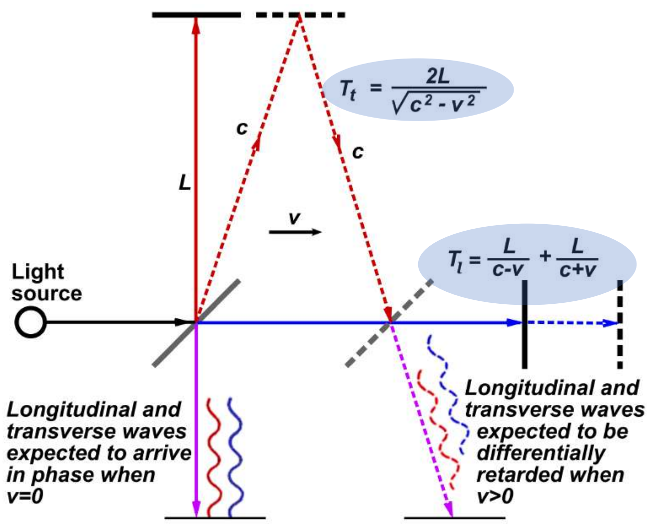
Michelson-Morley Experiment

The time difference between the two paths is

$$\Delta T = T_l - T_t = \frac{2L}{c}\left[\frac{1}{1 - v^2/c^2} - \frac{1}{\sqrt{1 - v^2/c^2}}\right]$$

$$= \frac{2L}{c}\left[1 + \frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2} + \dots\right]$$

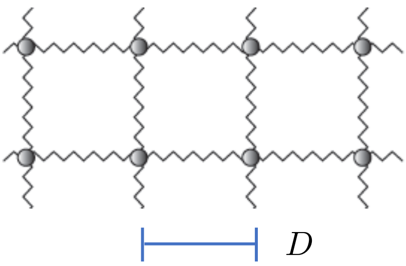
$$= \frac{L}{c}\cdot\frac{v^2}{c^2} + \dots$$



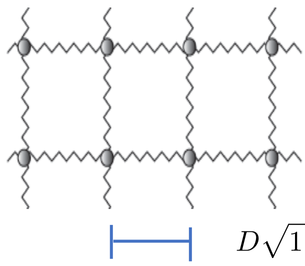
Time difference means **different paths**. An interference will appear.
Suppose the apparatus is rotated. Then the fringes will change. **No change!**

Lorentz theory (1853-1928)

The aether has an effect on matter, because **matter is held together by electrostatic forces**, and these changes depending on the aether's motion



D



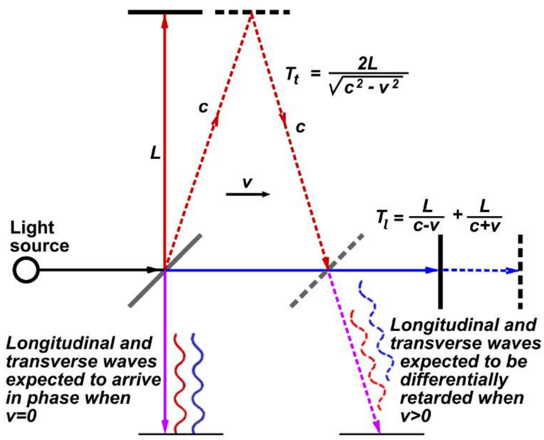
$D\sqrt{1-v^2/c^2}$

Contraction of objects along the direction of motion through the aether

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

Lorentz theory

This explains the null result of Michelson-Morley.



The longitudinal and vertical distances are not equal as initially assumed. Then

$$\Delta T = T_l - T_t = \frac{2}{c}\left[\frac{L}{1-v^2/c^2}\cdot\sqrt{1-c^2/c^2} - \frac{L}{\sqrt{1-v^2/c^2}}\right] = 0$$

New term

One can show that this holds for any position of the arms

Lorentz theory

To make everything consistent, also clocks have to run differently. Looking back at the experiment by Fizeau

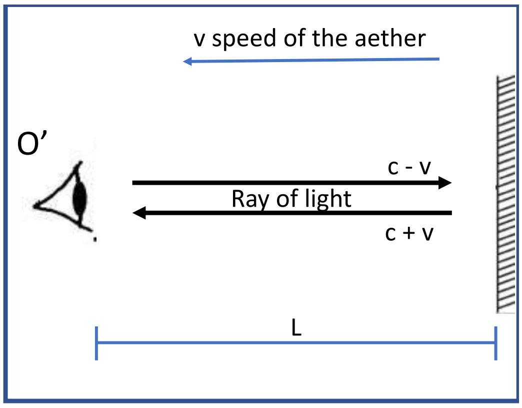
$$T = \frac{2L}{c} \cdot \frac{1}{1 - v^2/c^2}$$

We now know that lengths are contracted, so the correct answer is

$$T = \frac{2L_0}{c} \cdot \frac{1}{\sqrt{1 - v^2/c^2}}$$

The time difference is not 0, therefore this experiment could reveal the aether

Lorentz theory



Lorentz assumed that clock run slower when in motion with respect to the aether

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$ together with $L = L_0\sqrt{1 - \frac{v^2}{c^2}}$

Then when we are at rest with respect to the aether: $T_0 = \frac{2L_0}{c}$

When we are in motion: $T = \frac{2L}{c} \cdot \frac{1}{1 - v^2/c^2}$

The two expression, the one at rest and the one in motion, coincide. One cannot detect any difference.

Lorentz theory

The theory described by Lorentz works and is capable of describing the (null) outcomes of measurements.

But it also implies that we will **never be able to detect** the existence of **the aether**, because its effects cancel out. This is something completely different with respect to other media supporting waves.

Then Einstein came….

On the Electrodynamics of moving bodies (1905)

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

Electromagnetic phenomena are the same in all inertial frames

The description given by Maxwell's equation is not

On the Electrodynamics of moving bodies (1905)

Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the ***phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest.*** *They suggest rather that, as has already been shown to the first order of small quantities,* ***the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good.***

On the Electrodynamics of moving bodies (1905)

We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies. The introduction of a "luminiferous ether" will prove to be superfluous inasmuch…

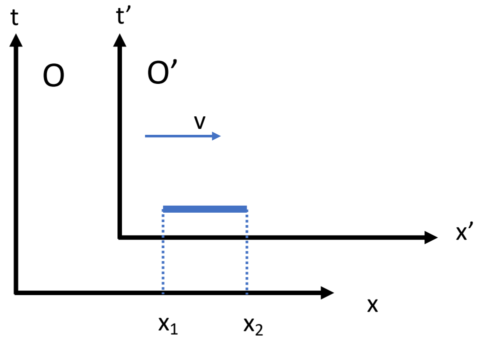
Lorentz Transformations

We now derive Lorentz transformation from the following assumptions:

- Homogeneity of spacetime
- Isotropy of spacetime
- The principle of relativity
- The constancy of the speed of light

We will se that – in a sense – Galileo could have discovered them (in a sense!)

Lorentz Transformations- homogeneity



The most general transformation from S to S' is:

$$
\begin{aligned}
t' &= T(t, x, y, z, v) \\
x' &= X(t, x, y, z, v) \\
y' &= Y(t, x, y, z, v) \\
z' &= Z(t, x, y, z, v)
\end{aligned}
$$

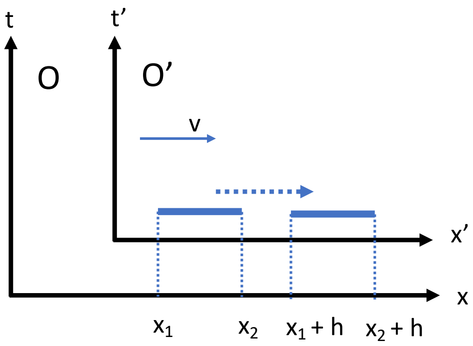
The length of a rod put on the x-axis in the frame S is

$$l = x_2 - x_1$$

and in the frame S' the length will generally be different:

$$l' = x'_2 - x'_1 = X(t, x_2, 0, 0, v) - X(t, x_1, 0, 0, v)$$

Lorentz Transformations - homogeneity



Homogeneity means, that if we move the left end of the rod in the frame S from x_1 to x_1+h, the right end will move to x_2+h giving the same length $l=(x_2+h)-(x_1+h)=x_2-x_1$ and that in the frame S' the new length $l'=X(t,x_2+h,0,0,v)-X(t,x_1+h,0,0,v)$ will also be the same as before:

$$X(t,x_2,0,0,v)-X(t,x_1,0,0,v)=X(t,x_2+h,0,0,v)-X(t,x_1+h,0,0,v)$$

so

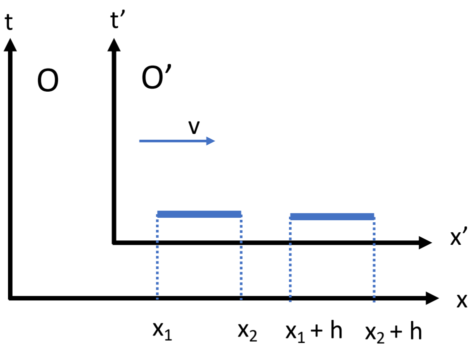
$$X(t,x_2+h,0,0,v)-X(t,x_2,0,0,v)=X(t,x_1+h,0,0,v)-X(t,x_1,0,0,v)$$

and dividing by h and taking a limit $h\to 0$:

$$\left.\frac{\partial X}{\partial x}\right|_{t,x_2,0,0}=\left.\frac{\partial X}{\partial x}\right|_{t,x_1,0,0}$$

but x_1 and x_2 are arbitrary, so $\frac{\partial X}{\partial x}$ is constant so $X(t,x,y,z,v)$ is linear with respect to x. Similar procedure shows, that $X(t,x,y,z,v)$ is linear with respect to y, z and t, and the same for Y, Z and T, which means, that

Lorentz Transformations - homogeneity



$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = A(v)\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$A(v)=\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

A inhomogeneous term B(v) can be added, which amounts to translations in space and time. We do not consider it.

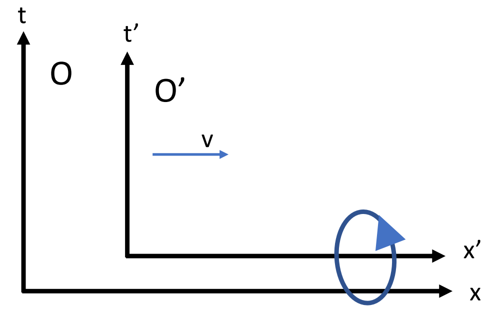
Lorentz Transformations - isotropy

The isotropy also implies, that since the only significant spacial direction is that of the (x, x')-axis – the direction of motion – the transformation $A(v)$ must be the same as if we first rotate about the (x, x')-axis, transform and then rotate back:

$$R(-\alpha)A(v)R(\alpha) = A(v)$$

where the $R(\alpha)$ is a matrix, that rotates the system around the x axis:

$$R(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$



Lorentz Transformations - isotropy

For each α, we have:

$$R(\alpha)A(v) = A(v)R(\alpha)$$

where

$$R(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & P(\alpha) \end{pmatrix}$$

$$P(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \mathbb{1}\cos\alpha + i\sigma_2 \sin\alpha = e^{i\alpha\sigma_2}$$

$$A(v) = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

and the σ_1, σ_2 and σ_3 are the Pauli matrices. Then

$$R(\alpha)A(v) - A(v)R(\alpha) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & P(\alpha) \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} - \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & P(\alpha) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & A_2(\mathbb{1} - P(\alpha)) \\ (P(\alpha) - \mathbb{1})A_3 & P(\alpha)A_4 - A_4P(\alpha) \end{pmatrix} = 0$$

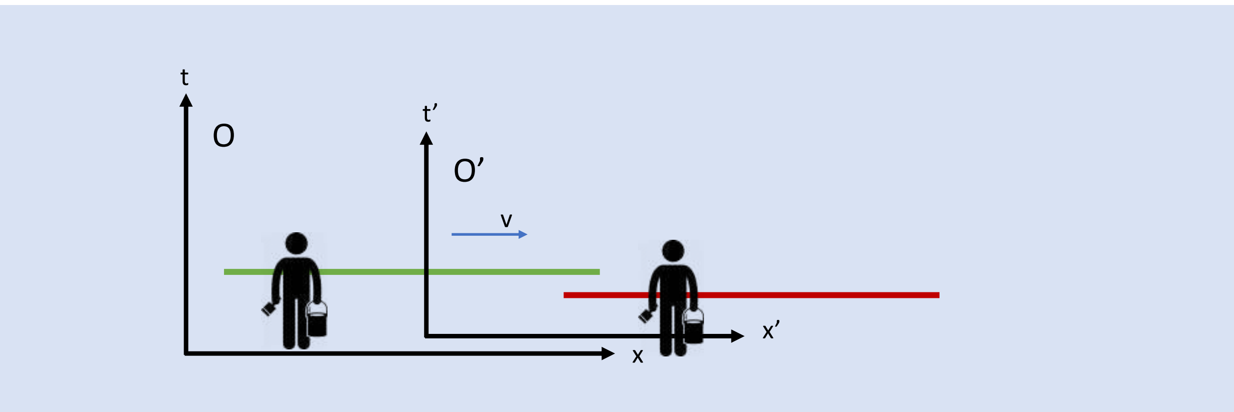
The parameter α is arbitrary, so $A_2 = A_3 = 0$

Lorentz Transformations- isotropy

So we end up with

$$A(v) = \begin{pmatrix} D & C & 0 & 0 \\ B & A & 0 & 0 \\ 0 & 0 & F & G \\ 0 & 0 & H & L \end{pmatrix}$$

Lorentz Transformations - relativity



Lorentz Transformations - relativity

Distances perpendicular to the direction of motion should not be affected:

y' = y

z' = z

Therefore

$$A(v) = \begin{pmatrix} D & C & 0 & 0 \\ B & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz Transformations

We are left with:

x' = Ax + Bt

t' = Cx + Dt

The inverse relations are:

$$\mathrm{x} = \frac{\mathrm{Dx'} - \mathrm{Bt'}}{\mathrm{AD} - \mathrm{BC}}$$

$$\mathrm{t} = \frac{\mathrm{At'} - \mathrm{Cx'}}{\mathrm{AD} - \mathrm{BC}}$$

Lorentz Transformations

Consider now the origin of O', which is $x' = 0$:

$Ax + Bt = 0 \rightarrow x/t = -B/A = v$ (velocity of O' with respect to O)

Then: $B = -vA$

So we have:

$x' = A(x - vt)$

$t' = Cx + Dt$

Lorentz Transformations

Now consider the origin of O, which is $x = 0$. From the inverse relations:

$Dx' - Bt' = Dx' + vAt' 0 \rightarrow x'/t' = -vA/D = -v$

(velocity of O with respect to O')

Then: $A = D$

Lorentz Transformations

So we have:

$x' = A(x - vt)$

$t' = A(t + Cx/A)$

The inverse relations are:

$$x = \frac{A(x'+vt')}{A^2+vAC}$$

$$t = \frac{A(t'-cx'/A)}{A^2+vAC}$$

Because of relativity, since the two frames are equivalent, any difference in the transformations can only be due to the fact that O sees O' moving with velocity v, while O' sees O' moving with velocity –v. Then:

$A^2 + vAC = 1$ or

$$C = -\frac{A^2-1}{vA}$$

Lorentz Transformations

So we end up with

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} A & -\frac{A^2-1}{vA} & 0 & 0 \\ -vA & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Now we use again relativity

For each u and v there exist w such that (relativity):

$$A(u)A(v) = A(w)$$

Lorentz Transformations

$$A(u)A(v) = \begin{pmatrix} A_u & -\frac{A_u^2-1}{uA_u} & 0 & 0 \\ -uA_u & A_u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_v & -\frac{A_v^2-1}{vA_v} & 0 & 0 \\ -vA_v & A_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A(w)$$

Multiplying the matrices:

$$A(u)A(v) = \begin{pmatrix} A_uA_v + (A_u^2-1)\frac{vA_v}{uA_u} & \dots & 0 & 0 \\ \dots & A_uA_v + (A_v^2-1)\frac{uA_u}{vA_v} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A(w) = \begin{pmatrix} A_w & -\frac{A_w^2-1}{wA_w} & 0 & 0 \\ -wA_w & A_w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz Transformations

so comparing the two expressions for A_w (the first and the second diagonal element) we get:

$$\frac{A_v^2-1}{v^2A_v^2} = \frac{A_u^2-1}{u^2A_u^2}$$

where the left hand side only depends on v, the right hand side only on u, thus both sides are equal to a constant K, that is independent of the frame of reference, because it doesn't depend on the coordinates or v, so we get (remember A(0)=1, so we take the positive square root)

$$A_v = \frac{1}{\sqrt{1-Kv^2}}$$

Lorentz Transformations

and we arrive at the expression for the transformation between S and S':

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-Kv^2}} & -\frac{Kv}{\sqrt{1-Kv^2}} & 0 & 0 \\ -\frac{v}{\sqrt{1-Kv^2}} & \frac{1}{\sqrt{1-Kv^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

which are the Lorentz transformations… almost.

Note that so far we did not use the constancy of the speed of light

Note that it must be: v < K. <u>There is a limit to the allowed speeds</u>

Lorentz Transformations – speed of light

Consider an electromagnetic wave

In O: $x^2 + y^2 + z^2 - c^2t^2 = 0$ (A)

In O': $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$ (B)

By using Lorentz transformations to go from (B) to (A):

$$\frac{1-c^2K^2v^2}{1-Kv^2}x^2 + y^2 + z^2 - c^2\frac{1-v^2/c^2}{1-Kv^2}t^2 - 2\frac{1-c^2K^2}{1-Kv^2}xt$$

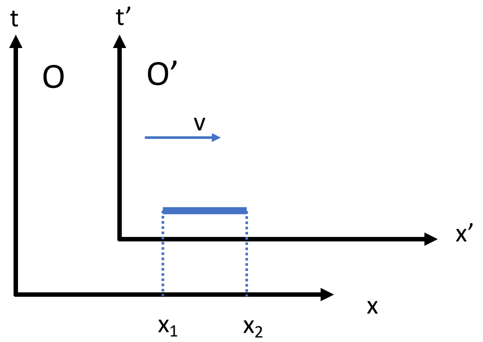
This should be equivalent to (A). This implies: <u>**$K = 1/c^2$**</u>

Lorentz Transformations

$$\begin{aligned} ct' &= \gamma\left(ct - \beta x\right) \\ x' &= \gamma\left(x - \beta ct\right) \\ y' &= y \\ z' &= z. \end{aligned}$$

$$\gamma = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1} \qquad \beta = \frac{v}{c},$$

Length contraction



In an inertial reference frame S, x_1 and x_2 shall denote the endpoints of an object in motion in this frame. There, its length L was measured according to the above convention by determining the simultaneous positions of its endpoints at $t_1 = t_2$. Now, the proper length of this object in S' shall be calculated by using the Lorentz transformation. Transforming the time coordinates from S into S' results in different times, but this is not problematic, as the object is at rest in S' where it does not matter when the endpoints are measured. Therefore, the transformation of the spatial coordinates suffices, which gives:

Length contraction

$$x_1' = \gamma (x_1 - vt_1) \quad \text{and} \quad x_2' = \gamma (x_2 - vt_2)$$

Since $t_1 = t_2$, and by setting $L = x_2 - x_1$ and $L_0' = x_2' - x_1'$, the proper length in S' is given by

$$L_0' = L \cdot \gamma. \qquad (1)$$

with respect to which the measured length in S is contracted by

$$L = L_0' / \gamma. \qquad (2)$$

According to the relativity principle, objects that are at rest in S have to be contracted in S' as well.
By exchanging the above signs and primes symmetrically, it follows:

$$L_0 = L' \cdot \gamma. \qquad (3)$$

Thus the contracted length as measured in S' is given by:

$$L' = L_0 / \gamma. \qquad (4)$$

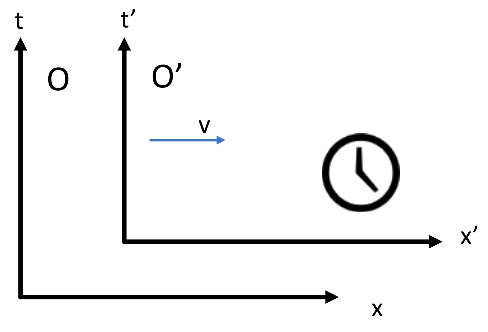
Moving objects are contracted

Length contraction - comments

As we saw, length contraction was first proposed by Lorentz. He assumed that when moving through the aether, the intermolecular forces change in such a way to cause the contraction of the object.

According to Einstein, the situation is entirely different. The contraction is a property of space, which affects objects living in it. In fact, in place of the rod we could have considered two “mathematical” points in spacetime.

Time dilation



Clock at rest in O'. It emits two ticks at times t_1' and t_2'. The time difference is:

$T' = t_2' - t_1'$

The two spacetime events are (t_1', x') and (t_2', x') – same space point since it is at rest.

Then according to Lorentz transformations:

$t_1 = \gamma(t_1' + (v/c^2)\, x')$

$t_2 = \gamma(t_2' + (v/c^2)\, x')$

$\Rightarrow$

$\boxed{T = t_2 - t_1 = \gamma\, T' > T'}$

Time dilation- comments

Moving clocks run slower. As for length contraction, this is not a feature of the mechanical working of the clock, is it a property of time.

Best experimental evidence: increase of the lifetime of radioactive particles moving at speed close to c.

A muon is a charged particle that decays into an electron or positron, a neutrino and an anti-neutrino:

$$\mu^+ \to e^+ + n_1 + n_2 \quad \text{or} \quad \mu^- \to e^- + n_1 + n_2$$

Muons occur in cosmic rays travelling through the atmosphere at speeds very close to that of light.[13] In 1941, Rossi and Hall measured the flux of muons in a laboratory at 6300 feet above sea level (top of Mount Washington) and in a laboratory at 2000 feet above sea level (bottom of Mount Washington).

At the top they measured 550 muons per hour. At the foot (simultaneously) they measured 422 muons per hour. The half-life of the muon is 1.56 microseconds.

Muon decay

From this information, one can calculate how long the muons spend travelling between the two laboratories and hence the speed of the muons *and the speed turns out to be much faster than the speed of light.* The muons travel a distance D at speed v taking time $T = D/v$; during this time the number $N(T)$ of muons remaining is given by

$$N(T) = N(0)\left(\tfrac{1}{2}\right)^{(T/T_{\text{half}})}.$$

Thus

$$v = \frac{D}{T} = \frac{D\log\frac{1}{2}}{T_{\text{half}}\log\left(N(T)/N(0)\right)} = \frac{4300\log\frac{1}{2}}{1.56\log(422/550)} = 7212$$

in units of feet per microsecond. The speed of light in these units is about 1000.

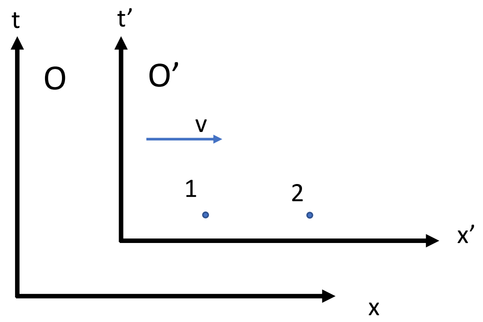
To put it another way, the observed flux of muons at the lower laboratory is far too high for particles covering the distance at less than the speed of light: many more should have decayed in the travel time.

What is the explanation? As we will see, it depends on whether we work in the rest frame of the muon or the rest frame of the laboratory, the two being in relative motion at close to the speed of light.

In the rest frame of the laboratory, the explanation is time dilation: time in the moving frame is dilated relative to time in the rest frame, which means that clocks are ticking slower, by a factor of γ, in the moving frame. Suppose the travel time is T seconds as measured (by distance/speed) in the laboratory frame. Then in this interval, only T/γ seconds have elapsed in the moving muon frame, so far fewer muons will decay, corresponding to a half-life of 1.56γ microseconds.

But how can this be explained in the muon frame, where the half-life is 1.56 microseconds? The explanation now is length contraction. In the rest frame of the muon, Mount Washington, which is zooming towards the muon at high speed, is only $6000/\gamma$ feet high, because lengths of moving rulers are contracted. Thus the time taken to cover this contracted distance is short: only T/γ seconds. There is little time for the muons to decay.

Relativity of simultaneity



$t_1' = t_2'$

The two events are simultaneous in O'.
Then in O:

$t_1 = \gamma(t_1' + (v/c^2)\, x_1')$

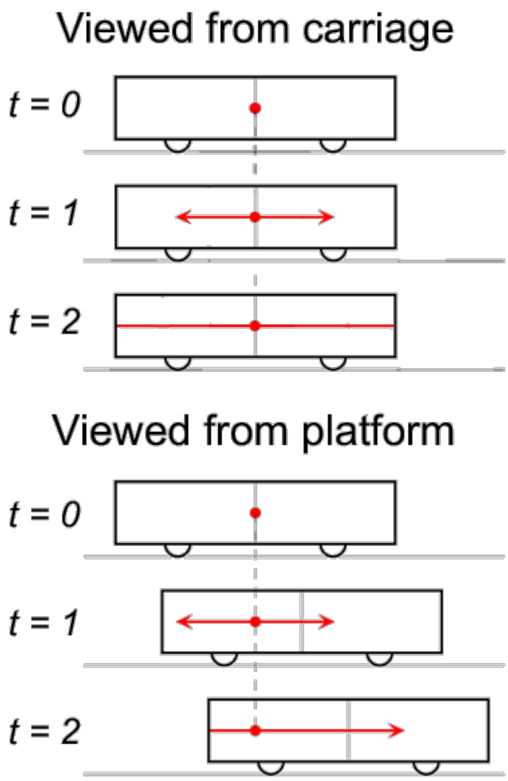
$t_2 = \gamma(t_2' + (v/c^2)\, x_2')$

$$t_2 - t_1 = \gamma(t_2' - t_1') + \gamma(v/c^2)(x_2' - x_1') = \gamma(v/c^2)(x_2' - x_1') \neq 0$$

The two events are not simultaneous in O

Relativity of simultaneity - comment

It is essentially a consequence of the constancy of the speed of light in all inertial frames.



The ladder & barn (non-)paradox

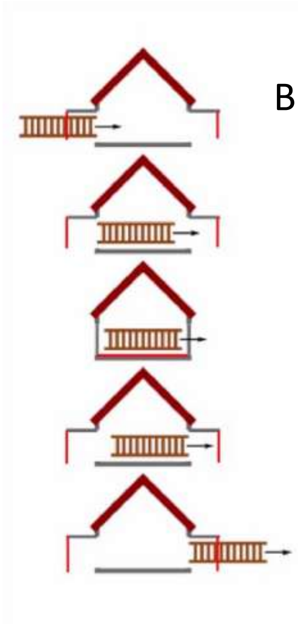
A builder runs towards a barn of length L carrying a ladder of length $2L$ at a speed[19] such that $\gamma = 2$ so that the length contraction factor is $\frac{1}{2}$.

- In the barn's rest frame, the moving ladder undergoes length contraction and has length L. It can therefore fit snugly in the barn.
- In the builder's rest frame, the barn is rushing towards the ladder and undergoes length contraction to $L/2$. There is no way the ladder can fit in.

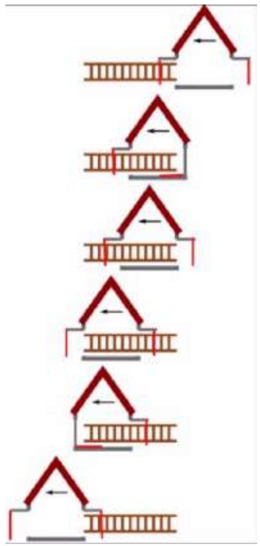
How can these two statements be reconciled?

[19] If $v = \sqrt{3}c/2$, then $\gamma^{-2} = 1 - 3/4$ and $\gamma = 2$.

The ladder & barn (non-)paradox



Barn's frame



Ladder's frame

The ladder & barn (non-)paradox

The answer stems, as is often the case with apparent paradoxes in relativity, from loose use of language. In this case, it is the use of the word 'fit'; what does it mean to say the ladder 'fits'' exactly into the barn? Clearly, we mean that the two events:

(i) front end of ladder hits back of barn; (ii) back end of ladder goes through the door

are simultaneous. But observers in different frames do not agree on simultaneity, so 'fit into' is a frame-dependent concept: we should not expect observers in different frames to agree so there is no paradox to account for. The two statements are true and compatible and that is really the end

The twins (non-)paradox

Twins Alice and Bob synchronise watches in an inertial frame and then Bob sets off at speed $\sqrt{3}c/2$, which corresponds to $\gamma = 2$. When Bob has been travelling for a time T according to Alice, he reaches Proxima Centauri[20] and turns round by means of accelerations that are very large in his frame and goes back to Alice at the same speed. Since Bob is in a moving frame, relative to Alice, his time runs slower by a factor of γ than Alice's, so he will only have aged by $2T \times \frac{1}{2}$ on the two legs of the journey. Thus when they meet up again, Alice has aged by $2T$ but Bob has aged only by T. *This is not the paradox: it is just a fact of life.*[21]

[21] In 1971, Hafele and Keating packed four atomic (caesium) clocks into suitcases and went round the Earth, in different directions, on commercial flights. When they returned, they found that the clocks were slightly behind a clock remaining at the first airport. The result was somewhat inconclusive. The calculations are complicated by the fact that the rate of the clocks is also affected by the gravitational field: clocks run slower in stronger fields, and in fact the two affects balance at $3R/2$ (where R is the radius of the Earth). Thus the heights of the aircraft had to be taken into account as well as their speeds, and it turns out that the two effects are of comparable magnitude, namely of the order of 100 nanoseconds.

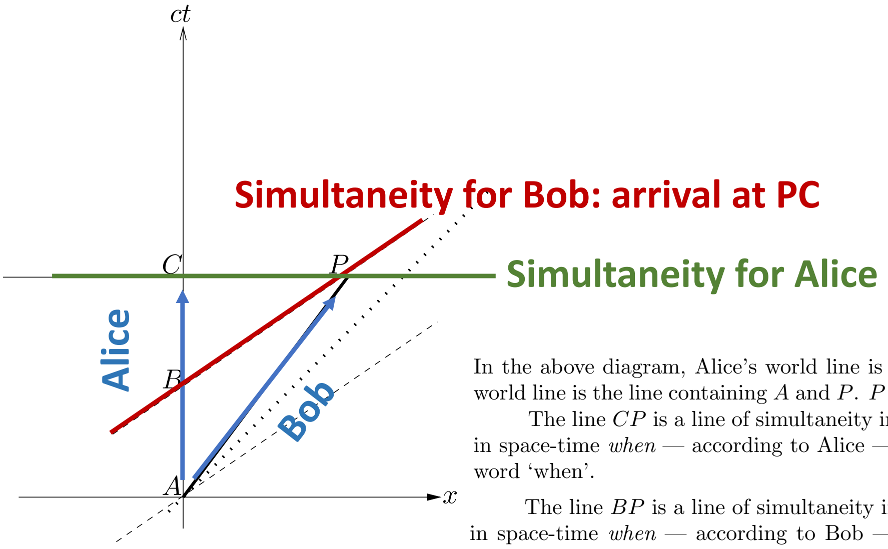
The twins (non-)paradox

The difficulty some people have with Alice and Bob is the apparent symmetry: surely exactly the same argument could be made, from Bob's point of view, to show that Alice would be the younger when they met again? But the same argument *cannot* be made for Bob because the situation is not symmetric: Alice's frame is inertial, whereas Bob has to accelerate to turn round: while he is accelerating, his frame is not inertial.

The twins (non-)paradox

BUT, some people might say, suppose we just consider the event of Bob's arrival at Proxima Centauri, so as not to worry about acceleration. Now the situation is symmetric. Surely from Alice's point of view, when Bob arrives he will have aged half as much as Alice, and from Bob's point of view, when he arrives, Alice will have aged half as much as Bob? The answer to this is a simple 'yes'. Surely, they would then say, this doesn't make sense?

The twins (non-)paradox

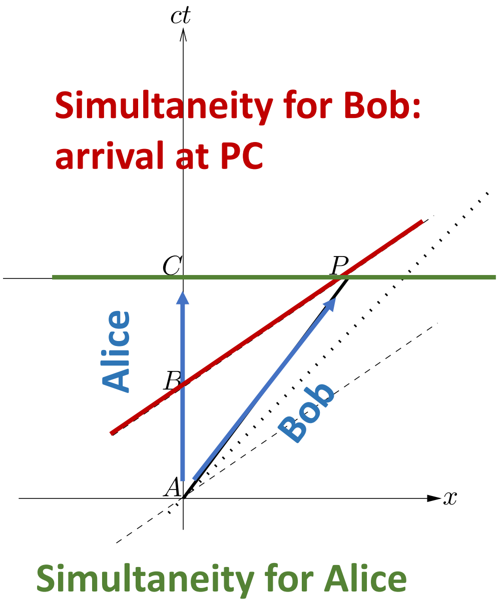


In the above diagram, Alice's world line is the ct (containing points A, B and C) axis and Bob's world line is the line containing A and P. P represents the event 'Bob arrives at Proxima Centauri'.

The line CP is a line of simultaneity in Alice's frame and C is the event 'Alice is at this point in space-time *when* — according to Alice — Bob arrives at Proxima Centauri'; the first use of the word 'when'.

The line BP is a line of simultaneity in Bob's frame and B is the event 'Alice is at this point in space-time *when* — according to Bob — he arrives Proxima Centauri'; the second use of the word 'when'. The two 'whens' don't mean the same thing, since one is a 'when' in Alice's frame the other is a 'when' in Bob's frame.

The twins (non-)paradox



We can do the calculation. Let us assume for simplicity that Bob sets off the moment he is born. The event C has coordinates $(cT, 0)$ in Alice's frame, and the event P has coordinates (cT, vT). In Bob's frame, the elapsed time T' is given by the Lorentz transformation:

$$T' = \gamma(T - v^2T/c^2) = T/\gamma = \tfrac{1}{2}T.$$

This is just the usual time dilation calculation. Thus Bob and Alice agree that Bob's age at Proxima Centauri is $\frac{1}{2}T$. In Alice's frame, Bob has aged half as much as Alice.

We now work out the coordinates of the event B, sticking with Alice's frame. The line of simultaneity, BP has equation $t' = \frac{1}{2}T$, i.e. (using a Lorentz transformation)

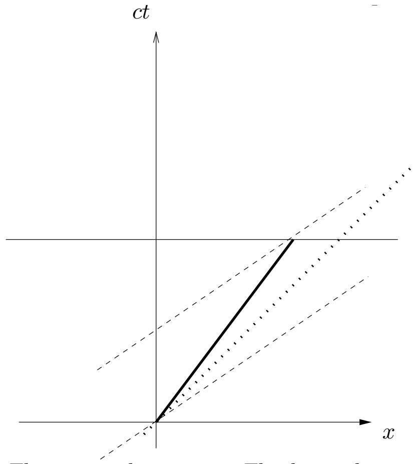
$$\gamma(t + vx/c^2) = \frac{1}{2}T$$

so the point B, for which $x = 0$, has coordinates $(\frac{1}{2}cT/\gamma, 0)$, i.e. $(\frac{1}{4}cT, 0)$. Alice's age when, according to Bob, he arrives at Proxima Centauri is therefore $\frac{1}{4}T$, which is indeed half of Bob's age. So no paradox there either.

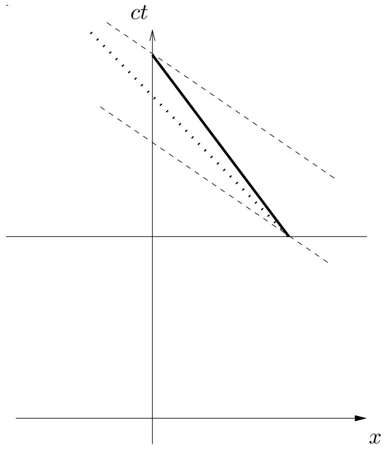
The twins (non-)paradox

BUT, some other people might say, suppose Bob does not turn round but just synchronises his watch at Proxima Centauri with that of another astronaut, Bob$'$, who is going at speed v in the opposite direction (like two trains passing at a station). Each leg of the journey is then symmetric, so why should Alice age faster or slower Bob and Bob$'$ during their legs of the journey?

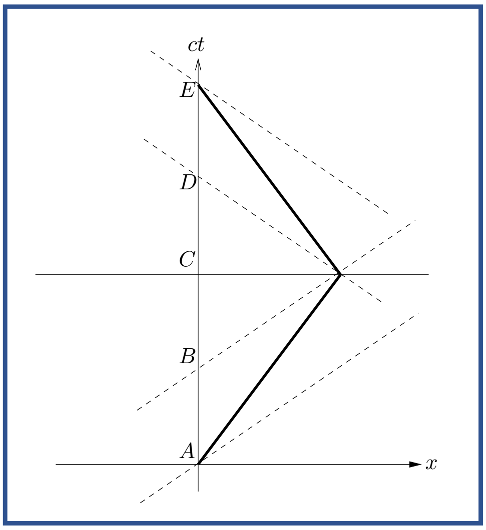
The twins (non-)paradox



The outward journey. The heavy line is Bob's world line. The dotted line through the origin is the light cone. The dashed lines are the lines of simultaneity in Bob's frame.



The return journey. The heavy line is the world line of Bob′. The dotted line through the turn-round event is the light cone. The dashed lines are the lines of simultaneity in the frame of Bob′.



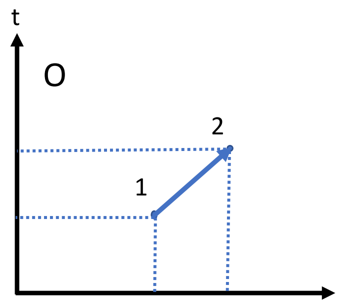
Alice. She: T + T. They: T/2 + T/2
Bobs. Them: T/2 + T/2. She: T/4 + T/4
However BD is missing, which accounts for another T/2 + T/2 for the Bobs

Proper time

The fact that the concept of time is frame dependent can be rather unsettling. It would be good to have some quantity that corresponds to time but does not vary at the whim of the observer. Such a quantity exists and is called *proper time*.

The proper time between two infinitesimally separated points (ct, x) and $(ct + cdt, x + dx)$ is given by

$$c^2 d\tau^2 = c^2 dt^2 - dx^2. \tag{6.18}$$



Proper time

The proper time between two infinitesimally separated points (ct, x) and $(ct + cdt, x + dx)$ is given by

$$c^2 d\tau^2 = c^2 dt^2 - dx^2. \tag{6.18}$$

Again, we note that if these points represent events on the world line of an observer then in the rest frame of the observer

$$d\tau = dt_{\text{rest}}$$

so infinitesimal proper time measures infinitesimal time displacements in the rest frame; ticks of the observer's clock. Comparing with (6.18) we see that in a general frame

$$\sqrt{dt^2 - dx^2/c^2} = d\tau = dt_{\text{rest}}$$

so

$$dt > dt_{\text{rest}}$$

which is time dilation.

By construction, the proper time is the same for all inertial frames, since:

$c^2dt^2 – dx^2 = c^2dt'^2 – dx'^2$

Relativistic kinematics- velocity

From Lorentz transformations

$$dx = \gamma_v (dx' + vdt'), \quad dy = dy', \quad dz = dz', \quad dt = \gamma_v \left(dt' + \frac{v}{c^2} dx'\right)$$

Then

$$\frac{dx}{dt} = \frac{\gamma_v (dx' + vdt')}{\gamma_v (dt' + \frac{v}{c^2} dx')}, \quad \frac{dy}{dt} = \frac{dy'}{\gamma_v (dt' + \frac{v}{c^2} dx')}, \quad \frac{dz}{dt} = \frac{dz'}{\gamma_v (dt' + \frac{v}{c^2} dx')},$$

$$u_x = \frac{dx}{dt} = \frac{\frac{dx'}{dt'} + v}{(1 + \frac{v}{c^2}\frac{dx'}{dt'})}, \quad u_y = \frac{dy}{dt} = \frac{\frac{dy'}{dt'}}{\gamma_v\ (1 + \frac{v}{c^2}\frac{dx'}{dt'})}, \quad u_z = \frac{dz}{dt} = \frac{\frac{dz'}{dt'}}{\gamma_v\ (1 + \frac{v}{c^2}\frac{dx'}{dt'})},$$

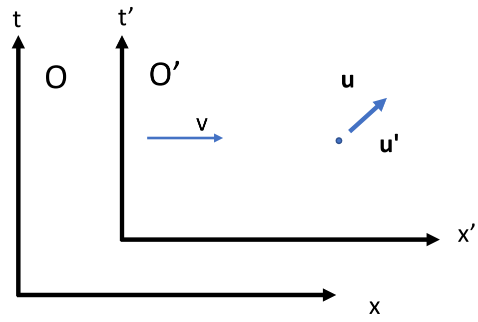
Relativistic kinematics- velocity

Therefore

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x} \qquad u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x},$$

$$u_y = \frac{\sqrt{1-\frac{v^2}{c^2}}u'_y}{1 + \frac{v}{c^2}u'_x} \qquad u'_y = \frac{\sqrt{1-\frac{v^2}{c^2}}u_y}{1 - \frac{v}{c^2}u_x}$$

$$u_z = \frac{\sqrt{1-\frac{v^2}{c^2}}u'_z}{1 + \frac{v}{c^2}u'_x} \qquad u'_z = \frac{\sqrt{1-\frac{v^2}{c^2}}u_z}{1 - \frac{v}{c^2}u_x}$$



Note: $u_x' = c \rightarrow u_x = c$

with no surprise

End first part