Special Relativity
Part 2

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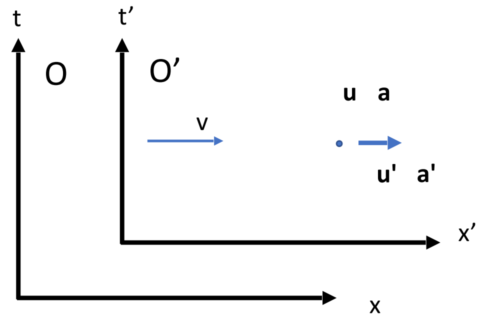
Relativistic kinematics - acceleration

We consider the simple case of acceleration along the x (x’) axis.



We start with the velocity at a given time:

The velocities u and u' in the two frames are related by

$$u' = \frac{u - v}{1 - uv/c^2} \equiv \frac{(c^2/v)(1 - v^2/c^2)}{1 - uv/c^2} - \frac{c^2}{v}$$

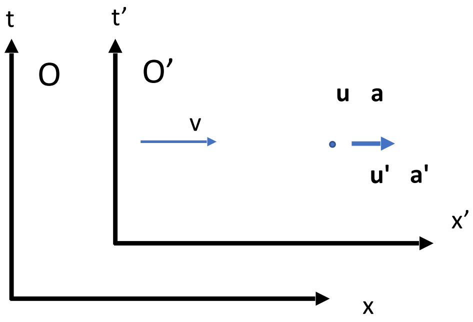
(the equivalent form is just a bit of algebra to obtain a useful expression). Differentiating this with respect to τ gives

$$\frac{du'}{d\tau} = \frac{1 - v^2/c^2}{(1 - uv/c^2)^2} \frac{du}{d\tau}. \qquad (6.22)$$

Relativistic kinematics - acceleration

The acceleration, a, in S is by definition du/dt and similarly for S' so

$$\begin{aligned} a' &= \frac{du'}{dt'} \\ &= \frac{du'}{d\tau} \bigg/ \frac{dt'}{d\tau} \\ &= \frac{1 - v^2/c^2}{(1 - uv/c^2)^2} \frac{du}{d\tau} \bigg/ \frac{dt'}{d\tau} \\ &= \frac{1 - v^2/c^2}{(1 - uv/c^2)^2} \frac{du}{d\tau} \bigg/ \gamma(1 - uv/c^2) \frac{dt}{d\tau} \\ &= \frac{(1 - v^2/c^2)^{\frac{3}{2}}}{(1 - uv/c^2)^3} a. \end{aligned}$$



Acceleration is not absolute anymore! (Should not be a surprise)

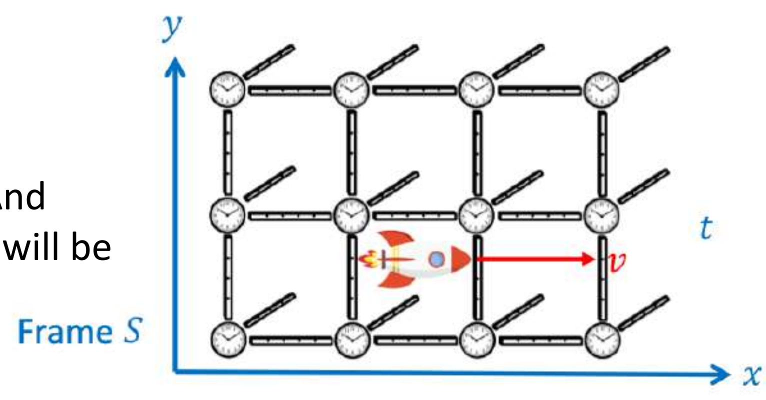
Relativistic kinematics – uniform acceleration

How do we define constant acceleration?
Initial guess:

dv/dt = constant = a ➔ $v = v_0 + at$

Does it make sense? No

Velocity will eventually exceed c. And moreover it will not be true that a will be constant in other frames

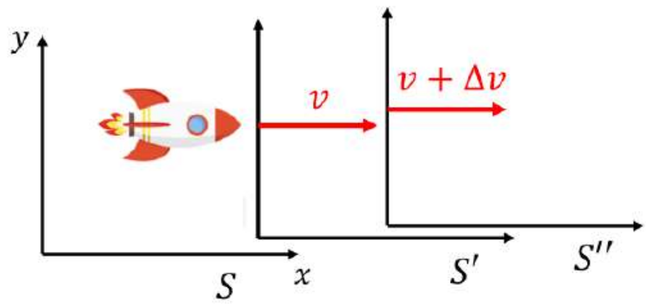


Relativistic kinematics – uniform acceleration

We define uniform acceleration as “feeling constant to the object being accelerated”. The accelerate observer can measure it with an accelerometer.

How doe we analyse this in terms of inertial frames?

We consider the **instantaneous reference frame**, where the object is as rest at that specific time.



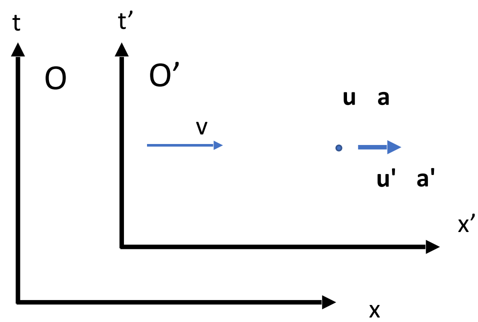
Relativistic kinematics – uniform acceleration

Let us consider the **instantaneous rest frame of the accelerating observer**: u’ = 0 and u = v. Then:

$$a = (1 - u^2/c^2)^{\frac{3}{2}} a'$$

Or:

$$a' = \left(1 - \frac{u^2}{c^2}\right)^{-3/2} a \;=\; \gamma^3 a \;=\; \frac{d}{dt}[\gamma u]$$



a’ is called the **proper acceleration** (as measured by an instantaneous rest frame).

Relativistic kinematics – uniform acceleration

$a' = \text{constant} \;\rightarrow\; \dfrac{d}{dt}[\gamma u] = \text{constant} = \alpha$

The solution is: $\gamma u = \alpha t$ (initial conditions: $u = 0$ for $t = 0$)

$$\frac{u}{\sqrt{1-\frac{u^2}{c^2}}} = \alpha t \quad \rightarrow \quad u = \frac{\alpha t}{\sqrt{1+\frac{\alpha^2 t^2}{c^2}}}$$

Classical answer

Relativistic correction

We see that u remain always smaller than c, and approaches c for large times.

Relativistic kinematics – uniform acceleration

$$x = x_0 + \int_0^t u\, dt' = x_0 + \frac{c^2}{\alpha}\left[\sqrt{1+\frac{\alpha^2 t^2}{c^2}} - 1\right] = \frac{c^2}{\alpha}\sqrt{1+\frac{\alpha^2 t^2}{c^2}}$$

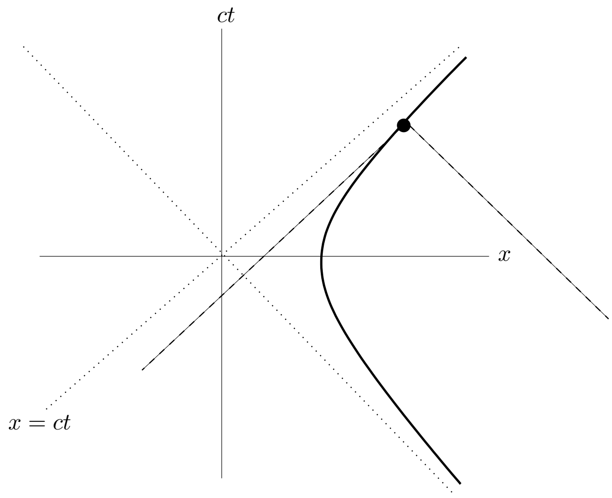
With the choice $x_0 = c^2/\alpha$

Then we have:

$$x^2 - (ct)^2 = (c^2/\alpha)^2$$

Relativistic kinematics – uniform acceleration

The diagram shows the trajectory. The dotted lines are the light cones. An event taking place within the dashed lines can influence an accelerated observer at the position shown, but events taking place outside the dashed lines would have to move faster than the speed of light to do so. As $\tau \to \infty$, the whole of the space-time to the left of the dotted line $x = ct$ would be inaccessible to the observer. This line is called the *Rindler event horizon* for the accelerated observer. In some ways, it performs the same function as the event horizon of a black hole. In particular, the observer has to accelerate to avoid falling through it and anything happening on the other side would be hidden to the observer. Of course, the accelerating observer could just stop accelerating whereas the observer in a black hole space-time can do nothing to affect the event horizon.



Relativistic kinematics – momentum

Classically we have

Conservation of mass:

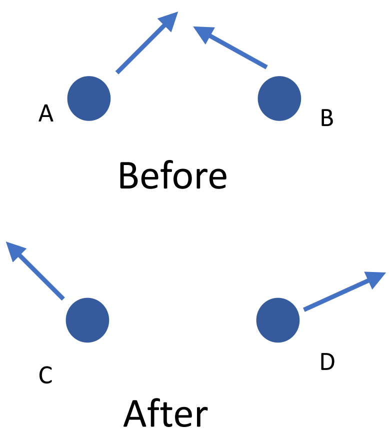
$$m_A + m_B = m_C + m_D$$

Conservation of momentum:

$$m_A \mathbf{u}_A + m_B \mathbf{u}_B = m_C \mathbf{u}_C + m_D \mathbf{u}_D$$

Conservation of energy:

$$\frac{1}{2} m_A \mathbf{u}_A^2 + \frac{1}{2} m_B \mathbf{u}_B^2 = \frac{1}{2} m_C \mathbf{u}_C^2 + \frac{1}{2} m_D \mathbf{u}_D^2$$



Relativistic kinematics – momentum

If these transformations hold true in a reference frame, Galilei's transformation laws make sure that they hold in any other (inertial) frame. In particular

$$\frac{1}{2}m_A\mathbf{u}_A^2 + \frac{1}{2}m_B\mathbf{u}_B^2 = \frac{1}{2}m_C\mathbf{u}_C^2 + \frac{1}{2}m_D\mathbf{u}_D^2 \quad \Rightarrow \quad (\mathbf{u} = \mathbf{u}' + \mathbf{v})$$

$$\frac{1}{2}m_A\mathbf{u}'^2_A + \frac{1}{2}m_B\mathbf{u}'^2_B = \frac{1}{2}m_C\mathbf{u}'^2_C + \frac{1}{2}m_D\mathbf{u}'^2_D - \frac{1}{2}\mathbf{v}^2\cancel{[m_A + m_B - m_C - m_D]}$$

conservation of mass

$$-\mathbf{v}\cancel{[m_A\mathbf{u}'_A + m_B\mathbf{u}'_B - m_C\mathbf{u}'_C - m_D\mathbf{u}'_D]}$$

conservation of momentum

Relativistic kinematics – momentum

$$m_A\mathbf{u}_A + m_B\mathbf{u}_B = m_C\mathbf{u}_C + m_D\mathbf{u}_D \quad \Rightarrow \quad (\mathbf{u} = \mathbf{u}' + \mathbf{v})$$

$$m_A\mathbf{u}'_A + m_B\mathbf{u}'_B = m_C\mathbf{u}'_C + m_D\mathbf{u}'_D - \mathbf{v}\cancel{[m_A + m_B - m_C - m_D]}$$

conservation of mass

This is how conservation properties are linked to each other

In a relativistic context, given the transformation properties of velocities, momentum as defined above is nor conserved in all inertial frames. If it is conserved in one frame, it is not in the others.
A new definition is needed!!

Relativistic kinematics – momentum

Two criteria for a new definition of momentum:

1. Momentum should be conserved in every inertial frame
2. It reduces to classical momentum for low velocities

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}}$$

It clearly reduces to the classical definition for low velocities. We now consider the issue of conservation

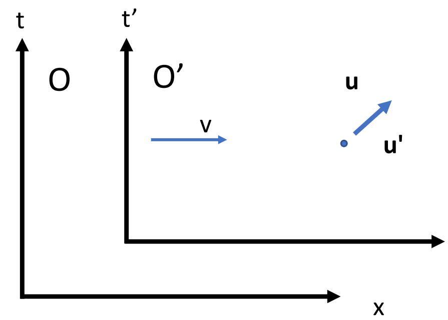
Relativistic kinematics – momentum

$$u'_x = \frac{u_x - v}{1-\frac{vu_x}{c^2}} \qquad u'_y = \frac{u_y}{\gamma\left(1-\frac{vu_x}{c^2}\right)} \qquad u'_z = \frac{u_z}{\gamma\left(1-\frac{vu_x}{c^2}\right)}$$

One can prove that:

$$\frac{1}{\sqrt{1-u^2/c^2}} = \gamma\,\frac{1+u'_x v/c^2}{\sqrt{1-u'^2/c^2}}$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$



Relativistic kinematics – momentum

Then:

$$p_x = \frac{m u_x}{\sqrt{1-u^2/c^2}} = m\left(\gamma\,\frac{\cancel{1+u'_x v/c^2}}{\sqrt{1-u'^2/c^2}}\right)\left(\frac{u'_x+v}{\cancel{1+v u'_x/c^2}}\right)$$

$$= \gamma\,\frac{m u'_x}{\sqrt{1-u'^2/c^2}} + \gamma\,\frac{m v}{\sqrt{1-u'^2/c^2}} = \gamma\, p'_x + \gamma\,\frac{v}{c^2}\frac{mc^2}{\sqrt{1-u'^2/c^2}}$$

$$= \gamma\left(p'_x + \frac{v}{c^2}E'\right) \qquad E' = \frac{mc^2}{\sqrt{1-u'^2/c^2}}$$

Relativistic kinematics – momentum

Then:

$$p_y = \frac{m u_y}{\sqrt{1-u^2/c^2}} = m\left(\gamma\,\frac{\cancel{1+u'_x v/c^2}}{\sqrt{1-u'^2/c^2}}\right)\left(\frac{u'_y}{\cancel{\gamma(1+v u'_x/c^2)}}\right)$$

$$= m\frac{u'_y}{\sqrt{1-u'^2/c^2}} = p'_y$$

And same for the z-component

Relativistic kinematics – momentum

To summarize we have:

$$\begin{cases} p_x = \gamma\left(p'_x + \dfrac{v}{c^2}E'\right) \\ p_y = p'_y \\ p_z = p'_z \end{cases} \qquad \begin{cases} p'_x = \gamma\left(p_x - \dfrac{v}{c^2}E\right) \\ p'_y = p_y \\ p'_z = p_z \end{cases}$$

With: $$E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

Relativistic kinematics – momentum

Coming back to conservation of momentum, conservation along the y and z directions is trivial. Along the x direction:

$$p_{xA} + p_{xB} = p_{xC} + p_{xD} \quad \Rightarrow$$

$$\gamma\left(p'_{xA} + \frac{v}{c^2}E'_A\right) + \gamma\left(p'_{xB} + \frac{v}{c^2}E'_B\right) = \gamma\left(p'_{xC} + \frac{v}{c^2}E'_C\right) + \gamma\left(p'_{xD} + \frac{v}{c^2}E'_D\right)$$

$$p'_{xA} + p'_{xB} = p'_{xC} + p'_{xD} - \frac{v}{c^2}\left(E'_A + E'_B - E'_C - E'_D\right)$$

Therefore momentum is conserved if also E is conserved.

Relativistic kinematics – energy

What is E?

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = mc^2 + \frac{1}{2}mu^2 + \frac{3}{8}m\frac{u^4}{c^2} + \dots$$

Constant term

Classical kinetic term

Relativistic corrections

E = total energy

T = E - mc^2
Kinetic energy

Relativistic kinematics – energy

The transformation properties of E are:

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = mc^2\gamma\frac{1+u'_x v/c^2}{\sqrt{1-u'^2/c^2}} = \gamma\left[E' + vp'_x\right]$$

It is easy to show that the following relation holds:

$$E^2 - p^2c^2 = m^2c^4$$

It takes the same value
in all reference frames

Relativistic kinematics – energy and momentum

If we want to have that momentum is conserved in all frames, then also the energy must be conserved. Then

$$\begin{aligned} p_{xA} + p_{xB} &= p_{xC} + p_{xD} \\ E_A + E_B &= E_C + E_D \end{aligned}$$

in one frame implies

$$\begin{aligned} p'_{xA} + p'_{xB} &= p'_{xC} + p'_{xD} - (v/c^2)[E'_A + E'_B - E'_C - E'_D] \\ E'_A + E'_B &= E'_C + E'_D - v[p'_{xA} + p'_{xB} - p'_{xC} - p'_{xD}] \end{aligned} \quad \Rightarrow \quad \begin{aligned} p'_{xA} + p'_{xB} &= p'_{xC} + p'_{xD} \\ E'_A + E'_B &= E'_C + E'_D \end{aligned}$$

in any other frame

Relativistic kinematics – energy and momentum

Conservation of energy and momentum in one frame implies conservation in all other frames.

But are they conserved?

Postulate: for every closed system (no external forces) energy an momentum are conserved.

It is experimentally verified (so far)

Relativistic kinematics – mass

And what about the mass?

The conservation of the mass is not necessary anymore - as it was in classical mechanics - to have conservation of energy and momentum.

And in fact in special relativity the mass is not conserved any more.

Relativistic kinematics – mass

Classically

$$m = m_1 + m_2 \qquad \mathbf{0} = m_1\mathbf{u}_1 + m_2\mathbf{u}_2 \qquad 0 = \frac{1}{2}(m_1 u_1^2 + m_2 u_2^2)$$

Impossible

Relativistic kinematics – mass

In special relativity

$$E = mc^2 = E_1 + E_2 = \frac{m_1 c^2}{\sqrt{1 - u_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1 - u_2^2/c^2}} \qquad \mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2$$

$$(m - m_1 - m_2)c^2 = m_1 c^2\left[\frac{m_1 c^2}{\sqrt{1 - u_1^2/c^2}} - 1\right] + m_2 c^2\left[\frac{m_2 c^2}{\sqrt{1 - u_2^2/c^2}} - 1\right] = T_1 + T_2$$

Relativistic kinematics – mass

$$\Delta m = \frac{T_1 + T_2}{c^2}$$

We see that this process is consistent with the theory: mass is lost in favour of kinetic energy.

Also, the amount of kinetic energy is equal to Δmc^2, which is very large

Relativistic kinematics – mass

From the conceptual point of view, what is remarkable is that all this comes from the structure of space and time, and this structure affects the properties of matter: how fat they can travel, how their mass and energy behaves.

Relativistic kinematics – massless particles

From the relation

$$E^2 - p^2c^2 \;=\; m^2c^4$$

Taking m = 0, we have: E = |**p**|c.

Relativity opens to the possibility of particles of zero mass. They have to travel at the sped of light.

Classically, without mass there is no momentum and no kinetic energy.

Relativistic dynamics

Newton's first law of inertia: ok

Newton's second law: $\mathbf{F} = \frac{d\mathbf{p}}{dt}$

It remains valid, provided that with **p** we use the relativistic momentum.

Remembering that $\mathbf{p} = m\gamma\mathbf{u}$ and that the proper acceleration is $\mathbf{a}' = d(\gamma\mathbf{u})/dt$, we see that **a constant force exerts a constant proper acceleration**, thus a hyperbolic motion.

Work & Energy

Work, as always, is the line integral of the force:

$$W \equiv \int \mathbf{F} \cdot d\mathbf{l}. \tag{12.62}$$

The **work-energy theorem** ("the net work done on a particle equals the increase in its kinetic energy") holds relativistically:

$$W = \int \frac{d\mathbf{p}}{dt} \cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{l}}{dt}\, dt = \int \frac{d\mathbf{p}}{dt} \cdot \mathbf{u}\, dt,$$

while

$$\begin{aligned} \frac{d\mathbf{p}}{dt} \cdot \mathbf{u} &= \frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}\right) \cdot \mathbf{u} \\ &= \frac{m\mathbf{u}}{(1-u^2/c^2)^{3/2}} \cdot \frac{d\mathbf{u}}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{dE}{dt}, \end{aligned} \tag{12.63}$$

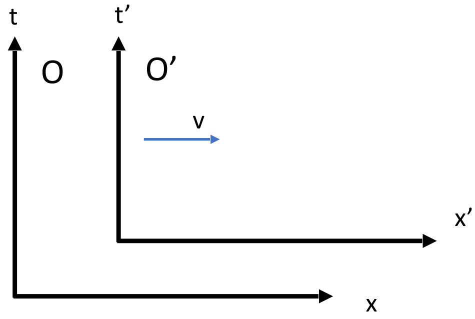
Transformation rules for the Force

Because $\mathbf{F}$ is the derivative of momentum with respect to *ordinary* time, it shares the ugly behavior of (ordinary) velocity, when you go from one inertial system to another: both the numerator *and the denominator* must be transformed. Thus,[18]

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma\, dt - \dfrac{\gamma\beta}{c}\, dx} = \frac{dp_y/dt}{\gamma\left(1 - \dfrac{\beta}{c}\dfrac{dx}{dt}\right)} = \frac{F_y}{\gamma(1 - \beta u_x/c)}, \tag{12.65}$$

and similarly for the z component:

$$\bar{F}_z = \frac{F_z}{\gamma(1 - \beta u_x/c)}.$$



Transformation rules for the Force

The x component is even worse:

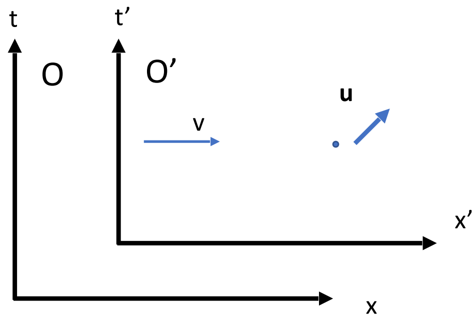
$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma\, dp_x - \gamma\beta\, dp^0}{\gamma\, dt - \dfrac{\gamma\beta}{c}\, dx} = \frac{\dfrac{dp_x}{dt} - \beta\dfrac{dp^0}{dt}}{1 - \dfrac{\beta}{c}\dfrac{dx}{dt}} = \frac{F_x - \dfrac{\beta}{c}\left(\dfrac{dE}{dt}\right)}{1 - \beta u_x/c}.$$

We calculated dE/dt in Eq. 12.63; putting that in,

$$\bar{F}_x = \frac{F_x - \beta(\mathbf{u}\cdot\mathbf{F})/c}{1 - \beta u_x/c}. \tag{12.66}$$

In one special case these equations are reasonably tractable: *If the particle is* (instantaneously) *at rest in* $\mathcal{S}$, so that $\mathbf{u} = 0$, then

$$\bar{\mathbf{F}}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \quad \bar{F}_\parallel = F_\parallel. \tag{12.67}$$

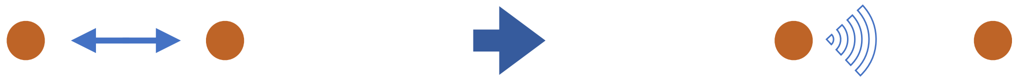


Relativistic dynamics

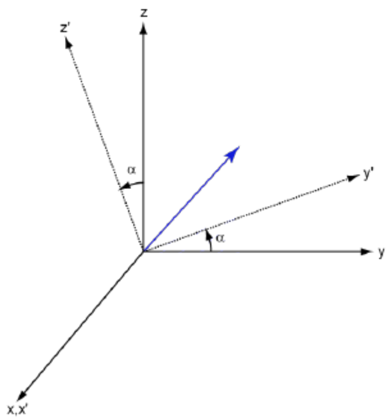
Newton's <u>third law</u>:

It does not extend to relativistic motion, because it is incompatible with the relativity of simultaneity. It holds only for contact interactions.

In relativity, forces are rep by **fields mediating the interaction**.



Geometry of spacetime - vectors



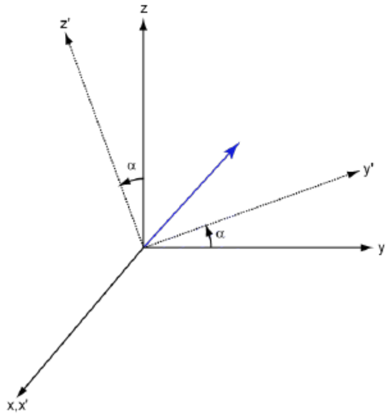
What does it mean that a vector is a vector?
It means that it has magnitude and a direction.
It can be expressed by its components, which however are not intrinsic, but relative to the reference frame.

Mathematically, a vector is expressed by three components (since space is three dimensional)

$$\mathbf{r} \leftrightarrow r^i$$

with i = 1,2,3

Geometry of spacetime - vectors



In another frame the components are

$$r'^i = O^i_j r^j$$

where O^i_j is an orthogonal rotation matrix.
We used Einstein's summation: repeated indices are summed.

The **components** of the vector **change** when changing the reference frame.

Let us see that the **length** and **direction** do **not change**

Geometry of spacetime - vectors

Let us introduce the metric $\eta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then we define

Scalar product: $\mathbf{p} \cdot \mathbf{q} \;=\; p^i \eta_{ij} q^j$

Norm: $\|\mathbf{p}\|^2 \;=\; p^i \eta_{ij} p^j$

Distance: $\|\mathbf{p} - \mathbf{q}\|$

These objects are defined by the metric tensor. It reflects the structure of Euclidean space

Geometry of spacetime - vectors

Now we can prove that the length of a vector is the same in every frame

$$\|\mathbf{r}'\|^2 \;=\; r'^i \eta_{ij} r'^j \;=\; O^i_k r^k \eta_{ij} O^j_\ell r^\ell \;=\; [O^i_k \eta_{ij} O^j_\ell] r^k r^\ell \;=\; [O^j_k O^j_\ell] r^k r^\ell$$

$$= \; [(O^k_j)^T O^j_\ell] r^k r^\ell \;=\; r^k \eta_{k\ell} r^\ell \;=\; \|\mathbf{r}\|^2$$

Same for the direction of a vector relative to another vector (scalar product)

Geometry of spacetime - Minkowski space

Minkowski: relativistic space has a different geometric structure. One should consider it has a **four-dimensional space** (space-time = space and time) with the following metric:

$$\eta_{\mu\nu} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- It is not Euclidean anymore. It is Minkowski space
- Space and time are put together (but not unified)
- Conventions with opposite signs are allowed

Geometry of spacetime - Minkowski space

In analogy with Euclidean geometry, we define **4-vectors**

$p \;\leftrightarrow\; p^{\mu}$

where the components p^{μ} depend on the reference frame and are related to those in another frame by **Lorentz transformations**: $p^{'\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu}$

Then we define again

Scalar product: $p \cdot q \;=\; p^{\mu}\eta_{\mu\nu}q^{\nu}$

'Norm': $p^2 \;=\; p^{\mu}\eta_{\mu\nu}p^{\nu}$ (not always positive)

Distance: $(p-q)^2$ (not always positive)

Geometry of spacetime - Minkowski space

We show that length and direction of 4-vectors do not change:

$$\|p\|^2 \;=\; p^{\mu}\eta_{\mu\nu}p^{\nu} \;\rightarrow\; \|p'\|^2 \;=\; p^{'\mu}\eta_{\mu\nu}p^{'\nu} \;=\; \Lambda^{\mu}{}_{\alpha}p^{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta}p^{\beta} \;=\; (\Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta})p^{\alpha}p^{\beta}$$

$$= \underline{(\Lambda^{T}\eta\Lambda)_{\alpha\beta}}\,p^{\alpha}p^{\beta} \;=\; \underline{\eta_{\alpha\beta}}\,p^{\alpha}p^{\beta} \;=\; \|p\|^2$$

Same for scalar product and distance

Let us show that $\Lambda^{T}\eta\Lambda = \eta$ (matrix identity)

Geometry of spacetime - Minkowski space

Let us rewrite $\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ $\beta = \dfrac{v}{c}$

Then:

$$\Lambda^T \eta \Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\gamma^2 + \beta\gamma^2 & 0 & 0 & 0 \\ 0 & \gamma^2 - \beta\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Geometry of spacetime - four vectors

Spacetime event: $x^{\mu} = (ct, \mathbf{x})$.
Its components transform with the Lorentz trnafotremahions, therefore it is a 4-vector.

The metric does not have a definite signature, therefore **three types of events are possible**.

Let us consider two events x_A and x_B and let $I = x_B - x_A$.

Geometry of spacetime - four vectors

1. Space-like separated events: $I^2 > 0$

Example: $I = (0, \Delta x, 0, 0)$: two events that occur simultaneously ($\Delta t = 0$) in frame O, at a distance Δx along the x axis. Then in another frame O':
$\Delta t' = \gamma \Delta t - \beta \Delta x / c = -\beta \Delta x / c \neq 0$
They are not simultaneous anymore.

The order of events depends on the reference frame. This does not conflict with causality because the two events cannot be connected. In fact the average speed would be $(\Delta x / \Delta t)^2 > c^2$ since $I > 0$

Geometry of spacetime - four vectors

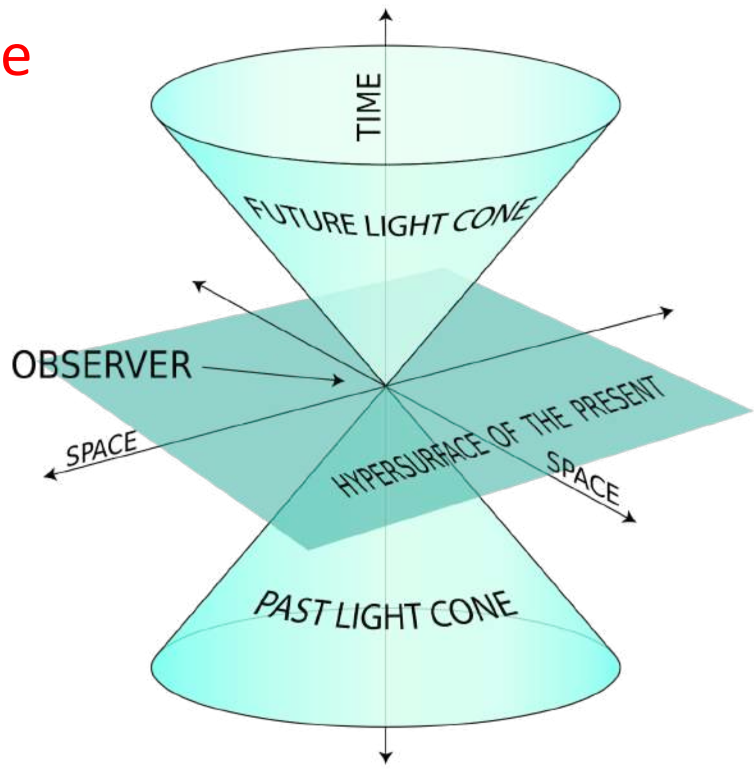
2. Time-like separated events: $I^2 < 0$

Example: $I = (\Delta t, 0, 0, 0)$: two events that occur in the same place ($\Delta x = 0$) in frame O, at different times. Then in another frame O':
$\Delta t' = \gamma \Delta t - \beta \Delta x / c = \gamma \Delta t$
The time ordering is preserved. No problem with causality, it is a fact that one occurs before the other.

3. Light-like separated events: $I^2 = 0$
Events that are connect by a ray of light

Geometry of spacetime - spacetime diagram

Subdivision of Minkowski spacetime with respect to an event in four disjoint sets. The light cone, the **absolute future**, the **absolute past**, and **elsewhere**.



Geometry of spacetime - four velocity

The velocity dx^μ/dt it is not a good definition of 4-velocity because it does not have the right transformation properties. A good definition is:

$$\eta^\mu = \frac{dx^\mu}{d\tau}$$

where τ is the proper time of the particle. The relation to the usual velocity is:

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \gamma c \quad \text{(nothing new) and} \qquad \boldsymbol{\eta} = \frac{d\mathbf{x}}{d\tau} = \gamma \mathbf{u}$$

It is easy to see that: $\eta^2 = -c^2$, which is invariant.

Geometry of spacetime - four momentum

The 4-momentum is defined as: $p^\mu = (E/c, \mathbf{p}) = m\eta^\mu = m dx^\mu / d\tau$

The previous calculations show that it is a 4-vector, i.e. that its components transform as: $p^{'\mu} = \Lambda^\mu{}_\nu p^\nu$

(it is not for granted that every object with 4 components si a 4-vector)

The length is: $p^2 = -E^2/c^2 + \mathbf{p}^2 = -m^2c^2$
which is invariant.

Geometry of spacetime - Minkowski force

The Minkowski force is defined as:

$$K^\mu = \frac{dp^\mu}{d\tau}$$

Then:

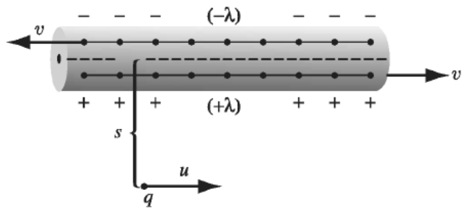
$$K^0 = \frac{1}{c}\frac{dE}{d\tau} \quad \text{and} \quad \mathbf{K} = \gamma\frac{d\mathbf{p}}{dt} = \gamma\mathbf{F}$$

Electromagnetism

To begin with, I'd like to show you why there *had* to be such a thing as magnetism, given electrostatics and relativity, and how, in particular, you can calculate the magnetic force between a current-carrying wire and a moving charge without ever invoking the laws of magnetism.[23] Suppose you had a string of positive charges moving along to the right at speed v. I'll assume the charges are close enough together so that we may treat them as a continuous line charge λ. Superimposed on this positive string is a negative one, $-\lambda$ proceeding to the left at the same speed v. We have, then, a net current to the right, of magnitude

$$I = 2\lambda v. \tag{12.76}$$

Meanwhile, a distance s away there is a point charge q traveling to the right at speed $u < v$ (Fig. 12.34a). Because the two line charges cancel, there is *no electrical force on q* in this system ($\mathcal{S}$).

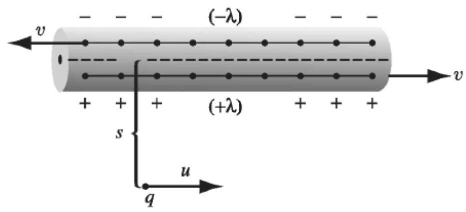


(a)

Electromagnetism

However, let's examine the same situation from the point of view of system $\bar{\mathcal{S}}$, which moves to the right with speed u (Fig. 12.34b). In this reference frame, q is at rest. By the Einstein velocity addition rule, the velocities of the positive and negative lines are now

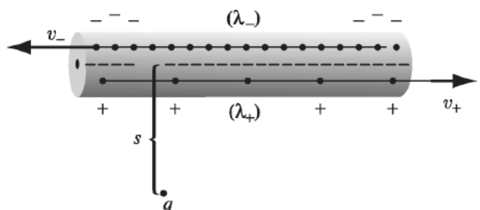
$$v_\pm = \frac{v \mp u}{1 \mp vu/c^2}. \tag{12.77}$$



(a)

Because v_- is greater than v_+, the Lorentz contraction of the spacing between negative charges is more severe than that between positive charges; *in this frame*, therefore, *the wire carries a net negative charge!* In fact,

$$\lambda_\pm = \pm(\gamma_\pm)\lambda_0, \tag{12.78}$$



(b)

where

$$\gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c^2}}, \tag{12.79}$$

Electromagnetism

Because v_- is greater than v_+, the Lorentz contraction of the spacing between negative charges is more severe than that between positive charges; *in this frame,* therefore, *the wire carries a net negative charge!* In fact,

$$\lambda_\pm = \pm(\gamma_\pm)\lambda_0, \tag{12.78}$$

where

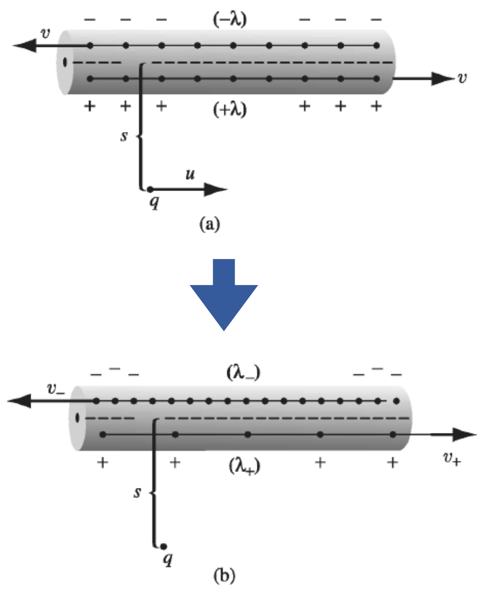
$$\gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c^2}}, \tag{12.79}$$

and λ_0 is the charge density of the positive line in its own rest system. That's not the same as λ, of course—in $\mathcal{S}$ they're already moving at speed v, so

$$\lambda = \gamma\lambda_0, \tag{12.80}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{12.81}$$



Electromagnetism

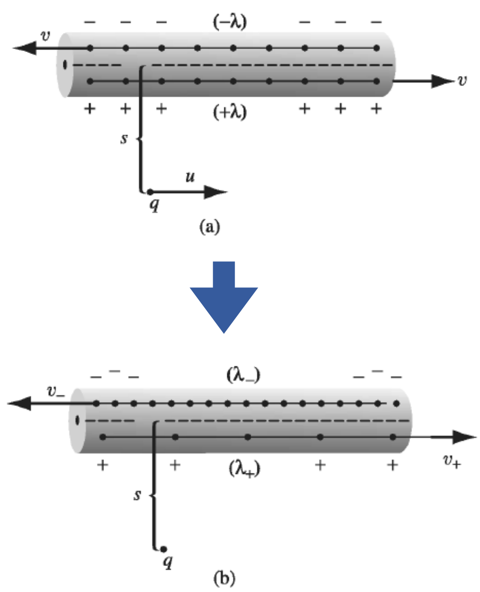
It takes some algebra to put $\gamma_\pm$ into simple form:

$$\gamma_\pm = \frac{1}{\sqrt{1 - \frac{1}{c^2}(v \mp u)^2(1 \mp vu/c^2)^{-2}}} = \frac{c^2 \mp uv}{\sqrt{(c^2 \mp uv)^2 - c^2(v \mp u)^2}}$$

$$= \frac{c^2 \mp uv}{\sqrt{(c^2 - v^2)(c^2 - u^2)}} = \gamma\frac{1 \mp uv/c^2}{\sqrt{1 - u^2/c^2}}. \tag{12.82}$$

The net line charge in $\bar{\mathcal{S}}$, then, is

$$\lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda uv}{c^2\sqrt{1 - u^2/c^2}}. \tag{12.83}$$



Electromagnetism

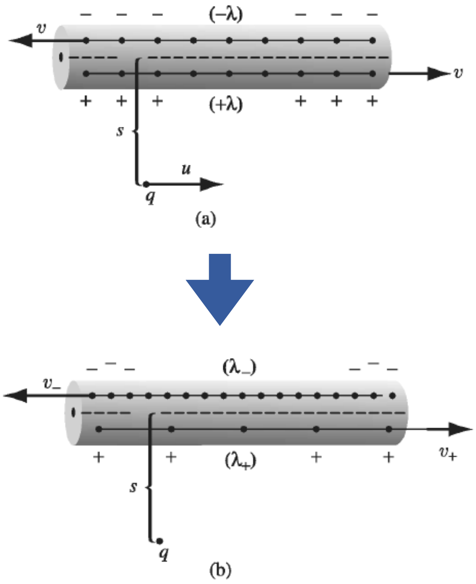
Conclusion: As a result of unequal Lorentz contraction of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

Now, a line charge λ_{tot} sets up an *electric* field

$$E = \frac{\lambda_{\text{tot}}}{2\pi\epsilon_0 s},$$

so there is an electrical force on q in $\bar{\mathcal{S}}$, to wit:

$$\bar{F} = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}. \tag{12.84}$$



Electromagnetism

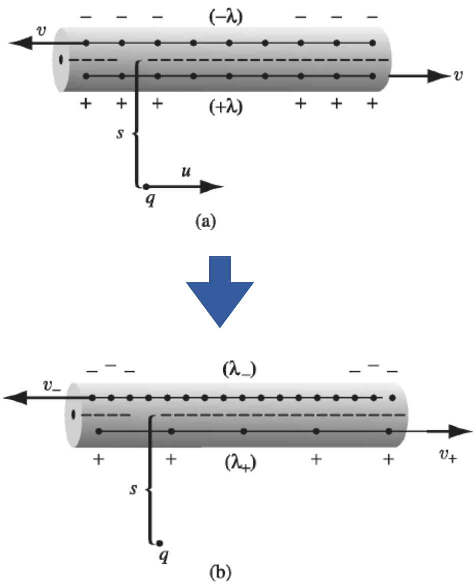
But if there's a force on q in $\bar{\mathcal{S}}$, there must be one in $\mathcal{S}$; in fact, we can calculate it by using the transformation rules for forces. Since q is at rest in $\bar{\mathcal{S}}$, and $\bar{F}$ is perpendicular to u, the force in $\mathcal{S}$ is given by Eq. 12.67:

$$F = \sqrt{1-u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}. \tag{12.85}$$

The charge is attracted toward the wire by a force that is purely electrical in $\bar{\mathcal{S}}$ (where the wire is charged, and q is at rest), but distinctly *non*electrical in $\mathcal{S}$ (where the wire is neutral). Taken together, then, electrostatics and relativity imply the existence of another force. This "other force" is, of course, *magnetic*. In fact, we can cast Eq. 12.85 into more familiar form by using $c^2 = (\epsilon_0\mu_0)^{-1}$ and expressing λv in terms of the current (Eq. 12.76):

$$F = -qu\left(\frac{\mu_0 I}{2\pi s}\right). \tag{12.86}$$

The term in parentheses is the magnetic field of a long straight wire, and the force is precisely what we would have obtained by using the Lorentz force law in system $\mathcal{S}$.



The Field Tensor

A (second-rank) tensor is an object with *two* indices, which transforms with *two* factors of Λ (one for each index):

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}. \tag{12.115}$$

A tensor (in 4 dimensions) has $4 \times 4 = 16$ components, which we can display in a 4×4 array:

$$t^{\mu\nu} = \left\{ \begin{array}{cccc} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{array} \right\}.$$

Like for 4-vectors, components change, but the generalization of length and direction remains invariant (for example $t^{\mu\nu}t_{\mu\nu}$)

The Field Tensor

However, the 16 elements need not all be different. For instance, a *symmetric* tensor has the property

$$t^{\mu\nu} = t^{\nu\mu} \quad \textbf{(symmetric tensor)}. \tag{12.116}$$

In this case there are 10 distinct components; 6 of the 16 are repeats ($t^{01} = t^{10}$, $t^{02} = t^{20}$, $t^{03} = t^{30}$, $t^{12} = t^{21}$, $t^{13} = t^{31}$, $t^{23} = t^{32}$). Similarly, an *anti*symmetric tensor obeys

$$t^{\mu\nu} = -t^{\nu\mu} \quad \textbf{(antisymmetric tensor)}. \tag{12.117}$$

Such an object has just 6 distinct elements—of the original 16, six are repeats (the same ones as before, only this time with a minus sign) and four are zero (t^{00}, t^{11}, t^{22}, and t^{33}). Thus, the general antisymmetric tensor has the form

$$t^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{array} \right\}.$$

The Field Tensor

Let us define
$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\}$$

Then it is not difficult to see that under a boost along the x direction

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y),$$
$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

This is the way **E** and **B** transform according to special relativity

The Field Tensor

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\}$$

The lesson here is that **E** and **B** are not independent quantities, but components of the same object, which is the electromagnetic tensor $F^{\mu\nu}$. What is **E** in one frame can be **B** in another frame - as we saw before - as it happens from any component of a vector/tensor.

The Field Tensor

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\}$$

Question: if **E** and **B** are components of a larger tensor, and can mix with each other, why did people think they were independent 3-vectors?

The Field Tensor

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\}$$

Answer: because they behave like independent three vectors under rotations

The Field Tensor

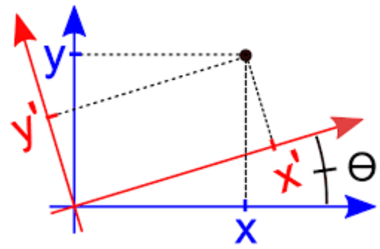
Le us consider a rotation along the z axis: $\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Then (c = 1):

$$E_x = F^{01} \rightarrow E'_x = F'^{01} = \Lambda^{0}{}_{\alpha}\Lambda^{1}{}_{\beta}F^{\alpha\beta}$$

$$= \Lambda^{0}{}_{0}\Lambda^{1}{}_{0}F^{01} + \Lambda^{0}{}_{0}\Lambda^{1}{}_{2}F^{02}$$

$$= E_x \cos\theta + E_y \sin\theta$$



which is how 3-vectors behave under rotations

Electrodynamics in relativistic notation

Charge density and current

$$\rho = \rho_0 \frac{1}{\sqrt{1-u^2/c^2}}, \quad \mathbf{J} = \rho_0 \frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}. \quad \Rightarrow \quad J^{\mu} = \rho_0 \eta^{\mu}$$

$$\boxed{J^{\mu} = (c\rho, J_x, J_y, J_z).}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad \Rightarrow \quad \boxed{\frac{\partial J^{\mu}}{\partial x^{\mu}} = 0,}$$

Electrodynamics in relativistic notation

Maxwell's equations

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0,}$$

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\} \qquad G^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{array} \right\}$$

Electrodynamics in relativistic notation

For $\mu = 0$

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$$

$$= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}(\nabla \cdot \mathbf{E})$$

$$= \mu_0 J^0 = \mu_0 c\rho,$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.$$

For $\mu = 1$ (and i)

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$$

$$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)_x$$

$$= \mu_0 J^1 = \mu_0 J_x.$$

Combining this with the corresponding results for $\mu = 2$ and $\mu = 3$ gives

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

Electrodynamics in relativistic notation

For $\mu = 0$

$$\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3}$$

$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \underline{\nabla \cdot \mathbf{B} = 0}$$

For $\mu = 1$

$$\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3}$$

$$= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = -\frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right)_x = 0.$$

Combining this with the corresponding results for $\mu = 2$ and $\mu = 3$ gives

$$\underline{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Electrodynamics in relativistic notation

Lorentz force law

$$\boxed{K^\mu = q\eta_\nu F^{\mu\nu}.}$$

For if $\mu = 1$, we have

$$K^1 = q\eta_\nu F^{1\nu} = q(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13})$$

$$= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(\frac{-E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}(B_z) + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right]$$

$$= \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]_x,$$

with a similar formula for $\mu = 2$ and $\mu = 3$. Thus,

$$\mathbf{K} = \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})], \qquad (12.129)$$

and therefore, referring back to Eq. 12.69,

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{u} \times \mathbf{B})],$$

Electrodynamics in relativistic notation

Relativistic potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{12.131}$$

As you might guess, V and $\mathbf{A}$ together constitute a 4-vector:

$$\boxed{A^\mu = (V/c, A_x, A_y, A_z).} \tag{12.132}$$

In terms of this **4-vector potential**, the field tensor can be written

$$\boxed{F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}.} \tag{12.133}$$

Electrodynamics in relativistic notation

Relativistic potentials

The potential formulation automatically takes care of the homogeneous Maxwell equation ($\partial G^{\mu\nu}/\partial x^\nu = 0$). As for the inhomogeneous equation ($\partial F^{\mu\nu}/\partial x^\nu = \mu_0 J^\mu$), that becomes

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right) = \mu_0 J^\mu. \tag{12.134}$$

This is an intractable equation as it stands. However, you will recall that the potentials are not uniquely determined by the fields—in fact, it's clear from Eq. 12.133 that you could add to A^μ the gradient of any scalar function λ:

$$A^\mu \longrightarrow A^{\mu\prime} = A^\mu + \frac{\partial \lambda}{\partial x_\mu}, \tag{12.135}$$

without changing $F^{\mu\nu}$. This is precisely the **gauge invariance**

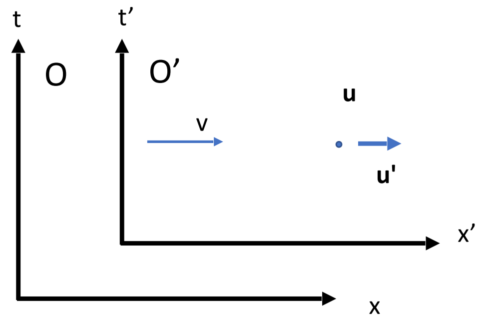
Example: Field generated by a moving charge

In O’ the particle is at rest:

$$\mathbf{E}' = \frac{q}{r'^3}(x', y', z') \qquad r' = \sqrt{x'^2 + y'^2 + z'^2}$$

$$\mathbf{B}' = 0$$

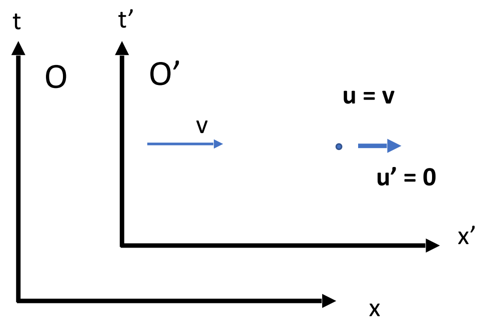
The charge is assumed to sit at the origin of O’.



We compute the fields when the charge passes at the origin of O:

Example: Field generated by a moving charge

In O: We need to express r’ in terms of r:



$$B_x = B'_x \qquad \Rightarrow \qquad B_x = 0$$

$$B_y = \gamma(B'_y - \beta E'_z/c) \qquad \Rightarrow \qquad B_y = -\gamma v E'_z/c^2$$

$$B_z = \gamma(B'_z - \beta E'_y/c) \qquad \Rightarrow \qquad B_z = \gamma v E'_y/c^2$$

$$E_x = E'_x \qquad \Rightarrow \qquad E_x = qx'/r'^3 = \gamma q x/r'^3$$

$$E_y = \gamma(E'_y + vB'_z) \qquad \Rightarrow \qquad E_y = \gamma E'_y = \gamma q y/r'^3$$

$$E_z = \gamma(E'_z - vB'_y) \qquad \Rightarrow \qquad E_z = \gamma E'_z = \gamma q z/r'^3$$

$$\mathbf{E} = \gamma q(x, y, z)/r'^3$$

$$\mathbf{B} = \mathbf{v} \times \mathbf{E}/c^2$$

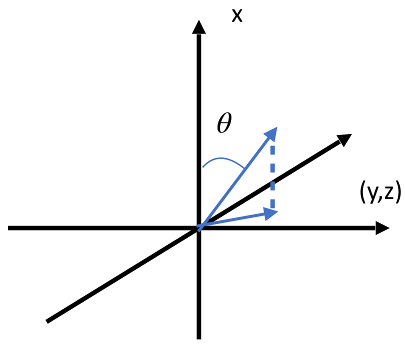
Example: Field generated by a moving charge

We need to express r' in terms of r:

$$r'^2 = \gamma^2 x^2 + y^2 + z^2 =$$

$$= \gamma^2 r^2 - (\gamma^2 - 1)(y^2 + z^2) =$$

$$= \gamma^2 r^2 [1 - (v^2/c^2)\sin^2\theta]$$



In conclusion:

$$\boxed{\mathbf{E} = \frac{q\mathbf{r}}{\gamma^2 r^3 [1 - (v^2/c^2)\sin^2\theta]^{3/2}}}$$

$$\boxed{\mathbf{B} = \frac{\mathbf{v} \times \mathbf{E}}{c^2}}$$