Quantum Foundations
2 – Bohmian Mechanics

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Completing Quantum Theory

There is a curious analogy between Quantum Mechanics and classical probability theory.

States. Quantum Mechanics assign a wave function to physical systems, probability theory a probability distribution.

Dynamics. In both cases it is linear: the Schrödinger equation in the first case, the Liouville equation in the second case.

Measurements. In both cases outcomes are random, and after a measurement the state changes suddenly: for Quantum Mechanics it is von Neumann's collapse, for probability theory it is the update of information.

[METTERE MODELLO DI QUBIT "CLASSICO", CON ANCHE STATI "ENTANGLED"]

In this light, it becomes quite natural to consider **the wave function as expressing our knowledge about the state of the system**, while the true state of the system is given by other variables, which for historical reasons have been called hidden variables.

In this new framework, macroscopic superpositions are not problematic anymore, because they simply refer to our ignorance about the state of the system: Schrödinger's cat is always dead or alive, even is its wave function is in a superposition of the two states. The collapse of the wave function is not problematic anymore: it is not a physical process, it only amounts to the change of our knowledge when we make a measurement. One can hope to reconstruct Quantum Mechanics without the measurement problem.

(See ref [1-8] of https://arxiv.org/pdf/1111.3328.pdf)

The formal structure of a hidden variable model we consider is the following.

There are extra variables λ which represent the true state of the system. In a classical analogy with a gas, they would be the position and velocities of all particles of the gas. They cannot be controlled with arbitrary precision.

When we prepare a system in a quantum state ψ, what we mean is that the hidden variables are distributed with some probability

$$p_{HV}(\lambda|\psi)$$

which depends in general on the state being prepared. It is analogous to the classical case of an ideal gas at thermal equilibrium, prepared for example with some definite temperature.

Suh a probability distribution characterizes the observer's knowledge of the system.

Measurements outcome are uniquely identified once the hidden variables are given. We consider only the case of deterministic models (nothing changes for a stochastic model). Assume that only discrete outcomes a_n are possible; the measurement outcome is denoted as

$$O_{HV}(M|\lambda) = a_n$$

Then the probability of obtaining outcome a_n is given by:

$$\mathbb{P}^M_{\mathrm{HV}}(a_n|\psi) = \int_{\Lambda_n} d\lambda \, p_{\mathrm{HV}}(\lambda|\psi)$$
$$\text{with: } \Lambda_n = \{\lambda : O_{\mathrm{HV}}(M|\lambda) = a_n\}$$

The hidden variable model reproduces Quantum Mechanics if:

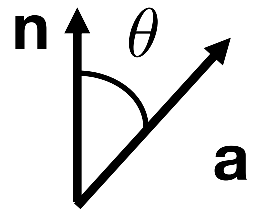
$$\mathbb{P}^M_{\mathrm{HV}}(a_n|\psi) = \mathbb{P}^M_{\mathrm{QM}}(a_n|\psi)$$

For any conceivable measurement M.

We give an **example** for a single ½ spin particle proposed by Bell (From "On the Einstein-Podolski-Rosen paradox", Physics 1, 195 (1964).)

First, some formulas regarding 1/2 spin states. By |n> we mean that the spin is along the positive direction n.

Suppose the initial spin is along direction **n**:



$$|\mathbf{n}\rangle = \cos\frac{\theta}{2}|\mathbf{a}\rangle + \sin\frac{\theta}{2}|-\mathbf{a}\rangle$$

Then the probabilities of finding the particle with spin up or down along direction a are:

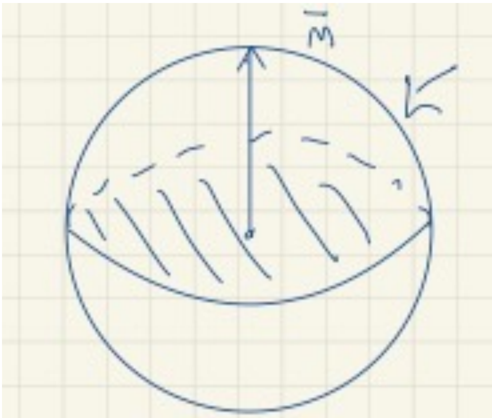
$$\mathbb{P}_{\rm QM}(+|\mathbf{a},\mathbf{n}) = \cos^2\frac{\theta}{2} \qquad \mathbb{P}_{\rm QM}(-|\mathbf{a},\mathbf{n}) = \sin^2\frac{\theta}{2}$$

And the expectation value of the spin measurement along direction a is:

$$\begin{aligned} \langle\sigma\cdot\mathbf{a}\rangle &\equiv \mathbb{P}_{\rm QM}(+|\mathbf{a},\mathbf{n}) - \mathbb{P}_{\rm QM}(-|\mathbf{a},\mathbf{n}) \\ &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta \end{aligned}$$

It is easy to construct a hidden variable model for this situation. Suppose the hidden variables **λ** form a unit vector on the 3D unit sphere. To prepare the spin in the quantum state |**n**> means that the hidden variables are uniformly distributed on the hemisphere **λ** · **n** > 0:

$$p_{\rm HV}(\lambda|\mathbf{n}) = \begin{cases} \frac{1}{2\pi} & \text{for } \lambda\cdot\mathbf{n} > 0 \\ 0 & \text{else} \end{cases}$$



Hemisphere on which λ has a non vanishing uniform disitrbution

The measurement interaction is such that the outcome of a spin **measurement** along direction **a** (which can be either +1 or -1) is **uniquely identifie**d by the relation:

$$O_{\text{HV}}(\mathbf{a}|\lambda,\mathbf{n}) = \text{sign}\lambda \cdot \mathbf{a}' = \pm 1$$

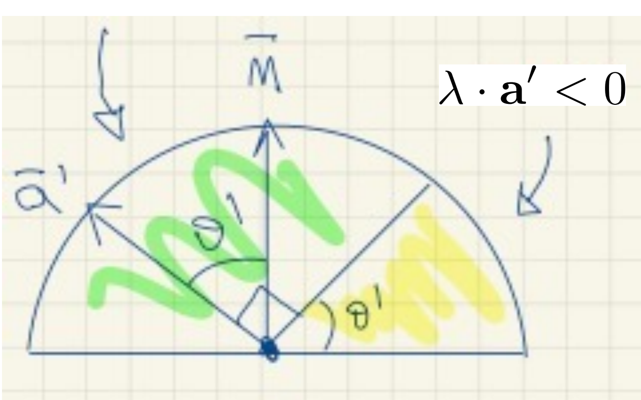
where **a**' is another unit vector (coplanar to **n** and **a**) forming and angle θ' with respect to **n** such that

$$\theta' = \pi \sin^2 \frac{\theta}{2}$$

Note that, according to the previous definition, **the hidden variable is both λ and the state vector**; this is an not accidental, as we will see. The probability of getting outcome -1 in a spin measurement along direction **a** is:

$$\begin{aligned} \mathbb{P}^{\mathbf{a}}_{\text{HV}}(-|\mathbf{n}) &= \int_{\Lambda_-} d\lambda\, p_{\text{HV}}(\lambda|\mathbf{n}), \qquad \text{with: } \Lambda_- = \{\lambda : O_{\text{HV}}(\mathbf{a}|\lambda,\mathbf{n}) = -1\} \\ &= \frac{2\theta'}{2\pi} = \sin^2\frac{\theta}{2} = \mathbb{P}^{\mathbf{a}}_{\text{QM}}(-|\mathbf{n}) \end{aligned}$$

$\lambda \cdot \mathbf{a}' > 0$



Area of the yellow spherical lune (fuso sferico) = 2θ'

A similar result holds for the probability of a positive outcome: the full statistics of measurement outcomes for a 1/2 spin particle has been recovered by averaging procedure over the distribution of the hidden variables.

Bell: “So in this simple case there is no difficulty in the view that the result of every measurement is determined by the value of an extra variable, and that the statistical features of quantum mechanics arise because the value of this variable in unknown”.

Note that, to recover quantum probabilities of successive measurement, i.e. the collapse of the wave function, the measurement process has to change the distribution of the hidden variables. Measurements are always invasive in a quantum context.

However, this probabilistic view of Quantum Mechanics is flawed: the wave function cannot be considered simply as expressing our ignorance about the true state of the system. The reason is the following.

Consider a classical particle in a box, whose position and momentum are unknown. We consider two observable quantities:

- The energy E of the particle.
- Whether as particle is or is not in a sub-volume V of the box.

There is a clear sense in which E in a property of the system, independent of our ignorance, while being or not being in V does depend on our ignorance. In fact, suppose that we prepare the particle so that it has definite energy E. Then the distribution $p_{CL}(x,p|E)$ of position and momenta (the hidden variables, in this case) is uniform over all points in phase space such that H(x,p) = E, and zero elsewhere. Let us call:

$$\mathcal{D}_E = \{(x,p) : p_{\mathrm{CL}}(x,p|E) \neq 0\}$$

Then:

$$E \neq E' \;\Rightarrow\; \mathcal{D}_E \cap \mathcal{D}_{E'} = \emptyset$$

This means that position and momentum identify uniquely the energy, which the is a property of the system. In the same sense, temperature of a gas is real, or pressure.

The same does not occur with the volume occupied by the particle: if $V \cap V' \neq \phi$, then obviously also the domains of the associated probability distributions and not disjoint. And in fact the position and momentum of the particle do not uniquely identify the sub-volume occupied by the particle: a particle can well be in two different (intersecting) sub-volumes of the box simultaneously.

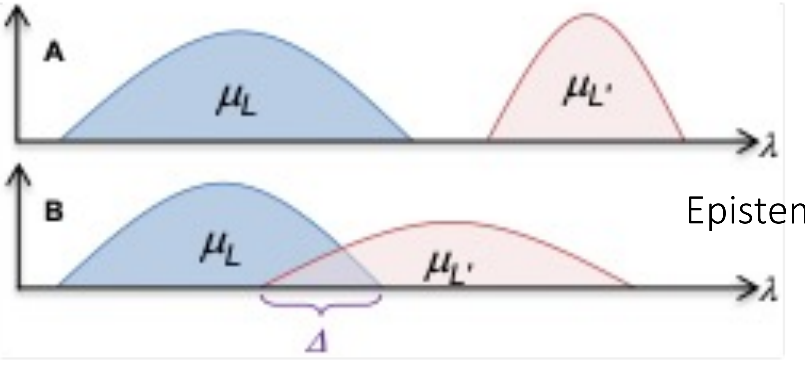
(rivedere def 4 e 5 di https://arxiv.org/pdf/0706.2661.pdf)

To summarize, given a property O, if

$$O \neq O' \;\Rightarrow\; \mathcal{D}_O \cap \mathcal{D}_{O'} = \emptyset$$

We say that O is a real (ontic, non-epistemic) property of the system. If the implication fails, then O is a subjective (epistemic) property.

Non-epistemic property



Epistemic property

We can transfer this to the quantum case. Given the wave function ψ, we supplement it with additional variables, historically called λ. The idea is that when a quantum system is prepared in a state ψ, the hidden variables – which we cannot control – are distributed with some probability $p_\psi(\lambda)$.

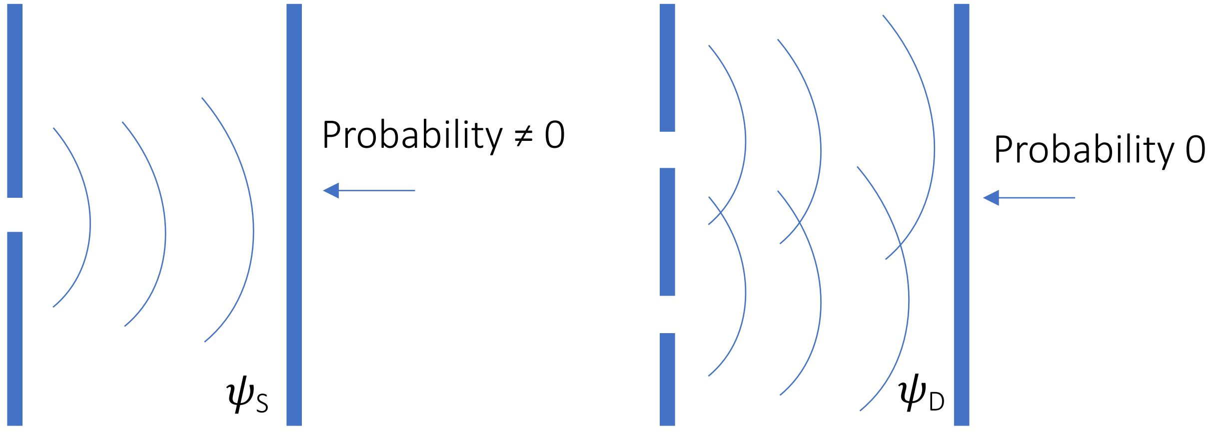
Similar to the classical case, we say that the wave function is real if

$$\psi \neq \phi \;\Rightarrow\; \mathcal{D}_\psi \cap \mathcal{D}_\phi = \emptyset$$

this means that the hidden variables uniquely identify the quantum state, which is then a real property of the system, like the energy. If this does not happen, the state is epistemic

We show that ψ is ontic, according to the definition above (see https://arxiv.org/pdf/1111.3328.pdf)

Consider a quantum particle and two preparation procedures: a single slit and a double slit. Let ψ_S be the wave function of the particle right before detection at the screen, assuming that before it passed through a single slit; similarly, let ψ_D be the wave function assuming that before it passed through the double slit.



There is (more than) one point on the screen where, if the state is ψ_S there is a non-zero probability of finding the particle, while if the state is ψ_D the probability is zero because of the interference between the two terms of the wave function.

Assume that ψ is epistemic, meaning that with some probability p the value of the hidden variables λ are such that they can refer to either ψ_S or ψ_D. In those cases, the measuring device is uncertain which of the four possible preparation methods was used, and on these occasions it runs the risk of giving an outcome that quantum theory predicts should occur with probability 0.

Then the wave function is real. This poses a problem, because it does not live in real space, but in configuration space (or more generally in Hilbert space). We will come to that.

The conclusion is that **the wave function** is not a mere manifestation of our ignorance about the true state of the system; it **is part of the fundamental description of the system**.

This is ultimately the idea of **de Broglie**: to a particle (having a definite position - the hidden variable) is associated a wave which conditions its motion. The wave is real, it is not a statistical effect.

This is reason why in Bell's example, the state vector was part of the hidden variables.

Bohmian Mechanics

States. The state of a system of N particles is described by its wave function $\psi = \psi(\mathbf{q}_1,\ldots,\mathbf{q}_N) = \psi(q)$, together with its actual configuration Q defined by the actual positions $\mathbf{Q}_1,\ldots,\mathbf{Q}_N$ of its particles.

Dynamics. The theory is then defined by two evolution equations: Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

for $\psi(t)$, where H is the nonrelativistic (Schrödinger) Hamiltonian, containing the masses of the particles and a potential energy term, and a first-order evolution equation for the actual configuration Q(t),

$$\frac{d\mathbf{Q}_k}{dt} = \frac{\hbar}{m_k}\mathrm{Im}\frac{\psi^*\partial_k\psi}{\psi^*\psi}(\mathbf{Q}_1,\ldots,\mathbf{Q}_N)$$

All the rest follows from these two axioms.

Example 1. Consider a free particle of mass m in a state given by a plane wave (though technically it is not a proper state)

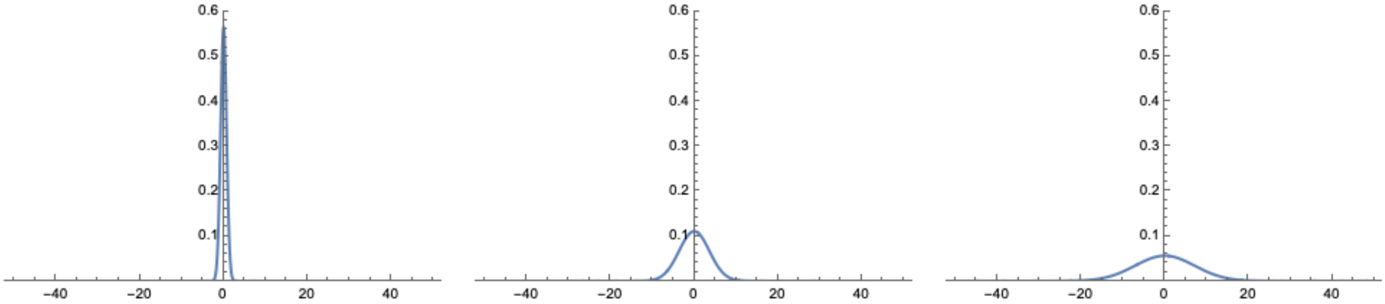
$$\psi(\mathbf{x},t) \propto e^{i(\mathbf{p}\cdot\mathbf{x}-\omega t)/\hbar} \qquad \omega = \mathbf{p}^2/2m$$

Then

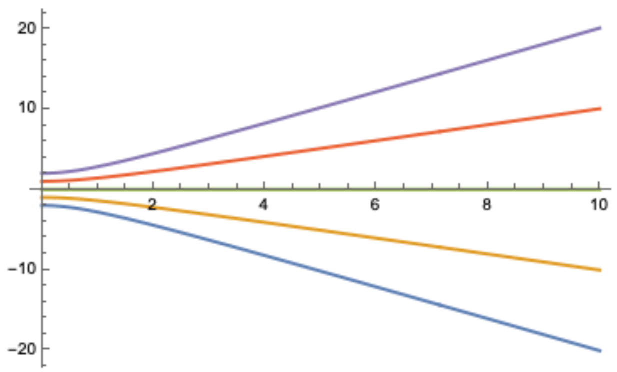
$$\dot{\mathbf{Q}}(t) = \frac{\hbar}{m}\mathrm{Im}\left[\frac{\nabla\psi(\mathbf{x},t)}{\psi(\mathbf{x},t)}\right] = \frac{\mathbf{p}}{m} \;\rightarrow\; \mathbf{Q}(t) = \mathbf{Q}(0) + \frac{\mathbf{p}}{m}t$$

Like for a classical particle

Example 2. A less trivial case is when the particle is initially in a Gaussian state.

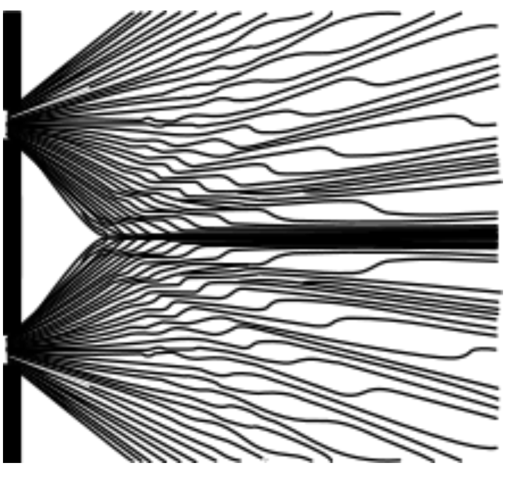


The Gaussian state expands over time.



The evolution of the trajectories depend on the initial condition. Although the particle is free, it does not move along a straight line. It is guided by the wave function, which expands over time, and carries the particle along.

Example 3. Even more interesting is the case of a superposition of two Gaussian states, which would be generated for example a the double slit experiment.



The Bohmian version of the double slit experiment shows that it perfectly legitimate to say that the particle has a definite position at any time, and that it passes only through only one slit, as any particle would do.

Determinism vs indeterminism: The Born rule. BM is a *deterministic* theory: once the initial positions of the particles and the initial wave function is known, their later values are uniquely determined. Indeterminism arise because of ignorance about the initial positions of the particles. The proof is the following.

Classical Hamiltonian system

The Hamiltonian H(q,p) guides the motion. q = {q_i, i = 1, … 3N} and p = {p_i, i = 1, … 3N}:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

The equations define a vector field in *phase space,* generated by the Hamiltonian H:

$$\mathbf{v}_{\mathrm{H}}(q,p) \equiv \left\{ \begin{array}{c} \dot{q} \\ \dot{p} \end{array} \right\} = \left\{ \begin{array}{c} \partial H/\partial p \\ -\partial H/\partial q \end{array} \right\}$$

which generates the *Hamiltonian flow*

$$T_t^{\mathrm{H}} \text{ such that: } (q_t, p_t) = T_t^{\mathrm{H}}(q,p)$$

Bohmian Mechanics

The wave function ψ(q, t) guides the motion of the particles. q = {q_i, i = 1, … 3N} :

$$\dot{Q}_i = \frac{\hbar}{m}\,\mathrm{Im}\frac{\partial_i \psi(q,t)}{\psi(q,t)}\bigg|_{q=Q(t)}$$

The equations define a vector field in *configuration space,* generate by the wave function ψ

$$\mathbf{v}_{\psi}(q,t) = \frac{\hbar}{m}\mathrm{Im}\frac{\nabla\psi(q,t)}{\psi(q,t)}$$

which generates the *Bohmian flow*

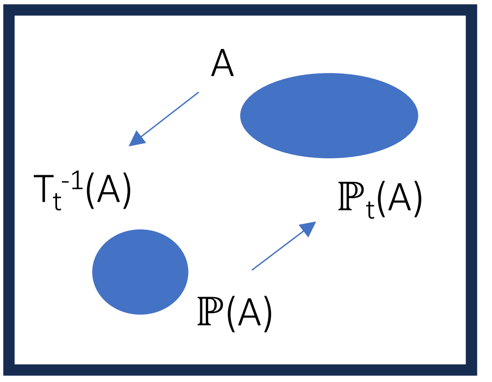
$$T_t^{\psi} \text{ such that: } Q(t) = T_t^{\mathrm{H}}(Q)$$

Let us consider e vector field not depending explicitly on time; the proof does not change if it does depend also on time. The flow generated by the vector field is solution of the equation

$$\frac{dT_t}{dt} = v(T_t), \quad T_0 = I$$

Consider a measure $\mathbb{P}$ on Ω = phase space or configuration space, which admits a density ρ(ω). Let us consider

$$\mathbb{P}_t(A) \equiv \mathbb{P}(T_t^{-1}(A)) \equiv \mathbb{P}(\{\omega | T_t(\omega) \in A\})$$



ρ(ω,t) is the density associated to $\mathbb{P}_t$. It satisfies the *continuity equation,* as we now show.

The above definition can be generalized to

$$\int f(\omega) d\mathbb{P}_t(\omega) = \int f(T_t(\omega)) d\mathbb{P}(\omega)$$

where f is a test function. If f = 1_A, then it reduces to the previous relation. Having $\mathbb{P}$ a density, we can also write

$$\int f(\omega)\rho(\omega,t) d\omega = \int f(T_t(\omega))\rho(\omega) d\omega$$

We take the time derivative

$$\begin{aligned}
\int f(\omega)\frac{\partial \rho(\omega,t)}{\partial t} d\omega &= \int \frac{dT_t(\omega)}{dt} \cdot \nabla f(T_t(\omega))\rho(\omega) d\omega && \text{Time derivative} \\
&= \int v(T_t(\omega)) \cdot \nabla f(T_t(\omega))\rho(\omega) d\omega && \text{Last Eq. page 12} \\
&= \int v(\omega) \cdot \nabla f(\omega)\rho(\omega,t) d\omega && \text{3rd Eq. page 13} \\
&= -\int (\nabla \cdot v(\omega)) f(\omega)\rho(\omega,t) d\omega && \text{Integration by parts}
\end{aligned}$$

We arrive at the continuity equation

$$\frac{\partial \rho(\omega,t)}{\partial t} + (\nabla \cdot v(\omega))\rho(\omega,t) = 0$$

This equation must be satisfied by the density of any probability measure. In the case of Bohmian Mechanics, let us consider

$$\rho_\psi(q,t) := |\psi(q,t)|^2$$

Then the continuity equation for the wave function ψ(q,t) solution of the Schrödinger equation coincides with the continuity equation for the density ρ_ψ. This means **that if particles' positions are initially distributed as |ψ(q,0)|^2, then at any later time they will be distributed as |ψ(q,t)|^2.**

Quantum equilibrium hypothesis: particles are initially distributed as |ψ(q,0)|^2.

This makes sure that the **predictions of Bohmian Mechanics are empirically equivalent to those of standard Quantum Mechanic**s, as long as measurement outcomes can be reduced to position measurements.

At this level, the wave function plays a **double role**:

- Dynamical: it guides the particles' motion.
- Statistical: it defines the probability distribution of particles' position.

This calls for an explanation. It will be provided later in terms of typicality.

The collapse of the wave function. In BM there is no collapse of the wave function. Then one has to justify why the system's wave function unavoidably changes during a measurement and seems to collapse as dictated by the von Neumann projection postulate.

The answer is in entanglement and the conditional wave function. For simplicity consider a universe made of 2 particles, with positions X_t and Y_t. Let ψ(x,y, t) be the wave function for the whole system, solution of the Schrödinger equation for a given initial condition.

BM naturally allows to associate a wave function to just one of the two particles, sat that at X_t, which is the **conditional wave function**

$$\phi_Y(x,t) := \psi(x, Y_t, t)$$

which is nothing by ψ(x,y,t) conditioned on the other particle being at Y_t.

The important facts about the conditional wave function are:

1. **It guides the motion of the particle** at X_t since, according to the guiding equation:

$$\frac{dX_t}{dt} = \frac{\hbar}{m}\mathrm{Im}\left.\frac{\nabla_x\psi(x,y,t)}{\psi(x,y,t)}\right|_{x=X_t,y=Y_t} = \frac{\hbar}{m}\mathrm{Im}\left.\frac{\nabla_x\phi_Y(x,t)}{\phi_Y(x,t)}\right|_{x=X_t}$$

So in this sense it **is the wave function of the subsystem** composed of one of the two particles. Its knowledge suffices to determine the motion of the particle

2. It is not normalized. To normalize, just take $\psi_Y(x,t) = \dfrac{\phi_Y(x,t)}{\|\phi_Y(x,t)\|}$

3. In general, it satisfies a **nonlinear differential equation** with respect to time, due to the time dependence of Y_t, which is also **random** since Y_t itself is a random variable (because of the initial conditions)

In general, it is very difficult to write explicitly the random differential equation satisfied by the conditional wave function. Two cases of interest can be easily understood.

Suppose that the **two particles are intendent from each other**, in the sense that the initial composite wave function is factorized and the Hamiltonian does not contain interaction terms. Then:

$$\psi(x,y,0) = \psi_1(x,0) \otimes \psi_2(y,0) \;\rightarrow\; \psi(x,y,t) = \psi_1(x,t) \otimes \psi_2(y,t)$$

where ψ_1 and ψ_1 are solution of the 1-particle Schrödinger equation. In this case **the normalized conditional wave function $\psi_Y(x,t)$ coincides with $\psi_1(x,t)$**, which satisfies the linear Schrödinger equation. The motion of the first particle is insensitive of the other one.

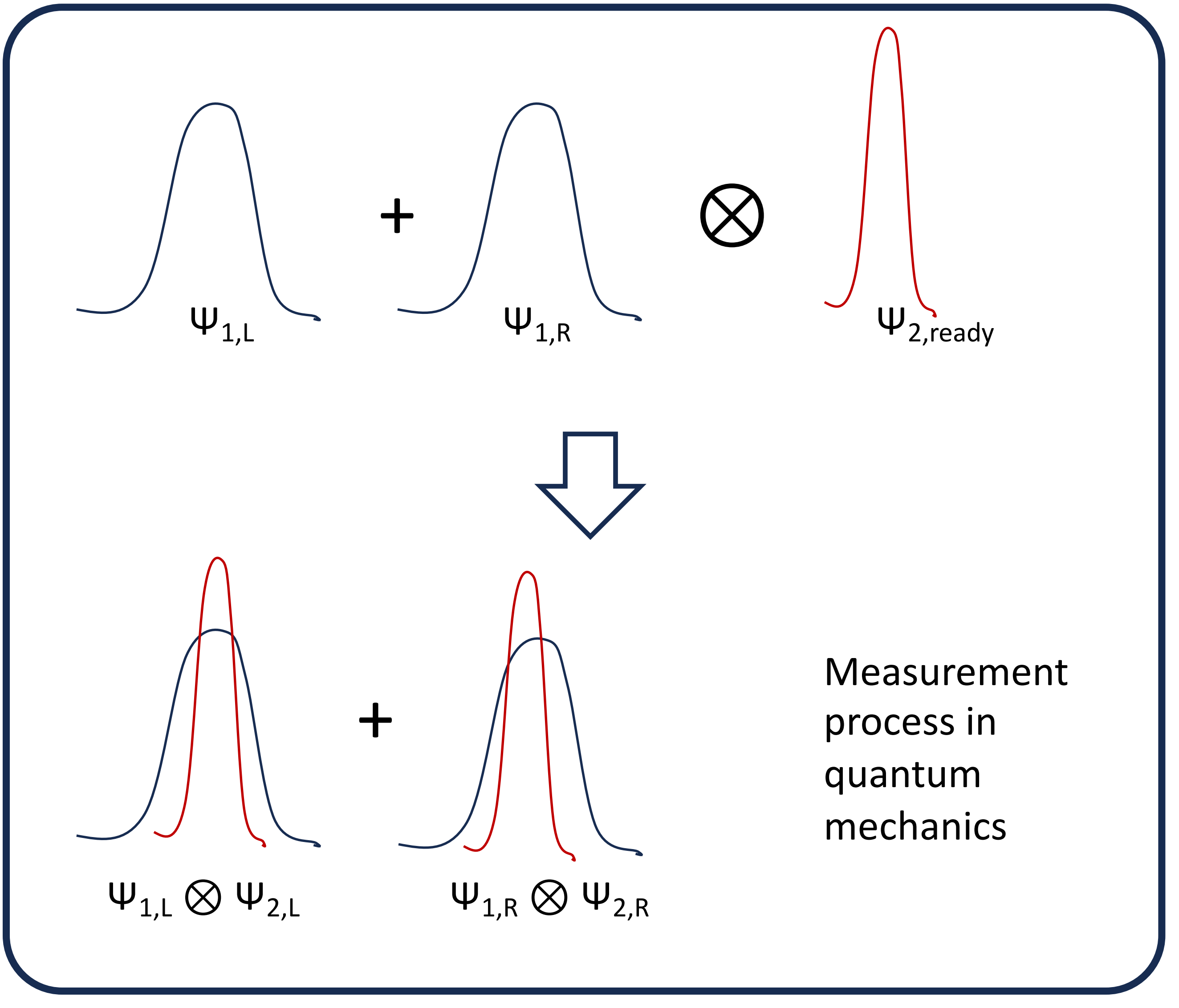
Opposite to this case, suppose that we have a **measurement situation**. The particle at X is the quantum systems, and that at Y is the pointer of a measurement device. The initial wave function is

$$\psi(x,y,0) = [\alpha_{\mathrm{L}}\psi_{1,\mathrm{L}}(x) + \alpha_{\mathrm{R}}\psi_{1,\mathrm{R}}(x)] \otimes \psi_{2,\mathrm{ready}}(y)$$

Meaning that the particle's initial wave function is in a superposition of a left and right state (well separated), while the pointer's wave function is in a ready state. The measurement interaction is such that the wave function changes, after some time, to

$$\psi(x,y,0) \rightarrow \psi(x,y,t) = \alpha_{\mathrm{L}}\psi_{1,\mathrm{L}}(x) \otimes \psi_{2,\mathrm{L}}(y) + \alpha_{\mathrm{R}}\psi_{1,\mathrm{R}}(x) \otimes \psi_{2,\mathrm{L}}(y)$$

which is an entangled state, where the pointer-s wave function is correlated to the particle's wave function.



Suppose that the initial conditions are such that at the end of the measurement Y belongs to the support of $\Psi_{2,L}$, which is assumed to be disjoint from that of $\Psi_{2,R}$. Then the conditiona wave function for the particle, given that the point is on the left, is

$$\phi_{Y\in L}(x,t) = \alpha_{\mathrm{L}}\psi_{1,\mathrm{L}}(x)\psi_{2,\mathrm{L}}(Y)$$

which upon normalization becomes

$$\psi_{Y\in L}(x,t) = \psi_{1,\mathrm{L}}(x)$$

The particles conditional wave function, which initially was in a superposition of two different states, has collapses depending on the pointer's position, in accordance to the von Neumann collapse postulate.

We can compute the probability for Y to belong to L, the support of $\Psi_{2,L}$. This is:

$$\begin{aligned}\mathbb{P}[Y \in L] &= \int_{Y\in L} dY \int_{\mathbb{R}} dX |\psi(X,Y,t)|^2 \\ &= |\alpha_{\mathrm{L}}|^2 \int_{Y\in L} dY |\psi_{2,\mathrm{L}}(Y)|^2 \int_{\mathbb{R}} dX |\psi_{1,\mathrm{L}}(X)|^2 \\ &= |\alpha_{\mathrm{L}}|^2\end{aligned}$$

which is the born rule. The whole phenomenology of quantum experiments is recovered.

In BM, **the conditional wave function behaves exactly like the system's wave function of standard quantum mechanics, both when the system is isolated (Schrödinger's evolution) and when it is subject to a measurement (collapse of the wave function).**

The difference is that this double behavior is not postulated: it is derived by considering the conditional wave function as ”part” of the global wave function, which always satisfies the Schrödinger's equation and never collapses.

SUMMARY

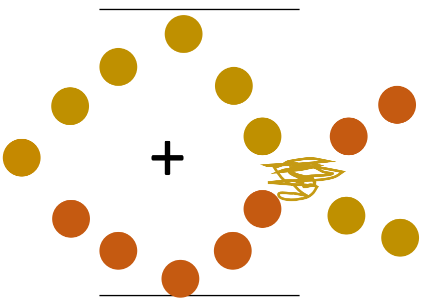
1. In BM the wave function of isolated systems evolves according to the Schrödinger's equation and particles are guided in their motion by the wave function.

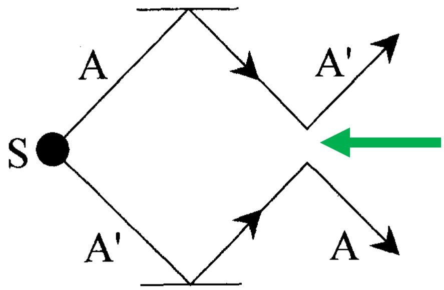
2. During (ideal) measurements (treating the device also as a quantum system, and conditioning e.g. over the position of its pointer, which makes the outcome of the measurement) the system's (conditional) wave function collapses randomly as prescribed by the von Neumann projection postulate, and with a probability given by the Born rule.

The Born rule and the collapse of the wave function are explained within the theory, not assumed.

Nonlocality in BM. This is a manifestly nonlocal theory. This should be quite evident, but it can be explicitly seen with the following example.

First of all, we notice that **in BM, trajectories in configuration space do not cross**.



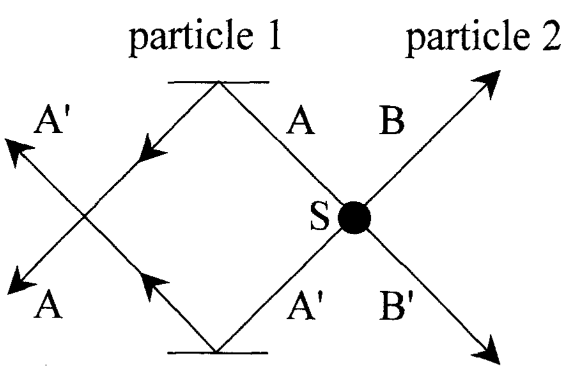


Interferernce of the two parts of the wave function, makes the trajectories bounch away from each other

If they did, there would be two values of the velocity field at that point, which cannot be. Then, consider the following initially entangled state:

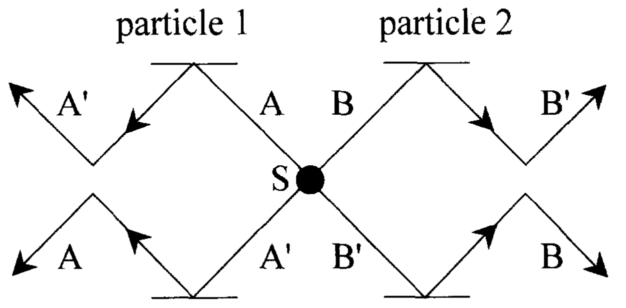
$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle_1|B\rangle_2 + |A'\rangle_1|B'\rangle_2\right).$$

with two mirrors of the left, which deflect the motion of the particle. The state evolves as follows



To respect the correlation, the trajectories must evolve es indicated in the picture, with an “intersection” on the left. This is not a problem, because this is a representation in 3D space; in configuration space there is no crossing.

Suppose now we place similar mirrors on the right. Trajectories must evolve as follows:



They cannot cross on both sides, otherwise it would amount to a crossing in configuration space. Therefore, to keep the quantum correlations, they can cross in neither side.

We see that the effect of having placed mirrors in the right has altered the trajectories on the left, no matter how far the two sides are. It is Bell's nonlocality fully displayed.

This poses a problem in connection to special relativity, which still has to be resolved.